MIDPOINT AND TRAPEZOID INEQUALITIES FOR DIFFERENTIABLE FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. In this paper we establish some midpoint and trapezoid norm inequalities for Gâteaux and Fréchet differentiable functions of selfadjoint operators in Hilbert spaces. Some examples for the class of functions

$$\mathcal{D}^{(1)}(0,\infty) := \left\{ f \mid \|Df(A)\| = \left\|f'(A)\right\| \text{ for all positive operators } A \right\},$$

where $Df(A)$ is the Fréchet derivative in A and $f'(A)$ is the operator function generated by f' and positive operator A, are also given. The case when f' is nonnegative and operator convex is also analyzed.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \le (\ge) (1-\lambda) f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \le B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \le f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [7] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \le r \le 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \le r \le 2$ or $-1 \le r \le 0$ and is operator concave on $(0,\infty)$ if $0 \le r \le 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

In [4] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f : I \to \mathbb{R}$

$$f\left(\frac{A+B}{2}\right) \le \int_0^1 f((1-s)A + sB)\,ds \le \frac{f(A)+f(B)}{2}, \tag{1.2}$$

where A, B are selfadjoint operators with spectra included in I.

For recent inequalities for operator convex functions see [1]-[6] and [8]-[17].

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Let $\mathcal{SA}_I(H)$ be the class of all selfadjoint operators with spectra in I. If A, $B \in \mathcal{SA}_I(H)$ and $t \in [0,1]$ the convex combination $(1-t)A + tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ is convex in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. If f is continuous function on I. By the continuous functional calculus of selfadjoint operator we conclude that $f((1-t)A + tB)$ is a selfadjoint operator with spectrum in I.

A continuous function $f : \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla f_A(B) := \lim_{s \to 0} \frac{f(A + sB) - f(A)}{s} \in \mathcal{B}(H). \tag{1.3}$$

If the limit (1.3) exists for all $B \in \mathcal{B}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $f \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $f \in \mathcal{G}(\mathcal{S})$.

If f is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators A, $B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A, B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$

We observe that A, $B \in [A, B]$ and $[A, B] \subset \mathcal{SA}_I(H)$.

In the recent paper [6] we obtained the following reverses of operator Hemite-Hadamard inequalities:

Theorem 1. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A, B])$, then*

$$\begin{aligned} 0 &\leq \int_0^1 f((1-t)A + tB)\,dt - f\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} 0 &\leq \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB)\,dt \\ &\leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \end{aligned} \tag{1.5}$$

Corollary 1. *With the assumption of Theorem 1 we have the norm inequalities*

$$\begin{aligned} &\left\| \int_0^1 f((1-t)A + tB)\,dt - f\left(\frac{A+B}{2}\right) \right\| \\ &\leq \frac{1}{8}\left\|\nabla f_B(B-A) - \nabla f_A(B-A)\right\| \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} &\left\| \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB)\,dt \right\| \\ &\leq \frac{1}{8}\left\|\nabla f_B(B-A) - \nabla f_A(B-A)\right\|. \end{aligned} \tag{1.7}$$

Motivated by the above results, we establish in this paper some midpoint and trapezoid norm inequalities for Gâteaux and Fréchet differentiable functions of selfadjoint operators. Some examples for the class of functions

$$\mathcal{D}^{(1)}(0,\infty) := \{f \mid \|Df(A)\| = \|f'(A)\| \text{ for all positive operators } A\},$$

where $Df(A)$ is the Fréchet derivative in A and $f'(A)$ is the operator function generated by f' and positive operator A, are also given. Finally, the case when f' is nonnegative and operator convex is also analyzed.

2. Midpoint Inequalities

We need the following preliminary results:

Lemma 1. *Let f be a continuous function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A). \tag{2.1}$$

Also we have for the lateral derivative that

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A) \tag{2.2}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A). \tag{2.3}$$

Proof. Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\begin{aligned} &\frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ &= \frac{f((1-t-h)A + (t+h)B) - f((1-t)A+tB)}{h} \\ &= \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A+tB)}{h}. \end{aligned} \tag{2.4}$$

Since $f \in \mathcal{G}([A,B])$, hence by taking the limit over $h \to 0$ in (2.4) we get

$$\begin{aligned} \varphi'_{(A,B)}(t) &= \lim_{h\to 0} \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ &= \lim_{h\to 0} \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A+tB)}{h} \\ &= \nabla g_{(1-t)A+tB}(B-A), \end{aligned}$$

which proves (2.1).

