WEIGHTED TRAPEZOID INEQUALITIES FOR DIFFERENTIABLE FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. In this paper we establish weighted trapezoid norm inequalities for Gâteaux and Fréchet differentiable functions of selfadjoint operators in Hilbert spaces. Some examples for the class of functions

$$\mathcal{D}^{(1)}(0,\infty) := \left\{ f \mid \|Df(A)\| = \left\| f'(A) \right\| \text{ for all positive operators } A \right\},$$

where $Df(A)$ is the Fréchet derivative in A and $f'(A)$ is the operator function generated by f' and positive operator A, are also given. The case when f' is nonnegative and operator convex and the weight is symmetric is also analyzed.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda) f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [8] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

In [4] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f : I \to \mathbb{R}$

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-s)A + sB)\, ds \leq \frac{f(A)+f(B)}{2}, \tag{1.2}$$

where A, B are selfadjoint operators with spectra included in I.

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From the operator convexity of the function f we have

$$f\left(\frac{A+B}{2}\right) \le \frac{1}{2}\left[f\left((1-s)A+sB\right)+f\left(sA+(1-s)B\right)\right] \le \frac{f(A)+f(B)}{2} \tag{1.3}$$

for all $s\in[0,1]$ and A, B selfadjoint operators with spectra included in I.

If $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric in the sense that $p(1-s)=p(s)$ for all $s\in[0,1]$, then by multiplying (1.3) with $p(s)$, integrating on $[0,1]$ and taking into account that

$$\int_0^1 p(s)\, f\left((1-s)A+sB\right)ds=\int_0^1 p(s)\, f\left(sA+(1-s)B\right)ds,$$

we get the weighted version of (1.2) for A, B selfadjoint operators with spectra included in I

$$\left(\int_0^1 p(s)\,ds\right) f\left(\frac{A+B}{2}\right) \le \int_0^1 p(s)\, f\left(sA+(1-s)B\right)ds \le \left(\int_0^1 p(s)\,ds\right)\frac{f(A)+f(B)}{2}, \tag{1.4}$$

which are the operator version of the well known *Féjer's inequalities* for scalar convex functions.

For recent inequalities for operator convex functions see [1]-[6] and [9]-[18].

Let $\mathcal{SA}_I(H)$ be the class of all selfadjoint operators with spectra in I. If A, $B\in\mathcal{SA}_I(H)$ and $t\in[0,1]$ the convex combination $(1-t)A+tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ is convex in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. If f is continuous function on I. By the continuous functional calculus of selfadjoint operator we conclude that $f((1-t)A+tB)$ is a selfadjoint operator with spectrum in I.

A continuous function $f:\mathcal{SA}_I(H)\to\mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A\in\mathcal{SA}_I(H)$ along the direction $B\in\mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla f_A(B):=\lim_{s\to 0}\frac{f(A+sB)-f(A)}{s}\in\mathcal{B}(H). \tag{1.5}$$

If the limit (1.5) exists for all $B\in\mathcal{B}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $f\in\mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $f\in\mathcal{G}(\mathcal{S})$.

If f is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators A, $B\in\mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B]:=\{(1-t)A+tB \mid t\in[0,1]\}.$$

We observe that A, $B\in[A,B]$ and $[A,B]\subset\mathcal{SA}_I(H)$.

In the recent paper [7] we obtained the following reverse of operator Féjer's second inequality:

Theorem 1. *Let f be an operator convex function on I and A, $B\in\mathcal{SA}_I(H)$, with $A\ne B$. If $f\in\mathcal{G}([A,B])$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and* *symmetric, namely $p(1-t) = p(t)$ for all $t \in [0,1]$, then we have the weighted trapezoid operator inequality*

$$0 \le \left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2} - \int_0^1 p(t)\, f((1-t)A+tB)\,dt \le \frac{1}{2}\left(\int_0^1 \left|t-\frac{1}{2}\right| p(t)\,dt\right)\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \tag{1.6}$$

By taking the norm in these inequalities, we get

$$\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2} - \int_0^1 p(t)\, f((1-t)A+tB)\,dt\right\| \le \frac{1}{2}\int_0^1 \left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right) p(t)\,dt\, \left\|\nabla f_B(B-A) - \nabla f_A(B-A)\right\| \tag{1.7}$$

for $A, B \in \mathcal{SA}_I(H)$ and f is an operator convex function on I.

Motivated by the above results, in this paper we establish weighted trapezoid norm inequalities for Gâteaux and Fréchet differentiable functions of selfadjoint operators in Hilbert spaces. Some examples for the class of functions

$$\mathcal{D}^{(1)}(0,\infty) := \{f \mid \|Df(A)\| = \|f'(A)\| \text{ for all positive operators } A\},$$

where $Df(A)$ is the Fréchet derivative in A and $f'(A)$ is the operator function generated by f' and positive operator A, are also given. The case when f' is nonnegative and operator convex and the weight is symmetric is also analyzed.

2. Weighted Trapezoid Inequalities

We need the following preliminary results:

Lemma 1. *Let f be a continuous function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A). \tag{2.1}$$

Also we have for the lateral derivative that

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A) \tag{2.2}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A). \tag{2.3}$$

Proof. Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} = \frac{f((1-t-h)A+(t+h)B) - f((1-t)A+tB)}{h} = \frac{f((1-t)A+tB+h(B-A)) - f((1-t)A+tB)}{h}. \tag{2.4}$$

Since $f\in\mathcal{G}([A,B])$, hence by taking the limit over $h\rightarrow 0$ in (2.4) we get

$$\begin{aligned}\varphi_{(A,B)}^{\prime}(t) &= \lim_{h\rightarrow 0}\frac{\varphi_{(A,B)}(t+h)-\varphi_{(A,B)}(t)}{h}\\ &= \lim_{h\rightarrow 0}\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\\ &= \nabla g_{(1-t)A+tB}(B-A),\end{aligned}$$

which proves (2.1).

