NORM INEQUALITIES FOR THE DIFFERENCE BETWEEN WEIGHTED AND INTEGRAL MEANS OF OPERATOR DIFFERENTIABLE FUNCTIONS

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Abstract. Let f be a continuous function on I and A, $B\in\mathcal{SA}_I(H)$, the convex set of selfadjoint operators with spectra in I. If $A\neq B$ and f, as an operator function, is Gâteaux differentiable on

$$[A,B]:=\{(1-t)A+tB\mid t\in[0,1]\},$$

while $p:[0,1]\to\mathbb{R}$ is Lebesgue integrable, then we have the inequalities

$$\begin{aligned}&\left\|\int_0^1 p(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\|\\ &\le\int_0^1\tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau\\ &\le\frac{1}{4}\int_0^1\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau\end{aligned}$$

Some particular examples of interest are also given.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\le(\ge)(1-\lambda)f(A)+\lambda f(B)\tag{1.1}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\le B$ with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$ imply $f(A)\le f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [9] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\le r\le1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\le r\le2$ or $-1\le r\le0$ and is operator concave on $(0,\infty)$ if $0\le r\le1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

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In [5] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f : I \to \mathbb{R}$

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f\left((1-s)A + sB\right) ds \leq \frac{f(A)+f(B)}{2}, \tag{1.2}$$

where A, B are selfadjoint operators with spectra included in I.

From the operator convexity of the function f we have

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq \frac{1}{2}\left[f\left((1-s)A+sB\right)+f\left(sA+(1-s)B\right)\right] \\ &\leq \frac{f(A)+f(B)}{2} \end{aligned} \tag{1.3}$$

for all $s \in [0,1]$ and A, B selfadjoint operators with spectra included in I.

If $p : [0,1] \to [0,\infty)$ is Lebesgue integrable and symmetric in the sense that $p(1-s) = p(s)$ for all $s \in [0,1]$, then by multiplying (1.3) with $p(s)$, integrating on $[0,1]$ and taking into account that

$$\int_0^1 p(s) f\left((1-s)A+sB\right) ds = \int_0^1 p(s) f\left(sA+(1-s)B\right) ds,$$

we get the weighted version of (1.2) for A, B selfadjoint operators with spectra included in I

$$\begin{aligned} \left(\int_0^1 p(s)\, ds\right) f\left(\frac{A+B}{2}\right) &\leq \int_0^1 p(s) f\left(sA+(1-s)B\right) ds \\ &\leq \left(\int_0^1 p(s)\, ds\right) \frac{f(A)+f(B)}{2}, \end{aligned} \tag{1.4}$$

which are the operator version of the well known *Féjer's inequalities* for scalar convex functions.

For recent inequalities for operator convex functions see [1]-[7] and [10]-[19].

Let $\mathcal{SA}_I(H)$ be the class of all selfadjoint operators with spectra in I. If A, $B \in \mathcal{SA}_I(H)$ and $t \in [0,1]$ the convex combination $(1-t)A+tB$ is a selfadjoint operator with the spectrum in I showing that $\mathcal{SA}_I(H)$ is convex in the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on H. If f is continuous function on I. By the continuous functional calculus of selfadjoint operator we conclude that $f((1-t)A+tB)$ is a selfadjoint operator with spectrum in I.

A continuous function $f : \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla f_A(B) := \lim_{s \to 0} \frac{f(A+sB)-f(A)}{s} \in \mathcal{B}(H). \tag{1.5}$$

If the limit (1.5) exists for all $B \in \mathcal{B}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $f \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $f \in \mathcal{G}(\mathcal{S})$.

If f is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators A, $B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B] := \{(1-t)A+tB \mid t \in [0,1]\}.$$

We observe that $A,\ B\in[A,B]$ and $[A,B]\subset\mathcal{SA}_I(H)$.