Also, we have

$$\begin{aligned} \varphi'_{(A,B)}(0+) &= \lim_{h\to 0+} \frac{\varphi_{(A,B)}(h) - \varphi_{(A,B)}(0)}{h} \\ &= \lim_{h\to 0+} \frac{f((1-h)A + hB) - f(A)}{h} \\ &= \lim_{h\to 0+} \frac{f(A + h(B-A)) - f(A)}{h} = \nabla f_A(B-A) \end{aligned}$$

since f is assumed to be Gâteaux differentiable in A. This proves (2.2).

The equality (2.3) follows in a similar way. □

We have the following midpoint norm inequality:

Theorem 2. *Let f be a continuous function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then*

$$\left\| \int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) \right\| \leq \int_{1/2}^1 (1-t)\left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt + \int_0^{1/2} t \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt =: M(f;A,B). \tag{2.5}$$

Proof. Using integration by parts formula for the Bochner integral, we have

$$\int_0^{1/2} t\varphi'_{(A,B)}(t)\,dt = \frac{1}{2}\varphi_{(A,B)}\left(\frac{1}{2}\right) - \int_0^{1/2} \varphi_{(A,B)}(t)\,dt = \frac{1}{2}f\left(\frac{A+B}{2}\right) - \int_0^{1/2} f((1-t)A+tB)\,dt \tag{2.6}$$

and

$$\int_{1/2}^1 (t-1)\varphi'_{(A,B)}(t)\,dt = \frac{1}{2}\varphi_{(A,B)}\left(\frac{1}{2}\right) - \int_{1/2}^1 f((1-t)A+tB)\,dt = \frac{1}{2}f\left(\frac{A+B}{2}\right) - \int_{1/2}^1 f((1-t)A+tB)\,dt. \tag{2.7}$$

If we add these two equalities and use Lemma 1 we get the following identity of interest

$$\int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) = \int_{1/2}^1 (1-t)\nabla f_{(1-t)A+tB}(B-A)\,dt - \int_0^{1/2} t\nabla f_{(1-t)A+tB}(B-A)\,dt. \tag{2.8}$$

Taking the norm and using the properties of the integral, we obtain

$$\begin{aligned} &\left\| \int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) \right\| \\ &\leq \left\| \int_{1/2}^1 (1-t)\nabla f_{(1-t)A+tB}(B-A)\,dt \right\| + \left\| \int_0^{1/2} t\nabla f_{(1-t)A+tB}(B-A)\,dt \right\| \\ &\leq \int_{1/2}^1 (1-t)\left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt + \int_0^{1/2} t\left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt, \end{aligned} \tag{2.9}$$

which proves (2.5). □

Remark 1. *It is well known that if f is a C^1-function defined on an open interval, then the operator function $f(X)$ is Fréchet differentiable and the derivative $Df(A)(B)$ equals the Gâteaux derivative $\nabla f_A(B)$. So for functions f that are of class C^1 on I we have the inequalities*

$$\left\| \int_0^1 f\left((1-t)A+tB\right)dt - f\left(\frac{A+B}{2}\right)\right\| \tag{2.10}$$
$$\le \int_{1/2}^1 (1-t)\left\|Df((1-t)A+tB)(B-A)\right\|dt$$
$$+\int_0^{1/2} t\left\|Df((1-t)A+tB)(B-A)\right\|dt$$
$$\le \|B-A\| \int_{1/2}^1 (1-t)\left\|Df((1-t)A+tB)\right\|dt$$
$$+\|B-A\|\int_0^{1/2} t\left\|Df((1-t)A+tB)\right\|dt,$$

for $A,\ B\in \mathcal{SA}_I(H)$.

Corollary 2. *With the assumptions of Theorem 2 and if*

$$\sup_{t\in[0,1]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|<\infty,$$

then

$$\left\| \int_0^1 f\left((1-t)A+tB\right)dt - f\left(\frac{A+B}{2}\right)\right\| \tag{2.11}$$
$$\le \frac{1}{8}\left(\sup_{t\in[0,1/2]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\| + \sup_{t\in[1/2,1]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|\right)$$
$$\le \frac{1}{4}\sup_{t\in[0,1]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|.$$

