REVERSE OPERATOR INEQUALITIES FOR CONVEX FUNCTIONS IN HILBERT SPACES

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Abstract. In this paper we obtain several operator inequalities providing upper bounds for the difference

$$C^{*}f(A)C-f(C^{*}AC)$$

for any convex function $f:I\to\mathbb{R}$, any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A)\subset I$ and any isometry $C\in\mathcal{B}(H)$. Some examples for convex and operator convex functions are also provided.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\leq B$ with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$ imply $f(A)\leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [9] and the references therein.

As examples of such functions, we note that $f(t)=t^{r}$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^{r}$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^{t}$ is neither operator convex nor operator monotone.

For recent inequalities for operator convex functions see [1]-[8] and [10]-[19].

The following Jensen's operator inequality is well know, see for instance [9, p. 10]:

Theorem 1. *Let H be a Hilbert space and f be a real valued continuous function on the interval I. Then f is operator convex on I if and only if*

$$f(C^{*}AC)\leq C^{*}f(A)C \tag{1.2}$$

for any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A)\subset I$ and any isometry $C\in\mathcal{B}(H)$, i.e. C satisfies the condition $C^{*}C=1_{H}$.

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It is known that there are convex functions f for which the inequality (1.2) does not hold, however one can obtain several operator inequalities providing upper bounds for the difference

$$C^{*}f(A)C-f(C^{*}AC)$$

for any convex function $f:I\to\mathbb{R}$, any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A)\subset I$ and any isometry $C\in\mathcal{B}(H)$. Some examples for convex and operator convex functions are also provided.

2. Main Results

We use the following result that was obtained in [4]:

Lemma 1. *If $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$, then*

$$0\leq\frac{(b-t)f(a)+(t-a)f(b)}{b-a}-f(t)\leq(b-t)(t-a)\frac{f'_{-}(b)-f'_{+}(a)}{b-a}\leq\frac{1}{4}(b-a)\left[f'_{-}(b)-f'_{+}(a)\right]\tag{2.1}$$

for any $t\in[a,b]$.

If the lateral derivatives $f'_{-}(b)$ and $f'_{+}(a)$ are finite, then the second inequality and the constant $1/4$ are sharp.

We have:

Theorem 2. *Let $f:[m,M]\to\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If $C\in\mathcal{B}(H)$ is an isometry, i.e. $C^{*}C=1_H$, then*

$$\begin{aligned}C^{*}f(A)C-f(C^{*}AC)&\leq\frac{f'_{-}(M)-f'_{+}(m)}{M-m}(M1_H-C^{*}AC)(C^{*}AC-m1_H)\\&\leq\frac{1}{4}(M-m)\left[f'_{-}(M)-f'_{+}(m)\right]1_H.\end{aligned}\tag{2.2}$$

Proof. Utilising the continuous functional calculus for a selfadjoint operator T with $0\leq T\leq 1_H$ and the convexity of f on $[m,M]$, we have

$$f(m(1_H-T)+MT)\leq f(m)(1_H-T)+f(M)T\tag{2.3}$$

in the operator order.

If we take in (2.3)

$$0\leq T=\frac{A-m1_H}{M-m}\leq 1_H,$$

then we get

$$\begin{aligned}&f\left(m\left(1_H-\frac{A-m1_H}{M-m}\right)+M\frac{A-m1_H}{M-m}\right)\\&\leq f(m)\left(1_H-\frac{A-m1_H}{M-m}\right)+f(M)\frac{A-m1_H}{M-m}.\end{aligned}\tag{2.4}$$

Observe that

$$\begin{aligned} & m\left(1_H-\frac{A-m1_H}{M-m}\right)+M\frac{A-m1_H}{M-m} \\ & =\frac{m\left(M1_H-A\right)+M\left(A-m1_H\right)}{M-m}=A \end{aligned}$$

and

$$\begin{aligned} & f\left(m\right)\left(1_H-\frac{A-m1_H}{M-m}\right)+f\left(M\right)\frac{A-m1_H}{M-m} \\ & =\frac{f\left(m\right)\left(M1_H-A\right)+f\left(M\right)\left(A-m1_H\right)}{M-m} \end{aligned}$$

and by (2.4) we get the following inequality of interest

$$f\left(A\right)\leq\frac{f\left(m\right)\left(M1_H-A\right)+f\left(M\right)\left(A-m1_H\right)}{M-m}. \tag{2.5}$$