In the recent paper we obtained the following result:

Theorem 1. *Let f be an operator convex function on I and $A,\ B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$ and $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that*

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds\le\frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau\in(0,1), \tag{1.6}$$

then we have the inequalities

$$\begin{aligned}&\left[\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\nabla f_A(B-A)\\&\le\int_0^1 p(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\\&\le\left[\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\nabla f_B(B-A).\end{aligned}\tag{1.7}$$

This inequality is equivalent to

$$\begin{aligned}&-\frac{1}{2}\left[\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right][\nabla f_A(B-A)-\nabla f_B(B-A)]\\&\le\int_0^1 p(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\\&-\left[\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\frac{\nabla f_B(B-A)+\nabla f_A(B-A)}{2}\\&\le\frac{1}{2}\left[\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right][\nabla f_A(B-A)-\nabla f_B(B-A)],\end{aligned}$$

which implies for $x\in H$, $\|x\|=1$ that

$$\begin{aligned}&\Bigg\langle\Bigg(\int_0^1 p(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\\&-\left[\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\frac{\nabla f_B(B-A)+\nabla f_A(B-A)}{2}\Bigg)x,x\Bigg\rangle\\&\le\frac{1}{2}\left[\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\langle[\nabla f_A(B-A)-\nabla f_B(B-A)]x,x\rangle\end{aligned}$$

and by taking the supremum over $x\in H$, $\|x\|=1$, we get the norm inequality

$$\begin{aligned}&\Bigg\|\int_0^1 p(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\\&-\left[\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\frac{\nabla f_B(B-A)+\nabla f_A(B-A)}{2}\Bigg)\Bigg\|\\&\le\frac{1}{2}\left[\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau\right]\|\nabla f_A(B-A)-\nabla f_B(B-A)\|\end{aligned}$$

provided that f is an operator convex function on I, $A,\ B\in\mathcal{SA}_I(H)$, with $A\neq B$, $f\in\mathcal{G}([A,B])$ and $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that the condition (1.6) holds.

Motivated by the above results, in this paper we establish norm inequalities for the difference between the weighted integral mean and the integral mean in the case of Gâteaux and Fréchet differentiable functions of selfadjoint operators in Hilbert spaces. Some examples for the class of functions

$$\mathcal{D}^{(1)}(0,\infty) := \{f \mid \|Df(A)\| = \|f'(A)\| \text{ for all positive operators } A\},$$

where $Df(A)$ is the Fréchet derivative in A and $f'(A)$ is the operator function generated by f' and positive operator A, are also given. The case when f' is nonnegative and operator convex and the weight is symmetric is also analyzed.

2. Norm Inequalities

We need the following preliminary result, see :

Lemma 1. *Let f be a continuous function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A). \tag{2.1}$$

Also we have for the lateral derivative that

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A) \tag{2.2}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A). \tag{2.3}$$

Proof. For the sake of completeness, we give here a short proof.

Let $t \in (0,1)$ and $h \neq 0$ small enough such that $t+h \in (0,1)$. Then

$$\begin{aligned} &\frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ &= \frac{f((1-t-h)A + (t+h)B) - f((1-t)A+tB)}{h} \\ &= \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A+tB)}{h}. \end{aligned} \tag{2.4}$$

Since $f \in \mathcal{G}([A,B])$, hence by taking the limit over $h \to 0$ in (2.4) we get

$$\begin{aligned} \varphi'_{(A,B)}(t) &= \lim_{h\to 0} \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h} \\ &= \lim_{h\to 0} \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A+tB)}{h} \\ &= \nabla g_{(1-t)A+tB}(B-A), \end{aligned}$$

which proves (2.1).

Also, we have

$$\begin{aligned} \varphi'_{(A,B)}(0+) &= \lim_{h\to 0+} \frac{\varphi_{(A,B)}(h) - \varphi_{(A,B)}(0)}{h} \\ &= \lim_{h\to 0+} \frac{f((1-h)A + hB) - f(A)}{h} \\ &= \lim_{h\to 0+} \frac{f(A + h(B-A)) - f(A)}{h} = \nabla f_A(B-A) \end{aligned}$$

since f is assumed to be Gâteaux differentiable in A. This proves (2.2).