Proof. We have

$$M(f;A,B) \le \sup_{t\in[0,1/2]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|\int_{1/2}^1 (1-t)\,dt$$
$$+\sup_{t\in[1/2,1]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|\int_0^{1/2} tdt$$
$$=\frac{1}{8}\sup_{t\in[0,1/2]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\| + \frac{1}{8}\sup_{t\in[1/2,1]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|$$
$$\le \frac{1}{4}\sup_{t\in[0,1]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|,$$

which proves the desired result. □

Remark 2. *For functions f that are of class C^1 on I we have the inequalities*

$$\left\| \int_0^1 f\left((1-t)A+tB\right)dt - f\left(\frac{A+B}{2}\right) \right\| \tag{2.12}$$
$$\le \frac{1}{8}\|B-A\|$$
$$\times \left(\sup_{t\in[0,1/2]} \|Df((1-t)A+tB)\| + \sup_{t\in[1/2,1]} \|Df((1-t)A+tB)\| \right)$$
$$\le \frac{1}{4}\|B-A\| \sup_{t\in[0,1]} \|Df((1-t)A+tB)\|$$

for $A,\ B\in \mathcal{SA}_I(H)$.

Corollary 3. *With the assumptions of Theorem 2 and if*

$$\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| dt < \infty,$$

then

$$\left\| \int_0^1 f\left((1-t)A+tB\right)dt - f\left(\frac{A+B}{2}\right) \right\| \le \frac{1}{2}\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| dt. \tag{2.13}$$

Proof. We have

$$M(f;A,B) \le \max_{t\in[0,1/2]}(1-t)\int_{1/2}^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| dt$$
$$+ \max_{t\in[1/2,1]} t \int_0^{1/2} \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| dt$$
$$= \frac{1}{2}\int_{1/2}^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| dt + \frac{1}{2}\int_0^{1/2} \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| dt$$
$$= \frac{1}{2}\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| dt,$$

which proves (2.13). □

Remark 3. *For functions f that are of class C^1 on I we have the inequalities*

$$\left\| \int_0^1 f\left((1-t)A+tB\right)dt - f\left(\frac{A+B}{2}\right) \right\| \le \frac{1}{2}\|B-A\| \int_0^1 \|Df((1-t)A+tB)\| dt \tag{2.14}$$

for $A,\ B\in \mathcal{SA}_I(H)$.

Corollary 4. *With the assumptions of Theorem 2 and if*

$$\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt < \infty,\ p>1,$$

then for $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *we have*

$$\begin{aligned}(2.15)\quad & \left\| \int_0^1 f\left((1-t)A+tB\right)dt - f\left(\frac{A+B}{2}\right)\right\| \\ & \le \frac{1}{2}\left(\frac{1}{2(q+1)}\right)^{1/q} \\ & \times\left[\left(\int_{1/2}^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p} + \left(\int_0^{1/2} \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p}\right] \\ & \le \frac{1}{2}\left(\frac{1}{q+1}\right)^{1/q}\left(\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p}.\end{aligned}$$

Proof. Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\begin{aligned} & M(f;A,B) \\ & \le \left(\int_{1/2}^1 (1-t)^q dt\right)^{1/q}\left(\int_{1/2}^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p} \\ & + \left(\int_0^{1/2} t^q dt\right)^{1/q}\left(\int_0^{1/2} \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p} \\ & = \left(\frac{\left(\frac{1}{2}\right)^{q+1}}{q+1}\right)^{1/q}\left(\int_{1/2}^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p} \\ & + \left(\frac{\left(\frac{1}{2}\right)^{q+1}}{q+1}\right)^{1/q}\left(\int_0^{1/2} \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p} \\ & = \frac{1}{2}\left(\frac{1}{2(q+1)}\right)^{1/q}\left(\int_{1/2}^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p} \\ & + \frac{1}{2}\left(\frac{1}{2(q+1)}\right)^{1/q}\left(\int_0^{1/2} \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p},\end{aligned}$$

which proves the first inequality in (2.15).