If we multiply (2.5) to the left with C^* and to the right with C we get

$$\begin{aligned} C^*f\left(A\right)C & \leq C^*\left[\frac{f\left(m\right)\left(M1_H-A\right)+f\left(M\right)\left(A-m1_H\right)}{M-m}\right]C \\ & =\frac{f\left(m\right)C^*\left(M1_H-A\right)C+f\left(M\right)C^*\left(A-m1_H\right)C}{M-m} \\ & =\frac{f\left(m\right)\left(MC^*C-C^*AC\right)+f\left(M\right)\left(C^*AC-mC^*C\right)}{M-m} \\ & =\frac{f\left(m\right)\left(M1_H-C^*AC\right)+f\left(M\right)\left(C^*AC-m1_H\right)}{M-m}, \end{aligned}$$

which implies that

$$\begin{aligned} & C^*f\left(A\right)C-f\left(C^*AC\right) \\ & \leq\frac{f\left(m\right)\left(M1_H-C^*AC\right)+f\left(M\right)\left(C^*AC-m1_H\right)}{M-m}-f\left(C^*AC\right). \end{aligned} \tag{2.6}$$

Since $m1_H\leq C^*AC\leq M1_H$, then by using (2.1) for $a=m$, $b=M$ and the continuous functional calculus, we have

$$\begin{aligned} & \frac{f\left(m\right)\left(M1_H-C^*AC\right)+f\left(M\right)\left(C^*AC-m1_H\right)}{M-m}-f\left(C^*AC\right) \\ & \leq\frac{f'_-\left(M\right)-f'_+\left(m\right)}{M-m}\left(M1_H-C^*AC\right)\left(C^*AC-m1_H\right) \\ & \leq\frac{1}{4}\left(M-m\right)\left[f'_-\left(M\right)-f'_+\left(m\right)\right]1_H. \end{aligned} \tag{2.7}$$

By making use of (2.6) and (2.7) we get the desired result (2.2). □

Corollary 1. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If $C\in\mathcal{B}(H)$ is an* *isometry, then*

$$0 \le C^* f(A) C - f(C^* AC) \le \frac{f'_-(M) - f'_+(m)}{M-m} (M1_H - C^* AC)(C^* AC - m1_H) \le \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right] 1_H. \tag{2.8}$$

We also have the following scalar inequality of interest:

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ and $t \in [0,1]$, then*

$$2\min\{t, 1-t\}\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right] \le (1-t) f(a) + t f(b) - f((1-t)a + tb) \le 2\max\{t,1-t\}\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right]. \tag{2.9}$$

The proof follows, for instance, by Corollary 1 from [5] for $n = 2$, $p_1 = 1-t$, $p_2 = t$, $t \in [0,1]$ and $x_1 = a$, $x_2 = b$.

Theorem 3. *Let $f : [m,M] \to \mathbb{R}$ be a convex function on $[m,M]$ and A a self-adjoint operator with the spectrum $\operatorname{Sp}(A) \subset [m,M]$. If $C \in \mathcal{B}(H)$ is an isometry, then*

$$\begin{aligned} &2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times \left(\frac{1}{2}(M-m)1_H - \left|C^*AC - \frac{1}{2}(m+M)1_H\right|\right) \\ &\le \frac{f(m)(M1_H - C^*AC) + f(M)(C^*AC - m1_H)}{M-m} - f(C^*AC) \\ &\le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times \left(\frac{1}{2}(M-m)1_H + \left|C^*AC - \frac{1}{2}(m+M)1_H\right|\right) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times \left(\frac{1}{2}(M-m)1_H - C^*\left|A - \frac{1}{2}(m+M)1_H\right|C\right) \\ &\le \frac{f(m)(M1_H - C^*AC) + f(M)(C^*AC - m1_H)}{M-m} - C^*f(A)C \\ &\le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times \left(\frac{1}{2}(M-m)1_H + C^*\left|A - \frac{1}{2}(m+M)1_H\right|C\right). \end{aligned} \tag{2.11}$$

Proof. We have from (2.9) that

$$2\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \tag{2.12}$$
$$\leq(1-t)f(m)+tf(M)-f((1-t)m+tM)$$
$$\leq 2\left(\frac{1}{2}+\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],$$

for all $t\in[0,1]$.