Also, we have

$$\begin{aligned}\varphi_{(A,B)}^{\prime}(0+) &= \lim_{h\rightarrow 0+}\frac{\varphi_{(A,B)}(h)-\varphi_{(A,B)}(0)}{h}\\ &= \lim_{h\rightarrow 0+}\frac{f((1-h)A+hB)-f(A)}{h}\\ &= \lim_{h\rightarrow 0+}\frac{f(A+h(B-A))-f(A)}{h}=\nabla f_{A}(B-A)\end{aligned}$$

since f is assumed to be Gâteaux differentiable in A. This proves (2.2).

The equality (2.3) follows in a similar way. □

We have:

Theorem 2. *Let f be a continuous function on I and A, $B\in\mathcal{SA}_{I}(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$ and $p:[0,1]\rightarrow[0,\infty)$ is Lebesgue integrable, then*

$$\begin{aligned}&\left\|\left(\int_{0}^{1}p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_{0}^{1}p(t)\,f((1-t)A+tB)\,dt\right\| \\ &\leq\frac{1}{2}\int_{0}^{1}\left|\int_{0}^{t}p(s)\,ds-\int_{t}^{1}p(s)\,ds\right|\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt \\ &\leq\frac{1}{2}\int_{0}^{1}p(s)\,ds\int_{0}^{1}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt.\end{aligned}\tag{2.5}$$

In particular, for for $p\equiv 1$ we get

$$\begin{aligned}&\left\|\frac{f(A)+f(B)}{2}-\int_{0}^{1}f((1-t)A+tB)\,dt\right\| \\ &\leq\int_{0}^{1}\left|t-\frac{1}{2}\right|\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt \\ &\leq\frac{1}{2}\int_{0}^{1}\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt.\end{aligned}\tag{2.6}$$

Proof. Using the integration by parts for Bochner's integral, we have

$$\begin{aligned}
&\int_0^1 \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt \\
&= \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi_{(A,B)}(t)\Big]_0^1 - \int_0^1 p(t)\,\varphi_{(A,B)}(t)\,dt \\
&= \left(\int_0^1 p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi_{(A,B)}(1) + \left(\frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi_{(A,B)}(0) \\
&- \int_0^1 p(t)\,\varphi_{(A,B)}(t)\,dt \\
&= \left(\frac{1}{2}\int_0^1 p(s)\,ds\right)\left[\varphi_{(A,B)}(0) + \varphi_{(A,B)}(1)\right] \\
&- \int_0^1 p(t)\,\varphi_{(A,B)}(t)\,dt \\
&= \left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2} - \int_0^1 p(t)\,\varphi_{(A,B)}(t)\,dt.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_0^1 \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^1 p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt \\
&= \int_0^1 \left(\int_0^t p(s)\,ds - \frac{1}{2}\int_0^t p(s)\,ds - \frac{1}{2}\int_t^1 p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt \\
&= \frac{1}{2}\int_0^1 \left(\int_0^t p(s)\,ds - \int_t^1 p(s)\,ds\right)\varphi'_{(A,B)}(t)\,dt.
\end{aligned}$$

Then we get the following identity of interest

$$\begin{aligned}
&\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2} - \int_0^1 p(t)\,f((1-t)A+tB)\,dt \\
&= \frac{1}{2}\int_0^1 \left(\int_0^t p(s)\,ds - \int_t^1 p(s)\,ds\right)\nabla f_{(1-t)A+tB}(B-A)\,dt.
\end{aligned} \tag{2.7}$$

Taking the norm and using the properties of the integral, we have

$$\begin{aligned}
&\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2} - \int_0^1 p(t)\,f((1-t)A+tB)\,dt\right\| \\
&\le \frac{1}{2}\int_0^1 \left\|\left(\int_0^t p(s)\,ds - \int_t^1 p(s)\,ds\right)\nabla f_{(1-t)A+tB}(B-A)\right\|dt \\
&= \frac{1}{2}\int_0^1 \left|\int_0^t p(s)\,ds - \int_t^1 p(s)\,ds\right|\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt,
\end{aligned}$$

which proves the first inequality in (2.5).

We also have

$$\begin{aligned}
&\int_0^1 \left|\int_0^t p(s)\,ds - \int_t^1 p(s)\,ds\right| \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| dt \\
&\le \int_0^1 \left[\left|\int_0^t p(s)\,ds\right| + \left|\int_t^1 p(s)\,ds\right|\right] \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| dt \\
&= \int_0^1 \left[\int_0^t p(s)\,ds + \int_t^1 p(s)\,ds\right] \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| dt \\
&= \int_0^1 p(s)\,ds \int_0^1 \left\|\nabla f_{(1-t)A+tB}(B-A)\right\| dt,
\end{aligned}$$

which proves the last part of (2.5). □

Remark 1. *It is well known that, if f is a C^1-function defined on an open interval, then the operator function $f(X)$ is Fréchet differentiable and the derivative $Df(A)(B)$ equals the Gâteaux derivative $\nabla f_A(B)$. So for functions f that are of class C^1 on I we have the inequalities*

$$\begin{aligned}
&\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2} - \int_0^1 p(t) f((1-t)A+tB)\,dt\right\| \\
&\le \frac{1}{2}\|B-A\| \int_0^1 \left|\int_0^t p(s)\,ds - \int_t^1 p(s)\,ds\right| \|Df((1-t)A+tB)\|\,dt \\
&\le \frac{1}{2}\|B-A\| \int_0^1 p(s)\,ds \int_0^1 \|Df((1-t)A+tB)\|\,dt
\end{aligned} \tag{2.8}$$

for $A, B \in \mathcal{SA}_I(H)$.