Using the power mean inequality for $p>1$

$$\left(\frac{a+b}{2}\right)^p \le \frac{a^p+b^p}{2},\ a,\ b>0,$$

namely

$$a+b\le 2^{\frac{p-1}{p}}\left(a^p+b^p\right)^{1/p},$$

we get

$$\left(\int_{1/2}^{1}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^{p}dt\right)^{1/p}+\left(\int_{0}^{1/2}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^{p}dt\right)^{1/p}$$
$$\leq 2^{\frac{p-1}{p}}\left[\int_{1/2}^{1}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^{p}dt+\int_{0}^{1/2}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^{p}dt\right]^{1/p}$$
$$=2^{\frac{p-1}{p}}\left(\int_{0}^{1}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^{p}dt\right)^{1/p}=2^{1/q}\left(\int_{0}^{1}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^{p}dt\right)^{1/p},$$

which proves the last part of (2.15). □

Remark 4. *For functions f that are of class C^1 on I we have the inequalities*

$$\left\|\int_{0}^{1}f((1-t)A+tB)\,dt-f\left(\frac{A+B}{2}\right)\right\| \tag{2.16}$$
$$\leq\frac{1}{2}\left(\frac{1}{2(q+1)}\right)^{1/q}\|B-A\|$$
$$\times\left[\left(\int_{1/2}^{1}\|Df((1-t)A+tB)\|^{p}dt\right)^{1/p}+\left(\int_{0}^{1/2}\|Df((1-t)A+tB)\|^{p}dt\right)^{1/p}\right]$$
$$\leq\frac{1}{2}\left(\frac{1}{q+1}\right)^{1/q}\|B-A\|\left(\int_{0}^{1}\|Df((1-t)A+tB)\|^{p}dt\right)^{1/p}$$

for $A,\ B\in\mathcal{SA}_I(H)$.

3. Trapezoid Inequalities

We have the following trapezoid norm inequality:

Theorem 3. *Let f be a continuous function on I and $A,\ B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then*

$$\left\|\frac{f(A)+f(B)}{2}-\int_{0}^{1}f((1-t)A+tB)\,dt\right\| \tag{3.1}$$
$$\leq\int_{0}^{1}\left|t-\frac{1}{2}\right|\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt=:T(f;A,B).$$

Proof. Using integration by parts formula for the Bochner integral, we have

$$\int_{0}^{1}\left(t-\frac{1}{2}\right)\varphi_{(A,B)}'(t)\,dt=\left.\left(t-\frac{1}{2}\right)\varphi_{(A,B)}(t)\right|_{0}^{1}-\int_{0}^{1}\varphi_{(A,B)}(t)\,dt$$
$$=\frac{1}{2}\left[\varphi_{(A,B)}(1)+\varphi_{(A,B)}(1)\right]-\int_{0}^{1}\varphi_{(A,B)}(t)\,dt,$$

which gives the following equality of interest

$$\frac{f(A)+f(B)}{2}-\int_{0}^{1}f((1-t)A+tB)\,dt \tag{3.2}$$
$$=\int_{0}^{1}\left(t-\frac{1}{2}\right)\nabla f_{(1-t)A+tB}(B-A)\,dt.$$

By taking the norm in (3.2) we get

$$\begin{aligned}&\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\|\\&\le\int_0^1\left\|\left(t-\frac{1}{2}\right)\nabla f_{(1-t)A+tB}(B-A)\right\|dt\\&=\int_0^1\left|t-\frac{1}{2}\right|\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt,\end{aligned}$$

which proves the desired inequality (3.1). □

Remark 5. *For functions f that are of class C^1 on I we have the inequalities*

$$\begin{aligned}&\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\|\\&\le\int_0^1\left|t-\frac{1}{2}\right|\left\|Df((1-t)A+tB)(B-A)\right\|dt\\&\le\|B-A\|\int_0^1\left|t-\frac{1}{2}\right|\left\|Df((1-t)A+tB)\right\|dt\end{aligned}\tag{3.3}$$

for $A,\ B\in\mathcal{SA}_I(H)$.

Corollary 5. *With the assumptions of Theorem 3 and if*

$$\sup_{t\in[0,1]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|<\infty,$$

then

$$\begin{aligned}&\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\|\\&\le\frac{1}{8}\left(\sup_{t\in[0,1/2]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|+\sup_{t\in[1/2,1]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|\right)\\&\le\frac{1}{4}\sup_{t\in[0,1]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|.\end{aligned}\tag{3.4}$$