The equality (2.3) follows in a similar way. □

We also need the following identity that is of interest in itself:

Lemma 2. *Let f be an operator convex function on I and $A,\ B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$ and $g:[0,1]\to\mathbb{C}$ is a Lebesgue integrable function, then we have the equality*

$$\begin{aligned}(2.5)\qquad &\int_0^1 g(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 g(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\\ &=\int_0^1\tau(1-\tau)\left(\frac{\int_\tau^1 g(s)\,ds}{1-\tau}-\frac{\int_0^\tau g(s)\,ds}{\tau}\right)\nabla f_{(1-\tau)A+\tau B}(B-A)\,d\tau.\end{aligned}$$

Proof. Integrating by parts in the Bochner's integral, we have

$$\begin{aligned}&\int_0^\tau t\varphi'_{(A,B)}(t)\,dt+\int_\tau^1(t-1)\varphi'_{(A,B)}(t)\,dt\\ &=\tau\varphi_{(A,B)}(\tau)-\int_0^\tau\varphi_{(A,B)}(t)\,dt-(\tau-1)\varphi_{(A,B)}(\tau)-\int_\tau^1\varphi_{(A,B)}(t)\,dt\\ &=\varphi_{(A,B)}(\tau)-\int_0^1\varphi_{(A,B)}(t)\,dt\end{aligned}$$

that holds for all $\tau\in[0,1]$.

If we multiply this identity by $g(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\begin{aligned}(2.6)\quad &\int_0^1 g(\tau)\varphi_{(A,B)}(\tau)\,d\tau-\int_0^1 g(\tau)\,d\tau\int_0^1\varphi_{(A,B)}(t)\,dt\\ &=\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\tau+\int_0^1 g(\tau)\left(\int_\tau^1(t-1)\varphi'_{(A,B)}(t)\,dt\right)d\tau.\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}(2.7)\qquad &\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\tau\\ &=\int_0^1\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right)\\ &=\left(\int_0^\tau g(s)\,ds\right)\left(\int_0^\tau t\varphi'_{(A,B)}(t)\,dt\right)\bigg|_0^1\\ &-\int_0^1\left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau\end{aligned}$$

$$\begin{aligned}&=\left(\int_0^1 g(s)\,ds\right)\left(\int_0^1 t\varphi'_{(A,B)}(t)\,dt\right)\\ &-\int_0^1\left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau\\ &=\int_0^1\left(\int_0^1 g(s)\,ds-\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau\\ &=\int_0^1\left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau\end{aligned}$$

and

$$\begin{aligned}(2.8)\quad & \int_0^1 g(\tau)\left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)d\tau \\ &= \int_0^1 \left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right) \\ &= \left(\int_\tau^1 (t-1)\,\varphi'_{(A,B)}(t)\,dt\right)\left(\int_0^\tau g(s)\,ds\right)\Bigg|_0^1 \\ &+ \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau \\ &= \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau,\end{aligned}$$

which proves the identity

$$\begin{aligned}(2.9)\quad & \int_0^1 g(\tau)\,\varphi_{(A,B)}(\tau)\,d\tau - \int_0^1 g(\tau)\,d\tau \int_0^1 \varphi_{(A,B)}(\tau)\,d\tau \\ &= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau \\ &+ \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau.\end{aligned}$$

Now, observe that

$$\begin{aligned}& \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(A,B)}(\tau)\,d\tau + \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(A,B)}(\tau)\,d\tau \\ &= \int_0^1 \tau\left(\int_\tau^1 g(s)\,ds\right)\varphi'_{(A,B)}(\tau)\,d\tau - \int_0^1 (1-\tau)\left(\int_0^\tau g(s)\,ds\right)\varphi'_{(A,B)}(\tau)\,d\tau \\ &= \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 g(s)\,ds}{1-\tau} - \frac{\int_0^\tau g(s)\,ds}{t}\right)\varphi'_{(A,B)}(\tau)\,d\tau\end{aligned}$$

and by (2.9) we obtain the desired equality (2.5). □

Remark 1. *It is well known that, if f is a C^1-function defined on an open interval, then the operator function $f(X)$ is Fréchet differentiable and the derivative $Df(A)(B)$ equals the Gâteaux derivative $\nabla f_A(B)$. So for functions f that are of class C^1 on I we have the equality*

$$\begin{aligned}(2.10)\quad & \int_0^1 g(\tau)\,f((1-\tau)A+\tau B)\,d\tau - \int_0^1 g(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau \\ &= \int_0^1 \tau(1-\tau)\left(\frac{\int_\tau^1 g(s)\,ds}{1-\tau} - \frac{\int_0^\tau g(s)\,ds}{\tau}\right)Df((1-\tau)A+\tau B)(B-A)\,d\tau.\end{aligned}$$

for $A, B \in \mathcal{SA}_I(H)$, where $g:[0,1]\to\mathbb{C}$ is a Lebesgue integrable function on $[0,1]$.