In particular, for $p \equiv 1$ we get

$$\begin{aligned}
&\left\|\frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt\right\| \\
&\le \|B-A\| \int_0^1 \left|t-\frac{1}{2}\right| \|Df((1-t)A+tB)\|\,dt \\
&\le \frac{1}{2}\|B-A\| \int_0^1 \|Df((1-t)A+tB)\|\,dt
\end{aligned} \tag{2.9}$$

for $A, B \in \mathcal{SA}_I(H)$.

Remark 2. *If p is symmetric, namely $p(1-t)=p(t)$ for all $t\in[0,1]$, then*

$$\int_0^t p(s)\,ds + \int_t^1 p(s)\,ds = \int_0^1 p(s)\,ds = 2\int_0^{1/2} p(s)\,ds,\ \ t\in(0,1),$$

which implies that

$$\begin{aligned}
&\left|\int_0^t p(s)\,ds - \int_t^1 p(s)\,ds\right| \\
&= \left|\int_0^t p(s)\,ds - 2\int_0^{1/2} p(s)\,ds + \int_0^t p(s)\,ds\right| \\
&= 2\left|\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds\right|.
\end{aligned}$$

By the inequality (2.5) we get

$$\begin{aligned}
&\left\| \left(\int_0^1 p(t)\,dt \right) \frac{f(A)+f(B)}{2} - \int_0^1 p(t) f((1-t)A+tB)\,dt \right\| \\
&\le \int_0^1 \left| \int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds \right| \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt \\
&= \int_0^{1/2} \left(\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds \right) \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt \\
&+ \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds \right) \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt.
\end{aligned} \tag{2.10}$$

If f is of class C^1 on I we have the inequalities

$$\begin{aligned}
&\left\| \left(\int_0^1 p(t)\,dt \right) \frac{f(A)+f(B)}{2} - \int_0^1 p(t) f((1-t)A+tB)\,dt \right\| \\
&\le \|B-A\| \int_0^{1/2} \left(\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds \right) \\
&\times \|Df((1-t)A+tB)\|\,dt \\
&+ \|B-A\| \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds \right) \\
&\times \|Df((1-t)A+tB)\|\,dt \\
&= \|B-A\| \int_0^{1/2} \left(\int_t^{1/2} p(s)\,ds \right) \|Df((1-t)A+tB)\|\,dt \\
&+ \|B-A\| \int_{1/2}^1 \left(\int_{1/2}^t p(s)\,ds \right) \|Df((1-t)A+tB)\|\,dt
\end{aligned} \tag{2.11}$$

for $A, B \in \mathcal{SA}_I(H)$. In this inequality, p is also symmetric.

Corollary 1. *With the assumptions of Theorem 2 and if*

$$\sup_{t\in[0,1]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| < \infty,$$

then

$$\begin{aligned}
&\left\| \left(\int_0^1 p(t)\,dt \right) \frac{f(A)+f(B)}{2} - \int_0^1 p(t) f((1-t)A+tB)\,dt \right\| \\
&\le \frac{1}{2} \sup_{t\in[0,1]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| \int_0^1 \left| \int_0^t p(s)\,ds - \int_t^1 p(s)\,ds \right| dt \\
&\le \frac{1}{2} \sup_{t\in[0,1]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| \int_0^1 p(s)\,ds.
\end{aligned} \tag{2.12}$$

In particular, for $p \equiv 1$ we get

$$\left\| \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \right\| \le \frac{1}{4} \sup_{t\in[0,1]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| . \tag{2.13}$$

Remark 3. *If f is of class C^1 on I we have the inequalities*

$$\begin{aligned} &\left\| \left(\int_0^1 p(t)\,dt\right) \frac{f(A)+f(B)}{2} - \int_0^1 p(t) f((1-t)A+tB)\,dt \right\| \\ &\le \frac{1}{2} \sup_{t\in[0,1]} \| Df((1-t)A+tB) \| \int_0^1 \left| \int_0^t p(s)\,ds - \int_t^1 p(s)\,ds \right| dt \\ &\le \frac{1}{2} \|B-A\| \sup_{t\in[0,1]} \| Df((1-t)A+tB) \| \int_0^1 p(s)\,ds \end{aligned} \tag{2.14}$$

for $A,\ B \in \mathcal{SA}_I(H)$.

In particular, for $p \equiv 1$ we get

$$\left\| \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \right\| \le \frac{1}{4} \|B-A\| \sup_{t\in[0,1]} \| Df((1-t)A+tB) \| \tag{2.15}$$

for $A,\ B \in \mathcal{SA}_I(H)$.

