SOME INEQUALITIES FOR WEIGHTED AND INTEGRAL MEANS OF OPERATOR DIFFERENTIABLE FUNCTIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let f be a continuous function on I and $A,\ B\in\mathcal{SA}_I(H)$, the convex set of selfadjoint operators with spectra in I. If $A\neq B$ and f, as an operator function, is Gâteaux differentiable on

$$[A,B]:=\{(1-t)A+tB\mid t\in[0,1]\},$$

while $p:[0,1]\to\mathbb{R}$ is Lebesgue integrable satisfying the condition

$$0\leq\int_0^\tau p(s)\,ds\leq\int_0^1 p(s)\,ds \text{ for all } \tau\in[0,1],$$

then

$$\left\|\int_0^1 p(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\|$$
$$\leq\sup_{t\in[0,1]}\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|\int_0^1\tau^2p(\tau)\,d\tau,$$

Some particular examples of interest are also given.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\leq B$ with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$ imply $f(A)\leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [9] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Operator convex functions, Integral inequalities, Hermite-Hadamard inequality, Féjer's inequalities.

RGMIA Res. REp. Coll. 22 (2019), Art. 106, 14 pp. Received 24/10/19

In [5] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f: I \to \mathbb{R}$

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f\left((1-s)A + sB\right) ds \leq \frac{f(A)+f(B)}{2}, \tag{1.2}$$

where A, B are selfadjoint operators with spectra included in I.

From the operator convexity of the function f we have

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq \frac{1}{2}\left[f\left((1-s)A+sB\right) + f\left(sA+(1-s)B\right)\right] \\ &\leq \frac{f(A)+f(B)}{2} \end{aligned} \tag{1.3}$$

for all $s \in [0,1]$ and A, B selfadjoint operators with spectra included in I.

If $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric in the sense that $p(1-s) = p(s)$ for all $s \in [0,1]$, then by multiplying (1.3) with $p(s)$, integrating on $[0,1]$ and taking into account that

$$\int_0^1 p(s) f\left((1-s)A+sB\right) ds = \int_0^1 p(s) f\left(sA+(1-s)B\right) ds,$$

we get the weighted version of (1.2) for A, B selfadjoint operators with spectra included in I

$$\begin{aligned} \left(\int_0^1 p(s)\, ds\right) f\left(\frac{A+B}{2}\right) &\leq \int_0^1 p(s) f\left(sA+(1-s)B\right) ds \\ &\leq \left(\int_0^1 p(s)\, ds\right) \frac{f(A)+f(B)}{2}, \end{aligned} \tag{1.4}$$

which are the operator version of the well known *Féjer's inequalities* for scalar convex functions.

For recent inequalities for operator convex functions see [1]-[7] and [10]-[19].

Let $\mathcal{SA}_I(H)$ be the class of all selfadjoint operators with spectra in I. If A, $B \in \mathcal{SA}_I(H)$ and $t \in [0,1]$ the convex combination $(1-t)A + tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ is convex in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. If f is continuous function on I. By the continuous functional calculus of selfadjoint operator we conclude that $f((1-t)A+tB)$ is a selfadjoint operator with spectrum in I.

A continuous function $f: \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla f_A(B) := \lim_{s \to 0} \frac{f(A+sB) - f(A)}{s} \in \mathcal{B}(H). \tag{1.5}$$

If the limit (1.5) exists for all $B \in \mathcal{B}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $f \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $f \in \mathcal{G}(\mathcal{S})$.

If f is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A, B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$

We observe that $A,\ B\in[A,B]$ and $[A,B]\subset\mathcal{SA}_I(H)$.

We say that the function $p:[0,1]\to\mathbb{R}$ is symmetric on $[0,1]$ if

$$p(1-t)=p(t)\ \text{ for all } t\in[0,1].$$

In the recent paper [8] we obtained among others the following operator bounds for the difference between the weighted integral mean and the integral mean:

Theorem 1. *Let f be an operator convex function on I and $A,\ B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$ and $p:[0,1]\to\mathbb{R}$ a Lebesgue integrable and symmetric function such that the condition*

$$0\le\int_0^\tau p(s)\,ds\le\int_0^1 p(s)\,ds \text{ for all } \tau\in[0,1], \tag{1.6}$$

holds, then we have

$$\begin{aligned}
&-\frac{1}{2}\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right] \\
&\le-\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&\times\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right] \\
&\le\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 f((1-\tau)A+\tau B)\,d\tau \\
&\le\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&\times\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right] \\
&\le\frac{1}{2}\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right].
\end{aligned}\tag{1.7}$$