Proof. We have

$$\begin{aligned}T(f;A,B)&=\int_0^1\left|t-\frac{1}{2}\right|\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt\\&=\int_0^{1/2}\left(\frac{1}{2}-t\right)\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt\\&+\int_{1/2}^1\left(t-\frac{1}{2}\right)\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt\\&\le\sup_{t\in[0,1/2]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|\int_0^{1/2}\left(\frac{1}{2}-t\right)dt\\&+\sup_{t\in[1/2,1]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|\int_{1/2}^1\left(t-\frac{1}{2}\right)\\&=\frac{1}{8}\left(\sup_{t\in[0,1/2]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|+\sup_{t\in[1/2,1]}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|\right),\end{aligned}$$

which proves (3.4). □

Remark 6. *For functions f that are of class C^1 on I we have the inequalities*

$$\begin{aligned}(3.5)\quad &\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\| \\ &\le \frac{1}{8}\|B-A\| \\ &\times\left(\sup_{t\in[0,1/2]}\|Df((1-t)A+tB)\|+\sup_{t\in[1/2,1]}\|Df((1-t)A+tB)\|\right) \\ &\le \frac{1}{4}\|B-A\|\sup_{t\in[0,1]}\|Df((1-t)A+tB)\|\end{aligned}$$

for $A,\ B\in\mathcal{SA}_I(H)$.

Corollary 6. *With the assumptions of Theorem 3 and if*

$$\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt<\infty,$$

then

$$\begin{aligned}(3.6)\quad &\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\| \\ &\le \frac{1}{2}\int_0^1\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt.\end{aligned}$$

Proof. If follows by the fact that

$$\begin{aligned}T(f;A,B) &= \int_0^1\left|t-\frac{1}{2}\right|\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt \\ &\le \frac{1}{2}\int_0^1\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt.\end{aligned}$$

□

Remark 7. *For functions f that are of class C^1 on I we have the inequalities*

$$\begin{aligned}(3.7)\quad &\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\| \\ &\le \frac{1}{2}\int_0^1\|Df((1-t)A+tB)(B-A)\|\,dt \\ &\le \frac{1}{2}\|B-A\|\int_0^1\|Df((1-t)A+tB)\|\,dt.\end{aligned}$$

Corollary 7. *With the assumptions of Theorem 3 and if*

$$\int_0^1\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt<\infty,\ \ p>1,$$

then for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\| \tag{3.8}$$
$$\le \frac{1}{2}\left(\frac{1}{2(q+1)}\right)^{1/q}$$
$$\times\left[\left(\int_{1/2}^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p}+\left(\int_0^{1/2} \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p}\right]$$
$$\le \frac{1}{2}\left(\frac{1}{q+1}\right)^{1/q}\left(\int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p}.$$

Proof. Let $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$T(f;A,B)=\int_0^{1/2}\left(\frac{1}{2}-t\right)\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt$$
$$+\int_{1/2}^1\left(t-\frac{1}{2}\right)\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt$$
$$\le\left(\int_{1/2}^1\left(t-\frac{1}{2}\right)^q dt\right)^{1/q}\left(\int_{1/2}^1\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p}$$
$$+\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)^q dt\right)^{1/q}\left(\int_0^{1/2}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p}$$
$$=\left(\frac{\left(\frac{1}{2}\right)^{q+1}}{q+1}\right)^{1/q}\left(\int_{1/2}^1\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p}$$
$$+\left(\frac{\left(\frac{1}{2}\right)^{q+1}}{q+1}\right)^{1/q}\left(\int_0^{1/2}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p}$$
$$=\frac{1}{2}\left(\frac{1}{2(q+1)}\right)^{1/q}\left(\int_{1/2}^1\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p}$$
$$+\frac{1}{2}\left(\frac{1}{2(q+1)}\right)^{1/q}\left(\int_0^{1/2}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^p dt\right)^{1/p},$$

which proves the first inequality in (3.8). The second part is obvious by Corollary 4. □

Remark 8. *For functions f that are of class C^1 on I we have the inequalities*

$$\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\| \tag{3.9}$$
$$\le \frac{1}{2}\left(\frac{1}{2(q+1)}\right)^{1/q}\|B-A\|$$
$$\times\left[\left(\int_{1/2}^1 \|Df((1-t)A+tB)\|^p\,dt\right)^{1/p}+\left(\int_0^{1/2}\|Df((1-t)A+tB)\|^p\,dt\right)^{1/p}\right]$$
$$\le \frac{1}{2}\left(\frac{1}{q+1}\right)^{1/q}\|B-A\|\left(\int_0^1\|Df((1-t)A+tB)\|^p\,dt\right)^{1/p}$$

for A, $B\in\mathcal{SA}_I(H)$.