The case of symmetric weights is as follows:

Corollary 2. *Let f be a continuous function on I and $A,\ B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric, then*

$$\begin{aligned} &\left\| \left(\int_0^1 p(t)\,dt\right) \frac{f(A)+f(B)}{2} - \int_0^1 p(t) f((1-t)A+tB)\,dt \right\| \\ &\le \frac{1}{2} \int_0^1 \left(\frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t)\,dt \sup_{t\in[0,1/2]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| \\ &+ \frac{1}{2} \int_0^1 \left(\frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t)\,dt \sup_{t\in[1/2,1]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| \\ &\le \int_0^1 \left(\frac{1}{2} - \left| t - \frac{1}{2} \right| \right) p(t)\,dt \sup_{t\in[0,1]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| , \end{aligned} \tag{2.16}$$

provided that

$$\sup_{t\in[0,1]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| < \infty.$$

In particular, for $p \equiv 1$ we get

$$\begin{aligned} &\left\| \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \right\| \\ &\le \frac{1}{8} \left[\sup_{t\in[0,1/2]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| + \sup_{t\in[1/2,1]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| \right] . \end{aligned} \tag{2.17}$$

Proof. From (2.10) we have

$$
\begin{aligned}
(2.18)\quad & \left\| \left(\int_0^1 p(t)\,dt \right) \frac{f(A)+f(B)}{2} - \int_0^1 p(t) f((1-t)A+tB)\,dt \right\| \\
& \le \int_0^{1/2} \left(\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds \right) \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt \\
& + \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds \right) \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt \\
& \le \sup_{t\in[0,1/2]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| \int_0^{1/2} \left(\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds \right) dt \\
& + \sup_{t\in[1/2,1]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds \right) dt \\
& \le \sup_{t\in[0,1]} \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| \\
& \times \left[\int_0^{1/2} \left(\int_0^{1/2} p(s)\,ds - \int_0^t p(s)\,ds \right) \right. \\
& \left. + \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds \right) \right].
\end{aligned}
$$

Now, observe that

$$
\begin{aligned}
& \int_{1/2}^1 \left(\int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds \right) dt \\
& = \int_{1/2}^1 \left(\int_0^t p(s)\,ds \right) dt - \frac{1}{2} \int_0^{1/2} p(s)\,ds \\
& = \left(\int_0^t p(s)\,ds \right) t \Big]_{1/2}^1 - \int_{1/2}^1 t p(t)\,dt - \frac{1}{2} \int_0^{1/2} p(s)\,ds \\
& = \int_0^1 p(s)\,ds - \frac{1}{2} \int_0^{1/2} p(s)\,ds - \int_{1/2}^1 t p(t)\,dt - \frac{1}{2} \int_0^{1/2} p(s)\,ds \\
& = \int_0^1 p(s)\,ds - \int_0^{1/2} p(s)\,ds - \int_{1/2}^1 t p(t)\,dt \\
& = \int_{1/2}^1 p(s)\,ds - \int_{1/2}^1 t p(t)\,dt = \int_{1/2}^1 (1-t) p(t)\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{1/2}\left(\int_0^{1/2} p(s)\,ds-\int_0^{t} p(s)\,ds\right)dt\\
&=\frac{1}{2}\int_0^{1/2} p(s)\,ds-\int_0^{1/2}\left(\int_0^{t} p(s)\,ds\right)dt\\
&=\frac{1}{2}\int_0^{1/2} p(s)\,ds-\left(\left(\int_0^{t} p(s)\,ds\right)t\Big]_0^{1/2}-\int_0^{1/2} tp(t)\,dt\right)\\
&=\frac{1}{2}\int_0^{1/2} p(s)\,ds-\frac{1}{2}\int_0^{1/2} p(s)\,ds+\int_0^{1/2} tp(t)\,dt=\int_0^{1/2} tp(t)\,dt.
\end{aligned}
$$

Also

$$
\begin{aligned}
&\int_0^{1}\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\
&=\int_0^{1/2}\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt+\int_{1/2}^{1}\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\
&=\int_0^{1/2}\left(\frac{1}{2}-\frac{1}{2}+t\right)p(t)\,dt+\int_{1/2}^{1}\left(\frac{1}{2}-t+\frac{1}{2}\right)p(t)\,dt\\
&=\int_0^{1/2} tp(t)\,dt+\int_{1/2}^{1}(1-t)\,p(t)\,dt.
\end{aligned}
$$

If we change the variable $s=1-t$, then

$$
\begin{aligned}
\int_0^{1/2} tp(t)\,dt&=-\int_1^{1/2}(1-s)\,p(1-s)\,ds=\int_{1/2}^{1}(1-s)\,p(1-s)\,ds\\
&=\int_{1/2}^{1}(1-s)\,p(s)\,ds,
\end{aligned}
$$

hence by (2.18) we get the desired result (2.16). $\square$

Remark 4. *If f is of class C^1 on I we have the inequality*

$$
\begin{aligned}
&\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)\,f((1-t)A+tB)\,dt\right\| \qquad (2.19)\\
&\leq\frac{1}{2}\|B-A\|\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\
&\times\sup_{t\in[0,1/2]}\|Df((1-t)A+tB)\|\\
&+\frac{1}{2}\|B-A\|\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\
&\times\sup_{t\in[1/2,1]}\|Df((1-t)A+tB)\|\\
&\leq\|B-A\|\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\sup_{t\in[0,1]}\|Df((1-t)A+tB)\|
\end{aligned}
$$

for $A,\ B\in\mathcal{SA}_I(H)$.

In particular, for $p \equiv 1$ we get

$$\left\| \left(\int_0^1 p(t)\,dt \right) \frac{f(A)+f(B)}{2} - \int_0^1 p(t)\, f((1-t)A+tB)\,dt \right\| \le \frac{1}{8} \|B-A\| \sup_{t\in[0,1/2]} \|Df((1-t)A+tB)\| + \frac{1}{8}\|B-A\| \sup_{t\in[1/2,1]} \|Df((1-t)A+tB)\| \tag{2.20}$$

for $A,\ B \in \mathcal{SA}_I(H)$.