If $x\in H$, $\|x\|=1$, then the inequality (1.7) implies

$$\begin{aligned}
&\left|\left\langle\left(\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right)x,x\right\rangle\right| \\
&\le\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left\langle\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right]x,x\right\rangle \\
&\qquad\le\frac{1}{2}\left\langle\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right]x,x\right\rangle
\end{aligned}$$

and by taking the supremum over $x\in H$, $\|x\|=1$ and observing that the involved operators are selfadjoint, we get the norm inequality

$$\begin{aligned}
&\left\|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \\
&\le\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left\|\nabla f_B(B-A)-\nabla f_A(B-A)\right\| \\
&\qquad\le\frac{1}{2}\left\|\nabla f_B(B-A)-\nabla f_A(B-A)\right\|,
\end{aligned}\tag{1.8}$$

provided that f is an operator convex function on I, A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$, $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition (1.6).

Motivated by the above results, in this paper we establish norm inequalities for the difference between the weighted integral mean and the integral mean in the case of Gâteaux and Fréchet differentiable functions of selfadjoint operators in Hilbert spaces. Some examples for the class of functions

$$\mathcal{D}^{(1)}(0,\infty) := \{f \mid \|Df(A)\| = \|f'(A)\| \text{ for all positive operators } A\},$$

where $Df(A)$ is the Fréchet derivative in A and $f'(A)$ is the operator function generated by f' and positive operator A, are also given. The case when f' is nonnegative and operator convex and the weight is symmetric is also analyzed.

2. Norm Inequalities

We need the following preliminary result, see :

Lemma 1. *Let f be a continuous function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A). \tag{2.1}$$

Also we have for the lateral derivative that

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A) \tag{2.2}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A). \tag{2.3}$$

Proof. For the sake of completeness, we give here a short proof.

Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\begin{aligned} &\frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ &= \frac{f((1-t-h)A + (t+h)B) - f((1-t)A+tB)}{h} \\ &= \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A+tB)}{h}. \end{aligned} \tag{2.4}$$

Since $f \in \mathcal{G}([A,B])$, hence by taking the limit over $h \to 0$ in (2.4) we get

$$\begin{aligned} \varphi'_{(A,B)}(t) &= \lim_{h\to 0} \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ &= \lim_{h\to 0} \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A+tB)}{h} \\ &= \nabla g_{(1-t)A+tB}(B-A), \end{aligned}$$

which proves (2.1).

Also, we have

$$\begin{aligned}\varphi'_{(A,B)}(0+) &= \lim_{h\to 0+}\frac{\varphi_{(A,B)}(h)-\varphi_{(A,B)}(0)}{h}\\ &= \lim_{h\to 0+}\frac{f((1-h)A+hB)-f(A)}{h}\\ &= \lim_{h\to 0+}\frac{f(A+h(B-A))-f(A)}{h}=\nabla f_A(B-A)\end{aligned}$$

since f is assumed to be Gâteaux differentiable in A. This proves (2.2).

The equality (2.3) follows in a similar way. □

We also need the following identity that is of interest in itself:

Lemma 2. *Let f be a continuous function on I and A, $B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$ and $g:[0,1]\to\mathbb{C}$ is a Lebesgue integrable function, then we have the equality*

$$\begin{aligned}(2.5)\qquad &\int_0^1 g(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 g(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\\ &=\int_0^1\left(\int_\tau^1 g(s)\,ds\right)\tau\nabla f_{(1-\tau)A+\tau B}(B-A)\,d\tau\\ &+\int_0^1\left(\int_0^\tau g(s)\,ds\right)(\tau-1)\nabla f_{(1-\tau)A+\tau B}(B-A)\,d\tau.\end{aligned}$$

Proof. Integrating by parts in the Bochner's integral, we have

$$\begin{aligned}&\int_0^\tau t\varphi'_{(A,B)}(t)\,dt+\int_\tau^1(t-1)\varphi'_{(A,B)}(t)\,dt\\ &=\tau\varphi_{(A,B)}(\tau)-\int_0^\tau\varphi_{(A,B)}(t)\,dt-(\tau-1)\varphi_{(A,B)}(\tau)-\int_\tau^1\varphi_{(A,B)}(t)\,dt\\ &=\varphi_{(A,B)}(\tau)-\int_0^1\varphi_{(A,B)}(t)\,dt\end{aligned}$$

that holds for all $\tau\in[0,1]$.