Utilising the continuous functional calculus for a selfadjoint operator T with $0\leq T\leq 1_H$ we get from (2.12) that

$$2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H-\left|T-\frac{1}{2}1_H\right|\right) \tag{2.13}$$
$$\leq(1-T)f(m)+Tf(M)-f((1-T)m+TM)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H+\left|T-\frac{1}{2}1_H\right|\right),$$

in the operator order.

If we take in (2.13)

$$0\leq T=\frac{A-m1_H}{M-m}\leq 1_H,$$

then, like in the proof of Theorem 2, we get

$$2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \tag{2.14}$$
$$\times\left(\frac{1}{2}(M-m)1_H-\left|A-\frac{1}{2}(m+M)1_H\right|\right)$$
$$\leq\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)1_H+\left|A-\frac{1}{2}(m+M)1_H\right|\right).$$

Since $m1_H\leq C^*AC\leq M1_H$, then by writing the inequality (2.14) for C^*AC instead of A we get (2.10).

If we multiply (2.14) to the left with C^* and to the right with C we get

$$2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times C^*\left(\frac{1}{2}(M-m)1_H-\left|A-\frac{1}{2}(m+M)1_H\right|\right)C$$
$$\leq C^*\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]C-C^*f(A)C$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times C^*\left(\frac{1}{2}(M-m)1_H+\left|A-\frac{1}{2}(m+M)1_H\right|\right)C,$$

which is equivalent to (2.11). $\square$

Corollary 2. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If $C\in\mathcal{B}(H)$ is an isometry, then*

$$\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\frac{1}{2}(M-m)1_H-C^*\left|A-\frac{1}{2}(m+M)1_H\right|C\right)\\
&\leq\frac{f(m)(M1_H-C^*AC)+f(M)(C^*AC-m1_H)}{M-m}-C^*f(A)C\\
&\leq\frac{f(m)(M1_H-C^*AC)+f(M)(C^*AC-m1_H)}{M-m}-f(C^*AC)\\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\frac{1}{2}(M-m)1_H+\left|C^*AC-\frac{1}{2}(m+M)1_H\right|\right).
\end{aligned}\tag{2.15}$$

We also have:

Corollary 3. *Let $f:[m,M]\to\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If $C\in\mathcal{B}(H)$ is an isometry, then*

$$\begin{aligned}
&C^*f(A)C-f(C^*AC)\\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\frac{1}{2}(M-m)1_H+\left|C^*AC-\frac{1}{2}(m+M)1_H\right|\right)\\
&\leq 2(M-m)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_H.
\end{aligned}\tag{2.16}$$

Proof. From (2.6) we have

$$\begin{aligned}
&C^*f(A)C-f(C^*AC)\\
&\leq\frac{f(m)(M1_H-C^*AC)+f(M)(C^*AC-m1_H)}{M-m}-f(C^*AC)
\end{aligned}$$

and from (2.11) we have

$$\begin{aligned}
&\frac{f(m)(M1_H-C^*AC)+f(M)(C^*AC-m1_H)}{M-m}-f(C^*AC)\\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\frac{1}{2}(M-m)1_H+\left|C^*AC-\frac{1}{2}(m+M)1_H\right|\right),
\end{aligned}$$

which produce the desired result (2.16). $\square$

Remark 1. *If $f:[m,M]\to\mathbb{R}$ is an operator convex function on $[m,M]$, A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$ and $C\in\mathcal{B}(H)$ is an isometry, then*

$$\begin{aligned}
0 &\le C^{*}f(A)C-f(C^{*}AC) \\
&\le 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_H+\left|C^{*}AC-\frac{1}{2}(m+M)1_H\right|\right) \\
&\le 2(M-m)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_H.
\end{aligned}\tag{2.17}$$

We also have [4]:

Lemma 3. *Assume that $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. If f' is K-Lipschitzian on $[a,b]$, then*

$$\begin{aligned}
&|(1-t)f(a)+tf(b)-f((1-t)a+tb)| \\
&\le\frac{1}{2}K(b-t)(t-a)\le\frac{1}{8}K(b-a)^2
\end{aligned}\tag{2.18}$$

for all $t\in[0,1]$.

The constants $1/2$ and $1/8$ are the best possible in (2.18).

Remark 2. *If $f:[a,b]\to\mathbb{R}$ is twice differentiable and $f''\in L_\infty[a,b]$, then*

$$\begin{aligned}
&|(1-t)f(a)+tf(b)-f((1-t)a+tb)| \\
&\le\frac{1}{2}\|f''\|_{[a,b],\infty}(b-t)(t-a)\le\frac{1}{8}\|f''\|_{[a,b],\infty}(b-a)^2,
\end{aligned}\tag{2.19}$$

where $\|f''\|_{[a,b],\infty}:=\operatorname{essup}_{t\in[a,b]}|f''(t)|<\infty$. The constants $1/2$ and $1/8$ are the best possible in (2.19).