Corollary 3. *Let f be a continuous function on I and $A,\ B \in \mathcal{SA}_I(H)$, with $A \ne B$. If $f \in \mathcal{G}([A,B])$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable, then*

$$\left\| \left(\int_0^1 p(t)\,dt \right) \frac{f(A)+f(B)}{2} - \int_0^1 p(t)\, f((1-t)A+tB)\,dt \right\| \le \frac{1}{2} \sup_{t\in[0,1]} \left| \int_0^t p(s)\,ds - \int_t^1 p(s)\,ds \right| \int_0^1 \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt. \tag{2.21}$$

If p is symmetric in $[0,1]$, then

$$\left\| \left(\int_0^1 p(t)\,dt \right) \frac{f(A)+f(B)}{2} - \int_0^1 p(t)\, f((1-t)A+tB)\,dt \right\| \le \sup_{t\in[0,1]} \left| \int_0^t p(s)\,ds - \int_0^{1/2} p(s)\,ds \right| \int_0^1 \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt. \tag{2.22}$$

In particular, for $p \equiv 1$ we get

$$\left\| \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \right\| \le \frac{1}{2} \int_0^1 \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt. \tag{2.23}$$

Proof. From (2.5) we have

$$\begin{aligned} &\left\| \left(\int_0^1 p(t)\,dt \right) \frac{f(A)+f(B)}{2} - \int_0^1 p(t)\, f((1-t)A+tB)\,dt \right\| \\ &\le \frac{1}{2} \int_0^1 \left| \int_0^t p(s)\,ds - \int_t^1 p(s)\,ds \right| \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt \\ &\le \frac{1}{2} \sup_{t\in[0,1]} \left| \int_0^t p(s)\,ds - \int_t^1 p(s)\,ds \right| \int_0^1 \left\| \nabla f_{(1-t)A+tB}(B-A) \right\| dt, \end{aligned}$$

which proves (2.21). □

Remark 5. *For functions f that are of class C^1 on I we have the inequalities*

$$\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)\,f((1-t)A+tB)\,dt\right\| \tag{2.24}$$
$$\leq \frac{1}{2}\|B-A\|\sup_{t\in[0,1]}\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|$$
$$\times\int_0^1\|Df((1-t)A+tB)\|\,dt$$

for $A,\ B\in\mathcal{SA}_I(H)$.

If p is symmetric in $[0,1]$, then

$$\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)\,f((1-t)A+tB)\,dt\right\| \tag{2.25}$$
$$\leq \|B-A\|\sup_{t\in[0,1]}\left|\int_0^t p(s)\,ds-\int_0^{1/2} p(s)\,ds\right|$$
$$\times\int_0^1\|Df((1-t)A+tB)\|\,dt.$$

In particular, for $p\equiv 1$ we get

$$\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\| \tag{2.26}$$
$$\leq\frac{1}{2}\|B-A\|\int_0^1\|Df((1-t)A+tB)\|\,dt$$

for $A,\ B\in\mathcal{SA}_I(H)$.

We also have:

Corollary 4. *With the assumptions of Theorem 2, we have for $r,\ q>1$ with $\frac{1}{r}+\frac{1}{q}=1$ that*

$$\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)\,f((1-t)A+tB)\,dt\right\| \tag{2.27}$$
$$\leq\frac{1}{2}\left(\int_0^1\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|^r\right)^{1/r}$$
$$\times\left(\int_0^1\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^q dt\right)^{1/q},$$

provided

$$\int_0^1\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^q dt<\infty.$$

If p *is symmetric on* $[0,1]$, *then*

$$\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)\,f((1-t)A+tB)\,dt\right\| \tag{2.28}$$
$$\le\left(\int_0^1\left|\int_0^t p(s)\,ds-\int_0^{1/2}p(s)\,ds\right|^r\right)^{1/r}$$
$$\times\left(\int_0^1\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^q dt\right)^{1/q}.$$

In particular, if $p\equiv 1$, *then we have*

$$\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\| \tag{2.29}$$
$$\le\frac{1}{2}\left(\frac{1}{r+1}\right)^{1/r}\left[\int_0^1\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^q dt\right]^{1/q}.$$

Proof. Using Hölder's integral inequality we have for $r,\ q>1$ with $\frac{1}{r}+\frac{1}{q}=1$ that

$$\int_0^1\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|dt$$
$$\le\left(\int_0^1\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|^r\right)^{1/r}$$
$$\times\left(\int_0^1\left\|\nabla f_{(1-t)A+tB}(B-A)\right\|^q dt\right)^{1/q},$$

which proves (2.28). □

Remark 6. *For functions* f *that are of class* C^1 *on* I *we have for* $r,\ q>1$ *with* $\frac{1}{r}+\frac{1}{q}=1$ *that*

$$\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)\,f((1-t)A+tB)\,dt\right\| \tag{2.30}$$
$$\le\frac{1}{2}\|B-A\|\left(\int_0^1\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|^r\right)^{1/r}$$
$$\times\left(\int_0^1\|Df((1-t)A+tB)\|^q dt\right)^{1/q},$$

for $A,\ B\in\mathcal{SA}_I(H)$.