If we multiply this identity by $g(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\begin{aligned}(2.6)\quad &\int_0^1 g(\tau)\varphi_{(A,B)}(\tau)\,d\tau-\int_0^1 g(\tau)\,d\tau\int_0^1\varphi_{(A,B)}(t)\,dt\\ &=\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\tau+\int_0^1 g(\tau)\left(\int_\tau^1(t-1)\varphi'_{(A,B)}(t)\,dt\right)d\tau.\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}(2.7)\quad &\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\tau\\ &=\int_0^1\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right)\\ &=\left(\int_0^\tau g(s)\,ds\right)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)\Bigg|_0^1-\int_0^1\left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^1 g(s)\,ds\right)\left(\int_0^1 t\varphi'_{(A,B)}(t)\,dt\right) - \int_0^1 \left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^1 g(s)\,ds - \int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 g(\tau)\left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)d\tau \\
&= \int_0^1 \left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right) \\
&= \left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)\left(\int_0^\tau g(s)\,ds\right)\Bigg|_0^1 \\
&+ \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau,
\end{aligned}
\tag{2.8}
$$

which, together with (2.6), prove the identity in (2.5). □

Remark 1. *It is well known that, if f is a C^1-function defined on an open interval, then the operator function $f(X)$ is Fréchet differentiable and the derivative $Df(A)(B)$ equals the Gâteaux derivative $\nabla f_A(B)$. So for functions f that are of class C^1 on I we have the equality*

$$
\begin{aligned}
&\int_0^1 g(\tau)\,f((1-\tau)A+\tau B)\,d\tau - \int_0^1 g(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau \\
&= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau Df((1-\tau)A+\tau B)(B-A)\,d\tau \\
&+ \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,Df((1-\tau)A+\tau B)(B-A)\,d\tau,
\end{aligned}
\tag{2.9}
$$

for A, $B \in \mathcal{SA}_I(H)$, where $g:[0,1]\to\mathbb{C}$ is a Lebesgue integrable function on $[0,1]$.

Theorem 2. *Let f be a continuous function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function satisfying the condition (1.6) then*

$$
\begin{aligned}
&\left\|\int_0^1 p(\tau)\,f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \\
&\le \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau \\
&+ \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau \\
&=: M(f,p;A,B).
\end{aligned}
\tag{2.10}
$$

Proof. By taking the norm in (2.5), we get

$$\begin{aligned}
&\left\|\int_0^1 p(\tau) f((1-\tau) A+\tau B) d\tau-\int_0^1 p(\tau) d\tau \int_0^1 f((1-\tau) A+\tau B) d\tau\right\| \\
&\leq\left\|\int_0^1\left(\int_\tau^1 p(s) ds\right) \tau \nabla f_{(1-\tau) A+\tau B}(B-A) d\tau\right\| \\
&+\left\|\int_0^1\left(\int_0^\tau p(s) ds\right)(\tau-1) \nabla f_{(1-\tau) A+\tau B}(B-A) d\tau\right\| \\
&\leq \int_0^1\left|\int_\tau^1 p(s) ds\right| \tau\left\|\nabla f_{(1-\tau) A+\tau B}(B-A)\right\| d\tau \\
&+\int_0^1\left|\int_0^\tau p(s) ds\right|(1-\tau)\left\|\nabla f_{(1-\tau) A+\tau B}(B-A)\right\| d\tau \\
&=\int_0^1\left(\int_\tau^1 p(s) ds\right) \tau\left\|\nabla f_{(1-\tau) A+\tau B}(B-A)\right\| d\tau \\
&+\int_0^1\left(\int_0^\tau p(s) ds\right)(1-\tau)\left\|\nabla f_{(1-\tau) A+\tau B}(B-A)\right\| d\tau,
\end{aligned}$$