We have:

Theorem 2. *Let f be an operator convex function on I and $A,\, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$ and $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function, then we have the inequality*

$$\begin{aligned}(2.11)\quad & \left\| \int_0^1 p(\tau) f((1-\tau)A + \tau B)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)A+\tau B)\, d\tau \right\| \\ & \leq \int_0^1 \tau(1-\tau) \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \left\| \nabla f_{(1-\tau)A+\tau B}(B-A) \right\| d\tau \\ & \leq \frac{1}{4} \int_0^1 \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \left\| \nabla f_{(1-\tau)A+\tau B}(B-A) \right\| d\tau\end{aligned}$$

Proof. If we take the norm in the equality (2.5) written for $p = g$, we have

$$\begin{aligned} & \left\| \int_0^1 p(\tau) f((1-\tau)A + \tau B)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)A+\tau B)\, d\tau \right\| \\ & \leq \int_0^1 \left\| \tau(1-\tau) \left(\frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right) \nabla f_{(1-\tau)A+\tau B}(B-A) \right\| d\tau \\ & = \int_0^1 \tau(1-\tau) \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \left\| \nabla f_{(1-\tau)A+\tau B}(B-A) \right\| d\tau\end{aligned}$$

and since $\tau(1-\tau) \leq \frac{1}{4}$ for $\tau \in [0,1]$, hence (2.11) is proved. □

Remark 2. *For functions f that are of class C^1 on I we have the inequality*

$$\begin{aligned}(2.12)\quad & \left\| \int_0^1 p(\tau) f((1-\tau)A + \tau B)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)A+\tau B)\, d\tau \right\| \\ & \leq \|B-A\| \int_0^1 \tau(1-\tau) \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \\ & \times \|Df((1-\tau)A+\tau B)\|\, d\tau \\ & \leq \frac{1}{4} \|B-A\| \int_0^1 \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \|Df((1-\tau)A+\tau B)\|\, d\tau\end{aligned}$$

for all $A,\, B \in \mathcal{SA}_I(H)$.

Corollary 1. *With the assumptions of Theorem 2 and if*

$$\sup_{\tau \in [0,1]} \left\| \nabla f_{(1-\tau)A+\tau B}(B-A) \right\| < \infty,$$

then

$$
\begin{aligned}
&\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \qquad (2.13)\\
&\le \sup_{\tau\in[0,1]}\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|\\
&\times\int_0^1 \tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|d\tau\\
&\le \sup_{\tau\in[0,1]}\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|\\
&\times\begin{cases}
\frac{1}{4}\int_0^1\left|\frac{\int_\tau^1 p(s)ds}{1-\tau}-\frac{\int_0^\tau p(s)ds}{\tau}\right|d\tau\\
\\
[\beta(r+1,r+1)]^{1/r}\left(\int_0^1\left|\frac{\int_\tau^1 p(s)ds}{1-\tau}-\frac{\int_0^\tau p(s)ds}{\tau}\right|^q d\tau\right)^{1/q}\\
\text{where } r,\ q>1,\ \frac{1}{r}+\frac{1}{q}=1;\\
\\
\frac{1}{6}\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)ds}{1-\tau}-\frac{\int_0^\tau p(s)ds}{\tau}\right|,
\end{cases}
\end{aligned}
$$

where β *is the Beta function*

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\ x,\ y>0.$$

Remark 3. *For functions* f *that are of class* C^1 *on* I *we have the inequality*

$$
\begin{aligned}
&\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \qquad (2.14)\\
&\le \|B-A\|\sup_{\tau\in[0,1]}\|Df((1-\tau)A+\tau B)\|\\
&\times\int_0^1 \tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|d\tau\\
&\le \|B-A\|\sup_{\tau\in[0,1]}\|Df((1-\tau)A+\tau B)\|\\
&\times\begin{cases}
\frac{1}{4}\int_0^1\left|\frac{\int_\tau^1 p(s)ds}{1-\tau}-\frac{\int_0^\tau p(s)ds}{\tau}\right|d\tau\\
\\
[\beta(r+1,r+1)]^{1/r}\left(\int_0^1\left|\frac{\int_\tau^1 p(s)ds}{1-\tau}-\frac{\int_0^\tau p(s)ds}{\tau}\right|^q d\tau\right)^{1/q}\\
\text{where } r,\ q>1,\ \frac{1}{r}+\frac{1}{q}=1;\\
\\
\frac{1}{6}\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)ds}{1-\tau}-\frac{\int_0^\tau p(s)ds}{\tau}\right|,
\end{cases}
\end{aligned}
$$

for all $A,\ B\in\mathcal{SA}_I(H)$.

Corollary 2. *With the assumptions of Corollary 1 and if* $p:[0,1]\to\mathbb{R}$ *is a Lebesgue integrable function such that*

$$\frac{1}{\tau}\int_0^\tau p(s)\,ds\le\frac{1}{1-\tau}\int_\tau^1 p(s)\,ds \text{ for all } \tau\in(0,1), \tag{2.15}$$

then we have

$$\begin{aligned}(2.16)\quad &\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\|\\ &\le \sup_{\tau\in[0,1]}\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|\int_0^1\left(\tau-\frac{1}{2}\right)p(\tau)\,d\tau.\end{aligned}$$

Proof. From (2.13) we have

$$\begin{aligned}(2.17)\quad &\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\|\\ &\le \sup_{\tau\in[0,1]}\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|\\ &\times\int_0^1\tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|d\tau\\ &=\sup_{\tau\in[0,1]}\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|\\ &\times\int_0^1\tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right)d\tau.\end{aligned}$$

Since, like in (2), we have the equality

$$\int_0^1\tau(1-\tau)\left(\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right)d\tau=\int_0^1\tau p(\tau)\,d\tau-\frac{1}{2}\int_0^1 p(\tau)\,d\tau,$$

hence by (2.17) we get (2.16). □

Remark 4. *For functions f that are of class C^1 on I and p satisfying the condition (2.15), we have the inequality*

$$\begin{aligned}(2.18)\quad &\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\|\\ &\le\|B-A\|\sup_{\tau\in[0,1]}\|Df((1-\tau)A+\tau B)\|\int_0^1\left(\tau-\frac{1}{2}\right)p(\tau)\,d\tau\end{aligned}$$

for all A, $B\in\mathcal{SA}_I(H)$.

Corollary 3. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}(2.19)\quad &\left\|\int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\|\\ &\le\sup_{\tau\in[0,1]}\left|\tau\int_0^1 p(s)\,ds-\int_0^\tau p(s)\,ds\right|\int_0^1\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau\\ &\le\frac{1}{4}\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\int_0^1\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau.\end{aligned}$$

Proof. We have

$$\begin{aligned}
&\int_0^1 \tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau\\
&\le \sup_{\tau\in[0,1]}\left[\tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\right]\int_0^1\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau\\
&=\sup_{\tau\in[0,1]}\left[\left|\tau\int_\tau^1 p(s)\,ds-(1-\tau)\int_0^\tau p(s)\,ds\right|\right]\int_0^1\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau\\
&=\sup_{\tau\in[0,1]}\left|\tau\int_0^1 p(s)\,ds-\int_0^\tau p(s)\,ds\right|\int_0^1\left\|\nabla f_{(1-\tau)A+\tau B}(B-A)\right\|d\tau.
\end{aligned}$$

Also

$$\begin{aligned}
&\sup_{\tau\in[0,1]}\left[\tau(1-\tau)\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\right]\\
&\le\frac{1}{4}\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|
\end{aligned}$$

and the inequalities in (2.19) are proved. □

Remark 5. *For functions f that are of class C^1 on I we have the inequalities*

$$\begin{aligned}
&\left\|\int_0^1 p(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \qquad (2.20)\\
&\le\|B-A\|\sup_{\tau\in[0,1]}\left|\tau\int_0^1 p(s)\,ds-\int_0^\tau p(s)\,ds\right|\\
&\times\int_0^1\|Df((1-\tau)A+\tau B)\|\,d\tau\\
&\le\frac{1}{4}\|B-A\|\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\\
&\times\int_0^1\|Df((1-\tau)A+\tau B)\|\,d\tau
\end{aligned}$$

for all A, $B\in\mathcal{SA}_I(H)$.

3. Examples for Some General Classes of Functions

Let f be a real function that is n-time differentiable on $(0,\infty)$, and let $f^{(n)}$ be its n-th derivative. Let f also denote the map induced by f on positive operators. Let $D^n f(A)$ be the n-th order Fréchet derivative of this map at the point A. For each A, the derivative $D^n f(A)$ is a n-linear operator on the space of all Hermitian operators. The norm of this operator is defined as

$$\|D^n f(A)\|:=\sup\{D^n f(A)(B_1,...,B_n)\mid\|B_1\|=...=\|B_n\|=1\}.$$

We consider the following class of functions defined on $(0,\infty)$ for a natural $n\ge 1$,

$$\mathcal{D}^{(n)}(0,\infty):=\left\{f\mid\|D^n f(A)\|=\left\|f^{(n)}(A)\right\|\text{ for all positive operators }A\right\}.$$

It is known (see for instance [10]) that every operator monotone function is in $\mathcal{D}^{(n)}(0,\infty)$ for all $n = 1, 2, \ldots$. Also the functions $f(t) = t^n$, $n = 2, 3, \ldots$, and $f(t) = \exp t$ are in $\mathcal{D}^{(1)}(0,\infty)$. None of these are operator monotone. Moreover, the power function $f(t) = t^p$ is in $\mathcal{D}^{(1)}(0,\infty)$ if p is in $(-\infty, 1]$ or in $[2,\infty)$, but not if p is in $\left(1, \sqrt{2}\right)$. Also that the functions $f(t) = \exp t$ and $f(t) = t^p$, $-\infty < p \le 1$, are in the class $\mathcal{D}^{(n)}(0,\infty)$ for all $n = 1, 2, \ldots$, and that for $p > 1$ the function $f(t) = t^p$ is in the class $\mathcal{D}^{(n)}(0,\infty)$ for all $n \ge [p+1]$, where $[\cdot]$ is the integer part (see for instance [10] and the references therein).

Proposition 1. *If $f \in \mathcal{D}^{(1)}(0,\infty)$, A, $B > 0$ and $p : [0,1] \to \mathbb{R}$ is Lebesgue integrable, then*

$$\begin{aligned}(3.1)\quad &\left\| \int_0^1 p(\tau) f((1-\tau)A + \tau B)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)A + \tau B)\, d\tau \right\| \\ &\le \|B - A\| \int_0^1 \tau(1-\tau) \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \\ &\times \|f'((1-\tau)A + \tau B)\|\, d\tau \\ &\le \frac{1}{4}\|B - A\| \int_0^1 \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \|f'((1-\tau)A + \tau B)\|\, d\tau\end{aligned}$$

and

$$\begin{aligned}(3.2)\quad &\left\| \int_0^1 p(\tau) f((1-\tau)A + \tau B)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)A + \tau B)\, d\tau \right\| \\ &\le \|B - A\| \sup_{\tau\in[0,1]} \left| \tau \int_0^1 p(s)\, ds - \int_0^\tau p(s)\, ds \right| \\ &\times \int_0^1 \|f'((1-\tau)A + \tau B)\|\, d\tau \\ &\le \frac{1}{4}\|B - A\| \sup_{\tau\in[0,1]} \left| \frac{\int_\tau^1 p(s)\, ds}{1-\tau} - \frac{\int_0^\tau p(s)\, ds}{\tau} \right| \\ &\times \int_0^1 \|f'((1-\tau)A + \tau B)\|\, d\tau.\end{aligned}$$

If p satisfies the condition (2.15), then we have the inequality

$$\begin{aligned}(3.3)\quad &\left\| \int_0^1 p(\tau) f((1-\tau)A + \tau B)\, d\tau - \int_0^1 p(\tau)\, d\tau \int_0^1 f((1-\tau)A + \tau B)\, d\tau \right\| \\ &\le \|B - A\| \sup_{\tau\in[0,1]} \|f'((1-\tau)A + \tau B)\| \int_0^1 \left(\tau - \frac{1}{2}\right) p(\tau)\, d\tau.\end{aligned}$$