Corollary 5. *If* p *is symmetric on* $[0,1]$, *then*

$$\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)\,f((1-t)A+tB)\,dt\right\| \tag{2.31}$$
$$\le\|B-A\|\left(\int_0^1\left|\int_0^t p(s)\,ds-\int_0^{1/2}p(s)\,ds\right|^r\right)^{1/r}$$
$$\times\left(\int_0^1\|Df((1-t)A+tB)\|^q dt\right)^{1/q}$$

for $A,\ B\in\mathcal{SA}_I(H)$.

In particular, if $p\equiv 1$*, then we have*

$$\left\Vert \frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\Vert \le \frac{1}{2}\Vert B-A\Vert\left(\frac{1}{r+1}\right)^{1/r}\left[\int_0^1 \Vert Df((1-t)A+tB)\Vert^q\,dt\right]^{1/q} \tag{2.32}$$

for $A,\ B\in\mathcal{SA}_I(H)$.

3. Examples for Some General Classes of Functions

Let f be a real function that is n-time differentiable on $(0,\infty)$, and let $f^{(n)}$ be its n-th derivative. Let f also denote the map induced by f on positive operators. Let $D^n f(A)$ be the n-th order Fréchet derivative of this map at the point A. For each A, the derivative $D^n f(A)$ is a n-linear operator on the space of all Hermitian operators. The norm of this operator is defined as

$$\Vert D^n f(A)\Vert := \sup\{D^n f(A)(B_1,...,B_n)\ \mid\ \Vert B_1\Vert=...=\Vert B_n\Vert=1\}.$$

We consider the following class of functions defined on $(0,\infty)$ for a natural $n\ge 1$,

$$\mathcal{D}^{(n)}(0,\infty):=\left\{f\ \mid\ \Vert D^n f(A)\Vert=\left\Vert f^{(n)}(A)\right\Vert \text{ for all positive operators } A\right\}.$$

It is known (see for instance [9]) that every operator monotone function is in $\mathcal{D}^{(n)}(0,\infty)$ for all $n=1,\ 2,\ldots$. Also the functions $f(t)=t^n$, $n=2,\ 3,\ \ldots$, and $f(t)=\exp t$ are in $\mathcal{D}^{(1)}(0,\infty)$. None of these are operator monotone. Moreover, the power function $f(t)=t^p$ is in $\mathcal{D}^{(1)}(0,\infty)$ if p is in $(-\infty,1]$ or in $[2,\infty)$, but not if p is in $\left(1,\sqrt{2}\right)$. Also that the functions $f(t)=\exp t$ and $f(t)=t^p$, $-\infty<p\le 1$, are in the class $\mathcal{D}^{(n)}(0,\infty)$ for all $n=1,\ 2,\ldots$, and that for $p>1$ the function $f(t)=t^p$ is in the class $\mathcal{D}^{(n)}(0,\infty)$ for all $n\ge[p+1]$, where $[\cdot]$ is the integer part (see for instance [9] and the references therein).

Proposition 1. *If* $f\in\mathcal{D}^{(1)}(0,\infty)$*,* $A,\ B>0$ *and* $p:[0,1]\to[0,\infty)$ *is Lebesgue integrable, then we have midpoint inequality*

$$\begin{aligned}\left\Vert\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)f((1-t)A+tB)\,dt\right\Vert &\le \frac{1}{2}\Vert B-A\Vert\int_0^1\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|\Vert f'((1-t)A+tB)\Vert\,dt\\ &\le \frac{1}{2}\Vert B-A\Vert\int_0^1 p(s)\,ds\int_0^1\Vert f'((1-t)A+tB)\Vert\,dt.\end{aligned} \tag{3.1}$$

In particular, for $p\equiv 1$ *we get*

$$\begin{aligned}\left\Vert\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\Vert &\le \Vert B-A\Vert\int_0^1\left|t-\frac{1}{2}\right|\Vert f'((1-t)A+tB)\Vert\,dt\\ &\le \frac{1}{2}\Vert B-A\Vert\int_0^1\Vert f'((1-t)A+tB)\Vert\,dt.\end{aligned} \tag{3.2}$$

The proof follows by (2.8).

If $f\in\mathcal{D}^{(1)}(0,\infty)$ and $A,\ B>0$, then we observe that all the inequalities from Remarks 2-6 hold for f' instead of Df. For instance, we have from (2.24) that

$$\begin{aligned}&\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)\,f((1-t)A+tB)\,dt\right\| \\ &\leq\frac{1}{2}\|B-A\|\sup_{t\in[0,1]}\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right| \\ &\times\int_0^1\|Df((1-t)A+tB)\|\,dt.\end{aligned}\tag{3.3}$$

provided $f\in\mathcal{D}^{(1)}(0,\infty)$, $A,\ B>0$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable.

If $f=\exp$, then

$$\begin{aligned}\int_0^1\|\exp((1-t)A+tB)\|\,dt&\leq\int_0^1\exp\|((1-t)A+tB)\|\,dt \\ &\leq\int_0^1\exp[(1-t)\|A\|+t\|B\|]\,dt \\ &=\begin{cases}\frac{\exp\|B\|-\exp\|A\|}{\|B\|-\|A\|} & \text{for } \|B\|\neq\|A\|, \\ \\ \exp\|A\| & \text{for } \|B\|=\|A\|\end{cases}\end{aligned}$$

and by (3.3) we get

$$\begin{aligned}&\left\|\left(\int_0^1 p(t)\,dt\right)\frac{\exp A+\exp B}{2}-\int_0^1 p(t)\exp((1-t)A+tB)\,dt\right\| \\ &\leq\frac{1}{2}\|B-A\|\sup_{t\in[0,1]}\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right| \\ &\times\begin{cases}\frac{\exp\|B\|-\exp\|A\|}{\|B\|-\|A\|} & \text{for } \|B\|\neq\|A\|, \\ \\ \exp\|A\| & \text{for } \|B\|=\|A\|,\end{cases}\end{aligned}\tag{3.4}$$

where $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and $A,\ B>0$.