which proves (2.10). □

Remark 2. *For functions f that are of class C^1 on I we have the inequality*

$$\begin{aligned}
&\left\|\int_0^1 p(\tau) f((1-\tau) A+\tau B) d\tau-\int_0^1 p(\tau) d\tau \int_0^1 f((1-\tau) A+\tau B) d\tau\right\| \\
&\leq\|B-A\| \int_0^1\left(\int_\tau^1 p(s) ds\right) \tau\|D f((1-\tau) A+\tau B)\| d\tau \\
&+\|B-A\| \int_0^1\left(\int_0^\tau p(s) ds\right)(1-\tau)\|D f((1-\tau) A+\tau B)\| d\tau
\end{aligned} \tag{2.11}$$

for $A, B \in \mathcal{SA}_I(H)$.

Corollary 1. *With the assumptions of Theorem 2 and if*

$$\sup_{t\in[0,1]}\left\|\nabla f_{(1-\tau) A+\tau B}(B-A)\right\|<\infty,$$

then

$$\begin{aligned}
&\left\|\int_0^1 p(\tau) f((1-\tau) A+\tau B) d\tau-\int_0^1 p(\tau) d\tau \int_0^1 f((1-\tau) A+\tau B) d\tau\right\| \\
&\leq \sup_{t\in[0,1]}\left\|\nabla f_{(1-\tau) A+\tau B}(B-A)\right\| \int_0^1 \tau^2 \breve{p}(\tau) d\tau,
\end{aligned} \tag{2.12}$$

where $\breve{p}(\tau):=\frac{1}{2}[p(\tau)+p(1-\tau)]$, $\tau\in[0,1]$.

Moreover, if is symmetric, then

$$\begin{aligned}
&\left\|\int_0^1 p(\tau) f((1-\tau) A+\tau B) d\tau-\int_0^1 p(\tau) d\tau \int_0^1 f((1-\tau) A+\tau B) d\tau\right\| \\
&\leq \sup_{t\in[0,1]}\left\|\nabla f_{(1-\tau) A+\tau B}(B-A)\right\| \int_0^1 \tau^2 p(\tau) d\tau.
\end{aligned} \tag{2.13}$$

Proof. We have

$$
\begin{aligned}
(2.14)\quad M(f,p;A,B) &\leq \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau \left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\| d\tau \\
&\quad + \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau) \left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\| d\tau \\
&\leq \sup_{t\in[0,1]} \left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\| \\
&\quad \times \left[\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau + \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\right].
\end{aligned}
$$

Using integration by parts, we have

$$
\begin{aligned}
0 &\leq \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau = \frac{1}{2}\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) d\left(\tau^2\right) \\
&= \frac{1}{2}\left[\left(\int_\tau^1 p(s)\,ds\right)\tau^2\Big|_0^1 + \int_0^1 \tau^2 p(\tau)\,d\tau\right] \\
&= \frac{1}{2}\int_0^1 \tau^2 p(\tau)\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\leq \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&= -\frac{1}{2}\left[\int_0^1 \left(\int_0^\tau p(s)\,ds\right) d\left((1-\tau)^2\right)\right] \\
&= -\frac{1}{2}\left[\left(\int_0^\tau p(s)\,ds\right)(1-\tau)^2\Big|_0^1 - \int_0^1 (1-\tau)^2 p(\tau)\,d\tau\right] \\
&= \frac{1}{2}\int_0^1 (1-\tau)^2 p(\tau)\,d\tau = \frac{1}{2}\int_0^1 \tau^2 p(1-\tau)\,d\tau
\end{aligned}
$$

and by (2.14) we get (2.12). $\square$

Remark 3. *For functions f that are of class C^1 on I we have the inequality*

$$
\begin{aligned}
(2.15)\quad &\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \\
&\leq \|B-A\| \sup_{t\in[0,1]} \|Df((1-\tau)A+\tau B)\| \int_0^1 \tau^2 \breve{p}(\tau)\,d\tau
\end{aligned}
$$

for $A,\ B \in \mathcal{SA}_I(H)$.