We have:

Theorem 4. *Let $f:[m,M]\to\mathbb{R}$ be a twice differentiable convex function on $[m,M]$ with $\|f''\|_{[m,M],\infty}:=\operatorname{essup}_{t\in[m,M]}f''(t)<\infty$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If $C\in\mathcal{B}(H)$ is an isometry, then*

$$\begin{aligned}
&C^{*}f(A)C-f(C^{*}AC) \\
&\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H-C^{*}AC)(C^{*}AC-m1_H) \\
&\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H.
\end{aligned}\tag{2.20}$$

Proof. From (2.19) and the continuous functional calculus, we get

$$\begin{aligned}
0&\le\frac{f(m)(M1_H-B)+f(M)(B-m1_H)}{M-m}-f(B) \\
&\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H-B)(B-m1_H) \\
&\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H
\end{aligned}\tag{2.21}$$

where B is a selfadjoint operator with the spectrum $\operatorname{Sp}(B)\subset[m,M]$.

If we take $m \leq B = C^*AC \leq M$ in (2.21) we get

$$
\begin{aligned}
0 &\leq \frac{f(m)(M1_H - C^*AC) + f(M)(C^*AC - m1_H)}{M-m} - f(C^*AC) \\
&\leq \frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H - C^*AC)(C^*AC - m1_H) \\
&\leq \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H .
\end{aligned} \tag{2.22}
$$

Since

$$
\begin{aligned}
&C^*f(A)C - f(C^*AC) \\
&\leq \frac{f(m)(M1_H - C^*AC) + f(M)(C^*AC - m1_H)}{M-m} - f(C^*AC),
\end{aligned}
$$

hence by (2.22) we get (2.20). □

Corollary 4. *Let $f : [m, M] \to \mathbb{R}$ be an operator convex function on $[m, M]$ and A a selfadjoint operator with the spectrum $\mathrm{Sp}(A) \subset [m, M]$. If $C \in \mathcal{B}(H)$ is an isometry, then*

$$
\begin{aligned}
0 &\leq C^*f(A)C - f(C^*AC) \\
&\leq \frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H - C^*AC)(C^*AC - m1_H) \\
&\leq \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H .
\end{aligned} \tag{2.23}
$$

3. Some Examples

We consider the exponential function $f(x) = \exp(\alpha x)$ with $\alpha \in \mathbb{R} \setminus \{0\}$. This function is convex but not operator convex on $\mathbb{R}$. If A is selfadjoint with $\mathrm{Sp}(A) \subset [m, M]$ for some $m < M$ and $C \in \mathcal{B}(H)$ is an isometry, then by (2.2), (2.16) and (2.20) we have

$$
\begin{aligned}
&C^*\exp(\alpha A)C - \exp(\alpha C^*AC) \\
&\leq \alpha\frac{\exp(\alpha M) - \exp(\alpha m)}{M-m}(M1_H - C^*AC)(C^*AC - m1_H) \\
&\leq \frac{1}{4}\alpha(M-m)[\exp(\alpha M) - \exp(\alpha m)]1_H,
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
&C^*\exp(\alpha A)C - \exp(\alpha C^*AC) \\
&\leq 2\left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_H + \left|C^*AC - \frac{1}{2}(m+M)1_H\right|\right) \\
&\leq 2(M-m)\left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha\frac{m+M}{2}\right)\right]1_H
\end{aligned} \tag{3.2}
$$

and

$$C^* f(A) C - f(C^*AC) \le \frac{1}{2}\alpha^2 \begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \times (M 1_H - C^*AC)(C^*AC - m 1_H) \le \frac{1}{8}\alpha^2 (M-m)^2 \begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \times 1_H. \tag{3.3}$$