4. Examples for Some General Classes of Functions

Let f be a real function that is n-time differentiable on $(0,\infty)$, and let $f^{(n)}$ be its n-th derivative. Let f also denote the map induced by f on positive operators. Let $D^n f(A)$ be the n-th order Fréchet derivative of this map at the point A. For each A, the derivative $D^n f(A)$ is a n-linear operator on the space of all Hermitian operators. The norm of this operator is defined as

$$\|D^n f(A)\| := \sup\{D^n f(A)(B_1,...,B_n) \mid \|B_1\|=...=\|B_n\|=1\}.$$

We consider the following class of functions defined on $(0,\infty)$ for a natural $n\ge 1$,

$$\mathcal{D}^{(n)}(0,\infty) := \left\{f \mid \|D^n f(A)\| = \left\|f^{(n)}(A)\right\| \text{ for all positive operators } A\right\}.$$

It is known (see for instance [8]) that every operator monotone function is in $\mathcal{D}^{(n)}(0,\infty)$ for all $n=1,2,....$ Also the functions $f(t)=t^n$, $n=2,3,...,$ and $f(t)=\exp t$ are in $\mathcal{D}^{(1)}(0,\infty)$. None of these are operator monotone. Moreover, the power function $f(t)=t^p$ is in $\mathcal{D}^{(1)}(0,\infty)$ if p is in $(-\infty,1]$ or in $[2,\infty)$, but not if p is in $\left(1,\sqrt{2}\right)$. Also that the functions $f(t)=\exp t$ and $f(t)=t^p$, $-\infty<p\le 1$, are in the class $\mathcal{D}^{(n)}(0,\infty)$ for all $n=1,2,...,$ and that for $p>1$ the function $f(t)=t^p$ is in the class $\mathcal{D}^{(n)}(0,\infty)$ for all $n\ge[p+1]$, where $[\cdot]$ is the integer part (see for instance [8] and the references therein).

Proposition 1. *If $f\in\mathcal{D}^{(1)}(0,\infty)$ and A, $B>0$, then we have midpoint inequality*

$$\left\|\int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right)\right\| \tag{4.1}$$
$$\le \|B-A\|\int_{1/2}^1 (1-t)\|f'((1-t)A+tB)\|\,dt$$
$$+\|B-A\|\int_0^{1/2} t\|f'((1-t)A+tB)\|\,dt,$$

and the trapezoid inequality

$$\left\| \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \right\| \leq \|B-A\| \int_0^1 \left| t-\frac{1}{2} \right| \|f'((1-t)A+tB)\|\,dt. \tag{4.2}$$

The proof follows by Remarks 1 and 5.

It is known that if A and B are commuting, i.e. $AB = BA$, then the exponential function satisfies the property

$$\exp(A)\exp(B) = \exp(B)\exp(A) = \exp(A+B).$$

Also, if A is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tA)\,dt = A^{-1}[\exp(bA) - \exp(aA)].$$

Moreover, if A and B are commuting and $B-A$ is invertible, then

$$\begin{aligned} \int_0^1 \exp((1-s)A+sB)\,ds &= \int_0^1 \exp(s(B-A))\exp(A)\,ds \\ &= \left(\int_0^1 \exp(s(B-A))\,ds \right) \exp(A) \\ &= (B-A)^{-1}[\exp(B-A) - I]\exp(A) \\ &= (B-A)^{-1}[\exp(B) - \exp(A)]. \end{aligned}$$

If we write the inequalities (4.1) and (4.2) for the exponential function, then we get the midpoint inequality

$$\begin{aligned} &\left\| \int_0^1 \exp((1-t)A+tB)\,dt - \exp\left(\frac{A+B}{2}\right) \right\| \\ &\leq \|B-A\| \int_{1/2}^1 (1-t)\|\exp((1-t)A+tB)\|\,dt \\ &+ \|B-A\| \int_0^{1/2} t\|\exp((1-t)A+tB)\|\,dt, \end{aligned} \tag{4.3}$$

and the trapezoid inequality

$$\left\| \frac{\exp(A)+\exp(B)}{2} - \int_0^1 \exp((1-t)A+tB)\,dt \right\| \leq \|B-A\| \int_0^1 \left| t-\frac{1}{2} \right| \|\exp((1-t)A+tB)\|\,dt \tag{4.4}$$

for all $A, B > 0$.