Corollary 2. *With the assumptions of Theorem 2 we have*

$$
\begin{aligned}
(2.16)\quad & \left\| \int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau \right\| \\
& \le \left(\sup_{t\in[0,1]} \left[\left(\int_\tau^1 p(s)\,ds \right) \tau \right] + \sup_{t\in[0,1]} \left[\left(\int_0^\tau p(s)\,ds \right) (1-\tau) \right] \right) \\
& \times \int_0^1 \left\| \nabla f_{(1-\tau)A+\tau B}(B-A) \right\| d\tau \\
& \le \sup_{t\in[0,1]} \left[\left(\int_\tau^1 p(s)\,ds \right) \tau + \left(\int_0^\tau p(s)\,ds \right) (1-\tau) \right] \\
& \times \int_0^1 \left\| \nabla f_{(1-\tau)A+\tau B}(B-A) \right\| d\tau.
\end{aligned}
$$

Remark 4. *For functions f that are of class C^1 on I we have the inequality*

$$
\begin{aligned}
(2.17)\quad & \left\| \int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau \right\| \\
& \le \|B-A\| \left(\sup_{t\in[0,1]} \left[\left(\int_\tau^1 p(s)\,ds \right) \tau \right] + \sup_{t\in[0,1]} \left[\left(\int_0^\tau p(s)\,ds \right) (1-\tau) \right] \right) \\
& \times \int_0^1 \| Df((1-\tau)A+\tau B) \| \,d\tau \\
& \le \|B-A\| \sup_{t\in[0,1]} \left[\left(\int_\tau^1 p(s)\,ds \right) \tau + \left(\int_0^\tau p(s)\,ds \right) (1-\tau) \right] \\
& \times \int_0^1 \| Df((1-\tau)A+\tau B) \| \,d\tau,
\end{aligned}
$$

for A, $B \in \mathcal{SA}_I(H)$.

We also have:

Corollary 3. *With the assumptions of Theorem 2 we have, for r, $q>1$ with $\frac{1}{r}+\frac{1}{q}=1$, that*

$$
\begin{aligned}
(2.18)\quad & \left\| \int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau \right\| \\
& \le \left[\left(\int_0^1 \left(\int_\tau^1 p(s)\,ds \right)^r \tau^r d\tau \right)^{1/r} + \int_0^1 \left(\left(\int_0^\tau p(s)\,ds \right)^r (1-\tau)^r d\tau \right)^{1/r} \right] \\
& \qquad \times \left(\int_0^1 \left\| \nabla f_{(1-\tau)A+\tau B}(B-A) \right\|^q d\tau \right)^{1/q} \\
& \le 2^{1/q} \left[\int_0^1 \left(\left(\int_\tau^1 p(s)\,ds \right)^r \tau^r + \left(\int_0^\tau p(s)\,ds \right)^r (1-\tau)^r \right) d\tau \right] \\
& \qquad \times \left(\int_0^1 \left\| \nabla f_{(1-\tau)A+\tau B}(B-A) \right\|^q d\tau \right)^{1/q}.
\end{aligned}
$$

Proof. By Hölder's integral inequality, we have for $r,\ q>1$ with $\frac{1}{r}+\frac{1}{q}=1$, that

$$
\begin{aligned}
&M(f,p;A,B) \qquad (2.19)\\
&\leq \left(\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)^r \tau^r d\tau\right)^{1/r} \left(\int_0^1 \left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|^q d\tau\right)^{1/q}\\
&+\left(\int_0^1 \left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r d\tau\right)^{1/r} \left(\int_0^1 \left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|^q d\tau\right)^{1/q}\\
&=\left[\left(\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)^r \tau^r d\tau\right)^{1/r}+\left(\int_0^1 \left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r d\tau\right)^{1/r}\right]\\
&\qquad\times\left(\int_0^1 \left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|^q d\tau\right)^{1/q},
\end{aligned}
$$

which proves the first inequality.