Corollary 6. *If $f\in\mathcal{D}^{(1)}(0,\infty)$ and f' is operator convex and nonnegative on $(0,\infty)$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable, then for $A,\ B>0$, we have the trapezoid inequality*

$$\begin{aligned}&\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)\,f((1-t)A+tB)\,dt\right\| \\ &\leq\frac{1}{2}\|B-A\|\left[\|f'(A)\|\int_0^1\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|(1-t)\,dt\right. \\ &\left.+\|f'(B)\|\int_0^1\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|tdt\right].\end{aligned}\tag{3.5}$$

Proof. Since f' is operator convex and nonnegative on $(0,\infty)$ then for $A,\ B>0$ we have

$$0\leq f'((1-t)A+tB)\leq(1-t)f'(A)+tf'(B)$$

for $t\in[0,1]$. By taking the norm, we get

$$\begin{aligned}\|f'((1-t)A+tB)\| &\le \|(1-t)f'(A)+tf'(B)\|\\ &\le (1-t)\|f'(A)\|+t\|f'(B)\|\end{aligned}$$

for $t\in[0,1]$.

Therefore,

$$\begin{aligned}&\int_0^1\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|\|f'((1-t)A+tB)\|\,dt\\ &\le \int_0^1\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|[(1-t)\|f'(A)\|+t\|f'(B)\|]\,dt\\ &=\|f'(A)\|\int_0^1\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|(1-t)\,dt\\ &+\|f'(B)\|\int_0^1\left|\int_0^t p(s)\,ds-\int_t^1 p(s)\,ds\right|tdt,\end{aligned}$$

which proves (3.5). □

The case of symmetric weights is as follows:

Corollary 7. *If $f\in\mathcal{D}^{(1)}(0,\infty)$ and f' is operator convex and nonnegative on $(0,\infty)$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, then for A, $B>0$, we have the trapezoid inequality*

$$\begin{aligned}&\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)f((1-t)A+tB)\,dt\right\| \\ &\le\frac{1}{2}\|B-A\|(\|f'(A)\|+\|f'(B)\|)\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt.\end{aligned}\tag{3.6}$$

In particular, for $p\equiv1$ we have (see also [9])

$$\begin{aligned}&\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\| \\ &\le\frac{1}{8}\|B-A\|(\|f'(A)\|+\|f'(B)\|).\end{aligned}\tag{3.7}$$

Proof. We have by (2.11) that

$$\begin{aligned}&\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)f((1-t)A+tB)\,dt\right\| \\ &\le\|B-A\|\int_0^{1/2}\left(\int_t^{1/2}p(s)\,ds\right)\|f'((1-t)A+tB)\|\,dt\\ &+\|B-A\|\int_{1/2}^1\left(\int_{1/2}^t p(s)\,ds\right)\|f'((1-t)A+tB)\|\,dt\\ &\le\|B-A\|\int_0^{1/2}\left(\int_t^{1/2}p(s)\,ds\right)[(1-t)\|f'(A)\|+t\|f'(B)\|]\,dt\\ &+\|B-A\|\int_{1/2}^1\left(\int_{1/2}^t p(s)\,ds\right)[(1-t)\|f'(A)\|+t\|f'(B)\|]\,dtdt\end{aligned}\tag{3.8}$$

$$
\begin{aligned}
&= \|B-A\| \left[\|f'(A)\| \int_0^{1/2} \left(\int_t^{1/2} p(s)\, ds \right) (1-t)\, dt \right. \\
&\left. + \|f'(B)\| \int_0^{1/2} \left(\int_t^{1/2} p(s)\, ds \right) t dt \right] \\
&+ \|B-A\| \left[\|f'(A)\| \int_{1/2}^{1} \left(\int_{1/2}^{t} p(s)\, ds \right) (1-t)\, dt \right. \\
&\left. + \|f'(B)\| \int_{1/2}^{1} \left(\int_{1/2}^{t} p(s)\, ds \right) t dt. \right.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\int_0^{1/2} \left(\int_t^{1/2} p(s)\, ds \right) (1-t)\, dt \\
&= -\frac{1}{2} \int_0^{1/2} \left(\int_t^{1/2} p(s)\, ds \right) d\left((1-t)^2 \right) \\
&= -\frac{1}{2} \left[\left(\int_t^{1/2} p(s)\, ds \right) (1-t)^2 \Big|_0^{1/2} + \int_0^{1/2} (1-t)^2\, p(t)\, dt \right] \\
&= -\frac{1}{2} \left[\int_0^{1/2} (1-t)^2\, p(t)\, dt - \int_0^{1/2} p(s)\, ds \right] \\
&= -\frac{1}{2} \left[\int_0^{1/2} \left[(1-t)^2 - 1 \right] p(t)\, dt \right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{1/2}^{1} \left(\int_{1/2}^{t} p(s)\, ds \right) (1-t)\, dt \\
&= -\frac{1}{2} \int_{1/2}^{1} \left(\int_{1/2}^{t} p(s)\, ds \right) d\left((1-t)^2 \right) \\
&= -\frac{1}{2} \left[\left(\int_{1/2}^{t} p(s)\, ds \right) (1-t)^2 \Big|_{1/2}^{1} - \int_{1/2}^{1} (1-t)^2\, p(t)\, dt \right] \\
&= \frac{1}{2} \int_{1/2}^{1} (1-t)^2\, p(t)\, dt = -\frac{1}{2} \int_{1/2}^{0} s^2 p(1-s)\, ds = \frac{1}{2} \int_0^{1/2} s^2 p(s)\, ds.
\end{aligned}
$$

If we add these equalities, we get

$$
\begin{aligned}
&-\frac{1}{2} \left[\int_0^{1/2} \left[(1-t)^2 - 1 \right] p(t)\, dt \right] + \frac{1}{2} \int_0^{1/2} t^2 p(t)\, dt \\
&= \frac{1}{2} \int_0^{1/2} \left(t^2 - (1-t)^2 + 1 \right) p(t)\, dt = \int_0^{1/2} t p(t)\, dt.
\end{aligned}
$$

Also, we have

$$\begin{aligned}
&\int_0^{1/2}\left(\int_t^{1/2} p(s)\,ds\right) tdt \\
&= \frac{1}{2}\int_0^{1/2}\left(\int_t^{1/2} p(s)\,ds\right) d\left(t^2\right) \\
&= \frac{1}{2}\left[\left(\int_t^{1/2} p(s)\,ds\right) t^2\bigg|_0^{1/2} + \int_0^{1/2} t^2 p(t)\,dt\right] = \frac{1}{2}\int_0^{1/2} t^2 p(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_{1/2}^{1}\left(\int_{1/2}^{t} p(s)\,ds\right) tdt \\
&= \frac{1}{2}\int_{1/2}^{1}\left(\int_{1/2}^{t} p(s)\,ds\right) d\left(t^2\right) \\
&= \frac{1}{2}\left[\left(\int_{1/2}^{t} p(s)\,ds\right) t^2\bigg|_{1/2}^{1} - \int_{1/2}^{1} t^2 p(t)\,dt\right] \\
&= \frac{1}{2}\left[\int_{1/2}^{1} p(s)\,ds - \int_{1/2}^{1} t^2 p(t)\,dt\right] \\
&= \frac{1}{2}\int_{1/2}^{1}\left(1-t^2\right) p(t)\,dt = -\frac{1}{2}\int_{1/2}^{0}\left(1-(1-s)^2\right) p(1-s)\,ds \\
&= \frac{1}{2}\int_0^{1/2}\left(2s-s^2\right) p(s)\,ds = \frac{1}{2}\int_0^{1/2}\left(2t-t^2\right) p(t)\,dt.
\end{aligned}$$

If we add these equalities, we get

$$\frac{1}{2}\int_0^{1/2} t^2 p(t)\,dt + \frac{1}{2}\int_0^{1/2}\left(2t-t^2\right) p(t)\,dt = \int_0^{1/2} tp(t)\,dt.$$

By (3.8) we then obtain

$$\begin{aligned}
(3.9)\qquad &\left\|\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2} - \int_0^1 p(t)\, f((1-t)A+tB)\,dt\right\| \\
&\leq \|B-A\|\left(\|f'(A)\| + \|f'(B)\|\right)\int_0^{1/2} tp(t)\,dt.
\end{aligned}$$

Finally, observe that

$$\begin{aligned}
&\frac{1}{2}\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\
&=\frac{1}{2}\int_0^{1/2}\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt+\frac{1}{2}\int_{1/2}^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\
&=\frac{1}{2}\int_0^{1/2}\left(\frac{1}{2}-\frac{1}{2}+t\right)p(t)\,dt+\frac{1}{2}\int_{1/2}^1\left(\frac{1}{2}-t+\frac{1}{2}\right)p(t)\,dt\\
&=\frac{1}{2}\int_0^{1/2}tp(t)\,dt+\frac{1}{2}\int_{1/2}^1(1-t)\,p(t)\,dt=\int_0^{1/2}tp(t)\,dt
\end{aligned}$$

and by (3.9) we get (3.6). □

Remark 7. *If we take* $p(t)=\left|t-\frac{1}{2}\right|$, $t\in[0,1]$, *then by (3.6) we have*

$$\begin{aligned}
&\left\|\frac{f(A)+f(B)}{8}-\int_0^1\left|t-\frac{1}{2}\right|f((1-t)A+tB)\,dt\right\|\\
&\leq\frac{1}{48}\|B-A\|\left(\|f'(A)\|+\|f'(B)\|\right).
\end{aligned}\tag{3.10}$$

Consider the function $f(x)=x^r$on $(0,\infty)$, where $0\leq r\leq 1$ or $2\leq r\leq 3$. If p is symmetric, then by (3.6) we get

$$\begin{aligned}
&\left\|\left(\int_0^1 p(t)\,dt\right)\frac{A^r+B^r}{2}-\int_0^1 p(t)\left((1-t)A+tB\right)^r dt\right\|\\
&\leq\frac{r}{2}\|B-A\|\left(\left\|A^{r-1}\right\|+\left\|B^{r-1}\right\|\right)\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt
\end{aligned}\tag{3.11}$$

for A, $B>0$.

For $p\equiv 1$ we get, see [9]

$$\begin{aligned}
&\left\|\frac{A^r+B^r}{2}-\int_0^1\left((1-t)A+tB\right)^r dt\right\|\\
&\leq\frac{r}{8}\|B-A\|\left(\left\|A^{r-1}\right\|+\left\|B^{r-1}\right\|\right)
\end{aligned}\tag{3.12}$$

for A, $B>0$, where $0\leq r\leq 1$ or $2\leq r\leq 3$.

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