If A and B are commuting and $B-A$ is invertible, then

$$\left\|(B-A)^{-1}[\exp(B)-\exp(A)]-\exp\left(\frac{A+B}{2}\right)\right\| \tag{4.5}$$
$$\leq\|B-A\|\int_{1/2}^{1}(1-t)\|\exp((1-t)A+tB)\|\,dt$$
$$+\|B-A\|\int_{0}^{1/2}t\|\exp((1-t)A+tB)\|\,dt,$$

and

$$\left\|\frac{\exp(A)+\exp(B)}{2}-(B-A)^{-1}[\exp(B)-\exp(A)]\right\| \tag{4.6}$$
$$\leq\|B-A\|\int_{0}^{1}\left|t-\frac{1}{2}\right|\|\exp((1-t)A+tB)\|\,dt.$$

Since, using the power series representation,

$$\begin{aligned}\|\exp((1-t)A+tB)\| &\leq \exp(\|(1-t)A+tB\|)\\ &\leq \exp((1-t)\|A\|+t\|B\|)\end{aligned}$$

for all selfadjoint operators A, B and $t\in[0,1]$, we have for $\|B\|\neq\|A\|$ that

$$\int_{1/2}^{1}(1-t)\|\exp((1-t)A+tB)\|\,dt$$
$$\leq\int_{1/2}^{1}(1-t)\exp((1-t)\|A\|+t\|B\|)\,dt$$
$$=\frac{\exp(\|B\|)-\exp\left(\frac{\|A\|+\|B\|}{2}\right)}{(\|B\|-\|A\|)^2}-\frac{\exp\left(\frac{\|A\|+\|B\|}{2}\right)}{2(\|B\|-\|A\|)},$$

$$\int_{0}^{1/2}t\|\exp((1-t)A+tB)\|\,dt$$
$$\leq\int_{0}^{1/2}t\exp((1-t)\|A\|+t\|B\|)\,dt$$
$$=\frac{\exp\left(\frac{\|A\|+\|B\|}{2}\right)}{2(\|B\|-\|A\|)}-\frac{\exp\left(\frac{\|A\|+\|B\|}{2}\right)-\exp(\|A\|)}{(\|B\|-\|A\|)^2}.$$

These imply that

$$\int_{1/2}^{1}(1-t)\|\exp((1-t)A+tB)\|\,dt$$
$$+\int_{0}^{1/2}t\|\exp((1-t)A+tB)\|\,dt$$
$$\leq\frac{\exp(\|B\|)-2\exp\left(\frac{\|A\|+\|B\|}{2}\right)+\exp(\|A\|)}{(\|B\|-\|A\|)^2}$$

and by (4.5) we get

$$\left\| (B-A)^{-1} [\exp(B) - \exp(A)] - \exp\left(\frac{A+B}{2}\right) \right\| \le \|B-A\| \frac{\exp(\|B\|) - 2\exp\left(\frac{\|A\|+\|B\|}{2}\right) + \exp(\|A\|)}{(\|B\| - \|A\|)^2} \tag{4.7}$$

if A and B are selfadjoint, commuting and $B-A$ is invertible with $\|B\| \neq \|A\|$.

Also, we have

$$\begin{aligned}
&\int_0^1 \left|t-\frac{1}{2}\right| \|\exp((1-t)A + tB)\| \, dt \\
&\le \int_0^1 \left|t-\frac{1}{2}\right| \exp((1-t)\|A\| + t\|B\|) \, dt \\
&= \frac{4\exp\left(\frac{\|A\|+\|B\|}{2}\right) + (\|B\| - \|A\| - 2)\exp(\|B\|) - (\|B\| - \|A\| + 2)\exp(\|A\|)}{2(\|B\| - \|A\|)^2} \\
&= \frac{\exp(\|B\|) - \exp(\|A\|)}{2(\|B\| - \|A\|)} + \frac{4\exp\left(\frac{\|A\|+\|B\|}{2}\right) - 2(\exp(\|B\|) + \exp(\|A\|))}{2(\|B\| - \|A\|)^2} \\
&= \frac{\exp(\|B\|) - \exp(\|A\|)}{2(\|B\| - \|A\|)} - \frac{\exp(\|B\|) - 2\exp\left(\frac{\|A\|+\|B\|}{2}\right) + \exp(\|A\|)}{(\|B\| - \|A\|)^2}
\end{aligned}$$

for $\|B\| \neq \|A\|$.

Using (4.6) we get

$$\begin{aligned}
&\left\| \frac{\exp(A) + \exp(B)}{2} - (B-A)^{-1} [\exp(B) - \exp(A)] \right\| \\
&\le \|B-A\| \left[\frac{\exp(\|B\|) - \exp(\|A\|)}{2(\|B\| - \|A\|)} \right. \\
&\left. - \frac{\exp(\|B\|) - 2\exp\left(\frac{\|A\|+\|B\|}{2}\right) + \exp(\|A\|)}{(\|B\| - \|A\|)^2} \right]
\end{aligned} \tag{4.8}$$

if A and B are selfadjoint, commuting and $B-A$ is invertible with $\|B\| \neq \|A\|$.