By the convexity of power function, we have

$$\left(\frac{a+b}{2}\right)^r \leq \frac{a^r+b^r}{2},\ r>1,\ a,\ b>0$$

namely

$$a+b\leq 2^{1-1/r}\left(a^r+b^r\right)^{1/r},\ r>1,\ a,\ b>0.$$

Therefore

$$
\begin{aligned}
&\left(\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)^r \tau^r d\tau\right)^{1/r}+\left(\int_0^1 \left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r d\tau\right)^{1/r}\\
&\leq 2^{1-1/r}\left[\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)^r \tau^r d\tau+\int_0^1 \left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r d\tau\right]^{1/r}\\
&=2^{1-1/r}\left[\int_0^1 \left(\left(\int_\tau^1 p(s)\,ds\right)^r \tau^r+\left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r\right) d\tau\right]
\end{aligned}
$$

and the last part of (2.18). □

Remark 5. *For functions f that are of class C^1 on I we have the inequality*

$$
\begin{aligned}
&\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \qquad (2.20)\\
&\leq \|B-A\|\left[\left(\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)^r \tau^r d\tau\right)^{1/r}+\int_0^1 \left(\left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r d\tau\right)^{1/r}\right]\\
&\qquad\times\left(\int_0^1 \|Df((1-\tau)A+\tau B)\|^q d\tau\right)^{1/q}\\
&\leq 2^{1/q}\|B-A\|\left[\int_0^1 \left(\left(\int_\tau^1 p(s)\,ds\right)^r \tau^r+\left(\int_0^\tau p(s)\,ds\right)^r (1-\tau)^r\right) d\tau\right]\\
&\qquad\times\left(\int_0^1 \|Df((1-\tau)A+\tau B)\|^q d\tau\right)^{1/q}
\end{aligned}
$$

for $A,\ B\in\mathcal{SA}_I(H)$.

We also have

Corollary 4. *With the assumptions of Theorem 2 we have*

$$\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \tag{2.21}$$
$$\le\int_0^1 p(s)\,ds\int_0^1\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau.$$

Proof. By the condition (1.6) we have

$$0\le\int_\tau^1 p(s)\,ds\le\int_0^1 p(s)\,ds,\ \ 0\le\int_0^\tau p(s)\,ds\le\int_0^1 p(s)\,ds$$

for $\tau\in[0,1]$.

Therefore

$$\begin{aligned}M(f,p;A,B)&\le\int_0^1 p(s)\,ds\int_0^1\tau\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau\\&\quad+\int_0^1 p(s)\,ds\int_0^1(1-\tau)\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau\\&=\int_0^1 p(s)\,ds\int_0^1\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau,\end{aligned}$$

which proves (2.21). □

Remark 6. *For functions f that are of class C^1 on I we have the inequality*

$$\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \tag{2.22}$$
$$\le\|B-A\|\int_0^1 p(s)\,ds\int_0^1\|Df((1-\tau)A+\tau B)\|\,d\tau$$

for $A,\ B\in\mathcal{SA}_I(H)$.

3. Examples for Some General Classes of Functions

Let f be a real function that is n-time differentiable on $(0,\infty)$, and let $f^{(n)}$ be its n-th derivative. Let f also denote the map induced by f on positive operators. Let $D^nf(A)$ be the n-th order Fréchet derivative of this map at the point A. For each A, the derivative $D^nf(A)$ is a n-linear operator on the space of all Hermitian operators. The norm of this operator is defined as

$$\|D^nf(A)\|:=\sup\{D^nf(A)(B_1,...,B_n)\ \mid\ \|B_1\|=...=\|B_n\|=1\}.$$

We consider the following class of functions defined on $(0,\infty)$ for a natural $n\ge1$,

$$\mathcal{D}^{(n)}(0,\infty):=\left\{f\ \mid\ \|D^nf(A)\|=\left\|f^{(n)}(A)\right\|\text{ for all positive operators }A\right\}.$$

It is known (see for instance [10]) that every operator monotone function is in $\mathcal{D}^{(n)}(0,\infty)$ for all $n=1,\ 2,....$ Also the functions $f(t)=t^n$, $n=2,\ 3,\ ...,$ and $f(t)=\exp t$ are in $\mathcal{D}^{(1)}(0,\infty)$. None of these are operator monotone. Moreover, the power function $f(t)=t^p$ is in $\mathcal{D}^{(1)}(0,\infty)$ if p is in $(-\infty,1]$ or in $[2,\infty)$, but not if p is in $\left(1,\sqrt{2}\right)$. Also that the functions $f(t)=\exp t$ and $f(t)=t^p$, $-\infty<p\le1$,

are in the class $\mathcal{D}^{(n)}(0,\infty)$ for all $n = 1, 2, ...$, and that for $p > 1$ the function $f(t) = t^p$ is in the class $\mathcal{D}^{(n)}(0,\infty)$ for all $n \geq [p+1]$, where $[\cdot]$ is the integer part (see for instance [10] and the references therein).