If $f=\exp$, then

$$\begin{aligned}\int_0^1 \|\exp((1-t)A+tB)\|\,dt &\le \int_0^1 \exp\|((1-t)A+tB)\|\,dt\\ &\le \int_0^1 \exp\left[(1-t)\|A\|+t\|B\|\right]dt\\ &=\begin{cases}\frac{\exp\|B\|-\exp\|A\|}{\|B\|-\|A\|} \text{ for } \|B\|\neq\|A\|,\\ \\ \exp\|A\| \text{ for } \|B\|=\|A\|\end{cases}\end{aligned}$$

and by (3.2) we have

(3.4)

$$\begin{aligned}&\left\|\int_0^1 p(\tau)\exp((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1\exp((1-\tau)A+\tau B)\,d\tau\right\|\\ &\le \|B-A\|\sup_{\tau\in[0,1]}\left|\tau\int_0^1 p(s)\,ds-\int_0^\tau p(s)\,ds\right|\\ &\quad\times\begin{cases}\frac{\exp\|B\|-\exp\|A\|}{\|B\|-\|A\|} \text{ for } \|B\|\neq\|A\|,\\ \\ \exp\|A\| \text{ for } \|B\|=\|A\|\end{cases}\\ &\le \frac{1}{4}\|B-A\|\sup_{\tau\in[0,1]}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\\ &\quad\times\begin{cases}\frac{\exp\|B\|-\exp\|A\|}{\|B\|-\|A\|} \text{ for } \|B\|\neq\|A\|,\\ \\ \exp\|A\| \text{ for } \|B\|=\|A\|\end{cases},\end{aligned}$$

where $p:[0,1]\to\mathbb{R}$ is Lebesgue integrable and $A,\ B>0$.

Further, if we assume more about the function f we have:

Proposition 2. *If* $f\in\mathcal{D}^{(1)}(0,\infty)$ *and* f' *is operator convex and nonnegative on* $(0,\infty)$ *and* $p:[0,1]\to\mathbb{R}$ *is Lebesgue integrable, then for* $A,\ B>0$*, we have*

$$\begin{aligned}&\left\|\int_0^1 p(\tau)f((1-\tau)A+\tau B)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)A+\tau B)\,d\tau\right\| \quad (3.5)\\ &\le\frac{1}{12}\|B-A\|\sup_{t\in(0,1)}\left|\frac{\int_\tau^1 p(s)\,ds}{1-\tau}-\frac{\int_0^\tau p(s)\,ds}{\tau}\right|\left[\|f'(A)\|+\|f'(B)\|\right].\end{aligned}$$

Proof. Since f' is operator convex and nonnegative on $(0,\infty)$ then for $A,\ B>0$ we have

$$0\le f'((1-t)A+tB)\le(1-t)f'(A)+tf'(B)$$

for $t\in[0,1]$. By taking the norm, we get

$$\begin{aligned}\|f'((1-t)A+tB)\|&\le\|(1-t)f'(A)+tf'(B)\|\\ &\le(1-t)\|f'(A)\|+t\|f'(B)\|\end{aligned}$$

for $t\in[0,1]$.

By the first inequality in (3.1) we have

$$\begin{aligned}
&\left\| \int_0^1 p(\tau) f((1-\tau)A+\tau B)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)A+\tau B)\,d\tau \right\| \\
&\leq \|B-A\| \sup_{t\in(0,1)} \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| \int_0^1 \tau(1-\tau) \\
&\times \|f'((1-\tau)A+\tau B)\|\,d\tau \\
&\leq \|B-A\| \sup_{t\in(0,1)} \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| \\
&\int_0^1 \tau(1-\tau)\left[(1-\tau)\|f'(A)\| + \tau\|f'(B)\|\right] \\
&= \frac{1}{12}\|B-A\| \sup_{t\in(0,1)} \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| \left[\|f'(A)\| + \|f'(B)\|\right],
\end{aligned}$$

which proves (3.5). □

Consider the function $f(x) = x^r$on $(0,\infty)$, where $0 \leq r \leq 1$ or $2 \leq r \leq 3$. Then by (3.5) we get

$$\begin{aligned}
&\left\| \int_0^1 p(\tau)((1-\tau)A+\tau B)^r\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 ((1-\tau)A+\tau B)^r\,d\tau \right\| \\
&\leq \frac{r}{12}\|B-A\| \sup_{t\in(0,1)} \left| \frac{\int_\tau^1 p(s)\,ds}{1-\tau} - \frac{\int_0^\tau p(s)\,ds}{\tau} \right| \left[\left\|A^{r-1}\right\| + \left\|B^{r-1}\right\|\right]
\end{aligned} \tag{3.6}$$

for A, $B > 0$.

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