If $f \in \mathcal{D}^{(1)}(0,\infty)$ and $A, B > 0$, then we observe that all the inequalities from Remarks 2-4 and Remarks 6-8 hold for f' instead of Df. We do not state them here.

However, if more assumptions are made, the inequalities (4.1) and (4.2) provide some other inequalities as well.

Corollary 8. *If $f \in \mathcal{D}^{(1)}(0,\infty)$ and f' is operator convex and nonnegative on $(0,\infty)$ then for $A, B > 0$, we have the midpoint inequality*

$$\left\| \int_0^1 f((1-t)A + tB) \, dt - f\left(\frac{A+B}{2}\right) \right\| \le \frac{1}{8} \|B-A\| (\|f'(A)\| + \|f'(B)\|) \tag{4.9}$$

and the trapezoid inequality

$$\left\| \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \right\| \le \frac{1}{8}\|B-A\|(\|f'(A)\|+\|f'(B)\|). \tag{4.10}$$

Proof. Since f' is operator convex and nonnegative on $(0,\infty)$ then for A, $B>0$ we have

$$0 \le f'((1-t)A+tB) \le (1-t)f'(A)+tf'(B)$$

for $t\in[0,1]$. By taking the norm, we get

$$\|f'((1-t)A+tB)\| \le \|(1-t)f'(A)+tf'(B)\| \le (1-t)\|f'(A)\|+t\|f'(B)\|$$

for $t\in[0,1]$.

Therefore,

$$\begin{aligned}
&\int_{1/2}^1 (1-t)\|f'((1-t)A+tB)\|\,dt + \int_0^{1/2} t\|f'((1-t)A+tB)\|\,dt\\
&\le \int_{1/2}^1 (1-t)[(1-t)\|f'(A)\|+t\|f'(B)\|]\,dt\\
&+\int_0^{1/2} t[(1-t)\|f'(A)\|+t\|f'(B)\|]\,dt\\
&=\|f'(A)\|\int_{1/2}^1 (1-t)^2\,dt+\|f'(B)\|\int_{1/2}^1(1-t)\,tdt\\
&+\|f'(A)\|\int_0^{1/2} t(1-t)\,dt+\|f'(B)\|\int_0^{1/2}t^2dt\\
&=\frac{1}{24}\|f'(A)\|+\frac{1}{12}\|f'(B)\|+\frac{1}{12}\|f'(A)\|+\frac{1}{24}\|f'(B)\|=\frac{1}{8}(\|f'(A)\|+\|f'(B)\|),
\end{aligned}$$

which, by (4.1), proves the inequality (4.9).

We also have

$$\begin{aligned}
&\int_0^1 \left|t-\frac{1}{2}\right|\|f'((1-t)A+tB)\|\,dt\\
&\le \int_0^1 \left|t-\frac{1}{2}\right|[(1-t)\|f'(A)\|+t\|f'(B)\|]\,dt\\
&=\|f'(A)\|\int_0^1\left|t-\frac{1}{2}\right|(1-t)\,dt+\|f'(B)\|\int_0^1\left|t-\frac{1}{2}\right|tdt=\frac{1}{8}(\|f'(A)\|+\|f'(B)\|),
\end{aligned}$$

which, by (4.1), proves the inequality (4.9). □

Remark 9. *The inequality (4.10) was obtained in* [8].

Consider the function $f(x)=x^r$on $(0,\infty)$, where $0\le r\le 1$ or $2\le r\le 3$. Then from Corollary 8 we get the midpoint inequality

$$\left\|\int_0^1 ((1-t)A+tB)^r\,dt-\left(\frac{A+B}{2}\right)^r\right\| \le \frac{r}{8}\|B-A\|\left(\left\|A^{r-1}\right\|+\left\|B^{r-1}\right\|\right) \tag{4.11}$$

and the trapezoid inequality

$$\left\| \frac{A^r + B^r}{2} - \int_0^1 ((1-t)A + tB)^r \, dt \right\| \le \frac{r}{8} \|B - A\| \left(\left\|A^{r-1}\right\| + \left\|B^{r-1}\right\| \right) \tag{4.12}$$

for A, $B > 0$, see also [8].

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