Proposition 1. *If $f \in \mathcal{D}^{(1)}(0,\infty)$, A, $B > 0$ and $p : [0,1] \to \mathbb{R}$ is Lebesgue integrable and such that the following condition*

$$0 \leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1], \tag{3.1}$$

holds. Then

$$\begin{aligned}&\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \\ &\leq \|B-A\| \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau \|f'((1-\tau)A+\tau B)\|\,d\tau \\ &+ \|B-A\| \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\|f'((1-\tau)A+\tau B)\|\,d\tau.\end{aligned} \tag{3.2}$$

We also have

$$\begin{aligned}&\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \\ &\leq \|B-A\| \sup_{t\in[0,1]} \|f'((1-\tau)A+\tau B)\| \int_0^1 \tau^2 \breve{p}(\tau)\,d\tau\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \\ &\leq \|B-A\| \int_0^1 p(s)\,ds \int_0^1 \|f'((1-\tau)A+\tau B)\|\,d\tau.\end{aligned} \tag{3.4}$$

If $f = \exp$, then

$$\begin{aligned}\int_0^1 \|\exp((1-t)A+tB)\|\,dt &\leq \int_0^1 \exp\|((1-t)A+tB)\|\,dt \\ &\leq \int_0^1 \exp[(1-t)\|A\| + t\|B\|]\,dt \\ &= \begin{cases} \frac{\exp\|B\|-\exp\|A\|}{\|B\|-\|A\|} \text{ for } \|B\| \neq \|A\|, \\ \exp\|A\| \text{ for } \|B\| = \|A\| \end{cases}\end{aligned}$$

and by (3.4) we get

$$\begin{aligned}&\left\|\int_0^1 p(\tau)\exp((1-\tau)A+\tau B)\,d\tau \right.\\ &\qquad\left. - \int_0^1 p(\tau)\,d\tau \int_0^1 \exp((1-\tau)A+\tau B)\,d\tau\right\| \\ &\leq \|B-A\| \int_0^1 p(s)\,ds \times \begin{cases} \frac{\exp\|B\|-\exp\|A\|}{\|B\|-\|A\|} \text{ for } \|B\| \neq \|A\|, \\ \exp\|A\| \text{ for } \|B\| = \|A\| \end{cases},\end{aligned} \tag{3.5}$$

where $p:[0,1]\to\mathbb{R}$ is Lebesgue integrable and satisfies the condition (3.1) while $A,\, B>0$.

Proposition 2. *If $f\in\mathcal{D}^{(1)}(0,\infty)$ and f' is operator convex and nonnegative on $(0,\infty)$ and $p:[0,1]\to\mathbb{R}$ is Lebesgue integrable satisfying condition (3.1), then for $A,\, B>0$, we have*

$$\begin{aligned}(3.6)\qquad &\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\|\\ &\leq\frac{1}{2}\|B-A\|\left[\|f'(A)\|+\|f'(B)\|\right]\int_0^1 p(s)\,ds.\end{aligned}$$

Proof. Since f' is operator convex and nonnegative on $(0,\infty)$ then for $A,\, B>0$ we have

$$0\leq f'((1-t)A+tB)\leq(1-t)f'(A)+tf'(B)$$

for $t\in[0,1]$. By taking the norm, we get

$$\begin{aligned}\|f'((1-t)A+tB)\|&\leq\|(1-t)f'(A)+tf'(B)\|\\ &\leq(1-t)\|f'(A)\|+t\|f'(B)\|\end{aligned}$$

for $t\in[0,1]$.