The function $f(x) = -\ln x$, $x > 0$ is operator convex on $(0,\infty)$. If A is selfadjoint with $\operatorname{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $C \in \mathcal{B}(H)$ is an isometry, then by (2.8), (2.17) and (2.23) we have

$$0 \le \ln(C^*AC) - C^* \ln(A) C \le \frac{1}{mM}(M 1_H - C^*AC)(C^*AC - m 1_H) \le \frac{1}{4mM}(M-m)^2 1_H, \tag{3.4}$$

$$0 \le \ln(C^*AC) - C^* \ln(A) C \le 2\ln\left(\frac{m+M}{2\sqrt{mM}}\right)\left(\frac{1}{2}(M-m) 1_H + \left| C^*AC - \frac{1}{2}(m+M) 1_H \right|\right) \le 2(M-m)\ln\left(\frac{m+M}{2\sqrt{mM}}\right) 1_H \tag{3.5}$$

and

$$0 \le \ln(C^*AC) - C^* \ln(A) C \le \frac{1}{2m^2}(M 1_H - C^*AC)(C^*AC - m 1_H) \le \frac{1}{8m^2}(M-m)^2 1_H. \tag{3.6}$$

We observe that if $M > 2m$ then the bound in (3.4) is better than the one from (3.6). If $M < 2m$, then the conclusion is the other way around.

The function $f(x) = x\ln x$, $x > 0$ is operator convex on $(0,\infty)$. If A is selfadjoint with $\operatorname{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $C \in \mathcal{B}(H)$ is an isometry, then by (2.8), (2.17) and (2.23) we have

$$0 \le C^* A \ln(A) C - C^*AC \ln(C^*AC) \le \frac{\ln(M) - \ln(m)}{M-m}(M 1_H - C^*AC)(C^*AC - m 1_H) \le \frac{1}{4}(M-m)[\ln(M) - \ln(m)] 1_H, \tag{3.7}$$

$$\begin{aligned} 0 &\le C^* A \ln(A) C - C^*AC \ln(C^*AC) \\ &\le 2\left[\frac{m\ln(m) + M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right] \\ &\times \left(\frac{1}{2}(M-m) 1_H + \left| C^*AC - \frac{1}{2}(m+M) 1_H \right|\right) \\ &\le 2(M-m)\left[\frac{m\ln(m) + M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right] 1_H \end{aligned} \tag{3.8}$$

and

$$0 \le C^* A \ln(A) C - C^* AC \ln(C^* AC) \le \frac{1}{2m}(M1_H - C^* AC)(C^* AC - m1_H) \le \frac{1}{8m}(M-m)^2 1_H. \tag{3.9}$$

Consider the power function $f(x) = x^r$, $x \in (0,\infty)$ and r a real number. If $r \in (-\infty, 0] \cup [1,\infty)$, then f is convex and for $r \in [-1,0] \cup [1,2]$ is operator convex. If we use the inequalities (2.2), (2.16) and (2.20) we have for $r \in (-\infty, 0] \cup [1,\infty)$ that

$$\begin{aligned} & C^* A^r C - (C^* AC)^r \\ & \le r\frac{M^{r-1} - m^{r-1}}{M-m}(M1_H - C^* AC)(C^* AC - m1_H) \\ & \le \frac{1}{4} r (M-m)\left[M^{r-1} - m^{r-1}\right] 1_H, \end{aligned} \tag{3.10}$$

$$\begin{aligned} & C^* A^r C - (C^* AC)^r \\ & \le 2\left[\frac{m^r + M^r}{2} - \left(\frac{m+M}{2}\right)^r\right] \\ & \times \left(\frac{1}{2}(M-m) 1_H + \left|C^* AC - \frac{1}{2}(m+M) 1_H\right|\right) \\ & \le 2(M-m)\left[\frac{m^r + M^r}{2} - \left(\frac{m+M}{2}\right)^r\right] 1_H \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} & C^* A^r C - (C^* AC)^r \\ & \le \frac{1}{2} r(r-1) \begin{cases} M^{r-2} \text{ for } r \ge 2 \\ m^{r-2} \text{ for } r \in (-\infty, 0] \cup [1,2) \end{cases} \\ & \times (M1_H - C^* AC)(C^* AC - m1_H) \\ & \le \frac{1}{8} r(r-1)(M-m)^2 \begin{cases} M^{r-2} \text{ for } r \ge 2 \\ m^{r-2} \text{ for } r \in (-\infty, 0] \cup [1,2) \end{cases} \times 1_H, \end{aligned} \tag{3.12}$$

where A is selfadjoint with $\operatorname{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $C \in \mathcal{B}(H)$ is an isometry.

If $r \in [-1,0] \cup [1,2]$, then we also have $0 \le C^* A^r C - (C^* AC)^r$ in the inequalities (3.10)-(3.12).

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