Using the inequality (3.4) we have

$$\begin{aligned}&\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\|\\ &\leq\|B-A\|\int_0^1 p(s)\,ds\int_0^1\|f'((1-\tau)A+\tau B)\|\,d\tau\\ &\leq\|B-A\|\int_0^1 p(s)\,ds\int_0^1\left[(1-t)\|f'(A)\|+t\|f'(B)\|\right]dt\\ &=\frac{1}{2}\|B-A\|\left[\|f'(A)\|+\|f'(B)\|\right]\int_0^1 p(s)\,ds\end{aligned}$$

and the inequality (3.6) is proved. □

Consider the function $f(x)=x^r$on $(0,\infty)$, where $0\leq r\leq 1$ or $2\leq r\leq 3$. Then by (3.6) we get

$$\begin{aligned}(3.7)\qquad &\left\|\int_0^1 p(\tau)((1-\tau)A+\tau B)^r\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1((1-\tau)A+\tau B)^r\,d\tau\right\|\\ &\leq\frac{r}{2}\|B-A\|\left[\left\|A^{r-1}\right\|+\left\|B^{r-1}\right\|\right]\int_0^1 p(s)\,ds\end{aligned}$$

for $A,\, B>0$.

References

[1] R. P. Agarwal and S. S. Dragomir, A survey of Jensen type inequalities for functions of selfadjoint operators in Hilbert spaces. *Comput. Math. Appl.* 5**9** (2010), no. 12, 3785–3812.

[2] V. Bacak, T. Vildan and R. Türkmen, Refinements of Hermite-Hadamard type inequalities for operator convex functions. *J. Inequal. Appl.* **2013**, 2013:262, 10 pp.

[3] V. Darvish, S. S. Dragomir, H. M. Nazari and A. Taghavi, Some inequalities associated with the Hermite-Hadamard inequalities for operator h-convex functions. *Acta Comment. Univ. Tartu. Math.* **21** (2017), no. 2, 287–297.

[4] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type.* Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[5] S. S. Dragomir, Hermite-Hadamard's type inequalities for operator convex functions. *Appl. Math. Comput.* **218** (2011), no. 3, 766–772.

[6] S. S. Dragomir, Some Hermite-Hadamard type inequalities for operator convex functions and positive maps. *Spec. Matrices* **7** (2019), 38–51. Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 80. [Online http://rgmia.org/papers/v19/v19a80.pdf].

[7] S. S. Dragomir, Reverses of operator Féjer's inequalities, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art. 91, 14 pp. [Online http://rgmia.org/papers/v22/v22a91.pdf].

[8] S. S. Dragomir, Some inequalities for weighted and integral means of operator convex functions, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art. 96, 15 pp. [Online http://rgmia.org/papers/v22/v22a96.pdf].

[9] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[10] A. G. Ghazanfari, Hermite-Hadamard type inequalities for functions whose derivatives are operator convex. *Complex Anal. Oper. Theory* **10** (2016), no. 8, 1695–1703.

[11] A. G. Ghazanfari, The Hermite-Hadamard type inequalities for operator s-convex functions. *J. Adv. Res. Pure Math.* **6** (2014), no. 3, 52–61.

[12] J. Han and J. Shi, Refinements of Hermite-Hadamard inequality for operator convex function. *J. Nonlinear Sci. Appl.* **10** (2017), no. 11, 6035–6041.

[13] B. Li, Refinements of Hermite-Hadamard's type inequalities for operator convex functions. *Int. J. Contemp. Math. Sci.* **8** (2013), no. 9-12, 463–467.

[14] G. K. Pedersen, Operator differentiable functions. *Publ. Res. Inst. Math. Sci.* **36** (1) (2000), 139-157.

[15] A. Taghavi, V. Darvish, H. M. Nazari and S. S. Dragomir, Hermite-Hadamard type inequalities for operator geometrically convex functions. *Monatsh. Math.* **181** (2016), no. 1, 187–203.

[16] M. Vivas Cortez, H. Hernández and E. Jorge, Refinements for Hermite-Hadamard type inequalities for operator h-convex function. *Appl. Math. Inf. Sci.* **11** (2017), no. 5, 1299–1307.

[17] M. Vivas Cortez, H. Hernández and E. Jorge, On some new generalized Hermite-Hadamard-Fejér inequalities for product of two operator h-convex functions. A*ppl. Math. Inf. Sci.* **11** (2017), no. 4, 983–992.

[18] S.-H. Wang, Hermite-Hadamard type inequalities for operator convex functions on the coordinates. *J. Nonlinear Sci. Appl.* **10** (2017), no. 3, 1116–1125

[19] S.-H. Wang, New integral inequalities of Hermite-Hadamard type for operator m-convex and (α, m)-convex functions. *J. Comput. Anal. Appl.* **22** (2017), no. 4, 744–753.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa