ON SOME REVERSE OPERATOR SUM INEQUALITIES FOR CONVEX FUNCTIONS IN HILBERT SPACES

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Abstract. In this paper we obtain several operator inequalities providing upper bounds for the difference

$$\sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right)$$

for any convex function $f : I \to \mathbb{R}$, any A_j selfadjoint operators with the spectrum $\mathrm{Sp}(A_j) \subset [m, M]$ and $C_j \in \mathcal{B}(H)$ with $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$. Some examples for convex and operator convex functions are also provided.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\mathrm{Sp}(A), \mathrm{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [9] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

For recent inequalities for operator convex functions see [1]-[8] and [10]-[19].

The following Jensen's operator inequality is well know, see for instance [9, p. 10]:

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Selfadjoint bounded linear operators, Functions of operators, Operator convex functions, Jensen's operator inequality.

Theorem 1. *Let H be a Hilbert space and f be a real valued continuous function on the interval I. Then f is operator convex on I if and only if*

$$f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \le \sum_{j=1}^{k} C_j^* f(A_j) C_j \tag{1.2}$$

for any A_j selfadjoint operators with the spectrum $\operatorname{Sp}(A_j) \subset [m, M]$ with $j = 1, ..., k$, and $C_j \in \mathcal{B}(H)$ for $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$,.

If P_j, with $j = 1, ..., k$ are projections satisfying the condition $\sum_{j=1}^{k} P_j = 1_H$ then by (1.2) for $C_j = P_j$ with $j = 1, ..., k$ we have

$$f\left(\sum_{j=1}^{k} P_j A_j P_j\right) \le \sum_{j=1}^{k} P_j f(A_j) P_j. \tag{1.3}$$

This inequality is also a sufficient condition for the function f to be operator convex on I, see [9, p. 10].

It is known that there are convex functions f for which the inequality (1.2) does not hold, however one can obtain several operator inequalities providing upper bounds for the difference

$$\sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right)$$

for any convex function $f : I \to \mathbb{R}$, any A_j selfadjoint operators with the spectrum $\operatorname{Sp}(A_j) \subset [m, M]$ with $j = 1, ..., k$, and $C_j \in \mathcal{B}(H)$ for $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$. Some examples for convex and operator convex functions are also provided.

2. Main Results

We use the following result that was obtained in [4]:

Lemma 1. *If $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$, then*

$$\begin{aligned} 0 &\le \frac{(b-t) f(a) + (t-a) f(b)}{b-a} - f(t) \\ &\le (b-t)(t-a) \frac{f'_-(b) - f'_+(a)}{b-a} \le \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right] \end{aligned} \tag{2.1}$$

for any $t \in [a, b]$.

If the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $1/4$ are sharp.

We have:

Theorem 2. *Let $f : [m, M] \to \mathbb{R}$ be a convex function on $[m, M]$ and A_j selfadjoint operators with the spectrum $\operatorname{Sp}(A_j) \subset [m, M]$ for $j = 1, ..., k$. If $C_j \in \mathcal{B}(H)$ for*

$j=1,...,k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$, then

$$\sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \tag{2.2}$$
$$\leq \frac{f'_-(M) - f'_+(m)}{M-m}\left(M 1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m 1_H\right)$$
$$\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right] 1_H.$$

If P_j, with $j=1,...,k$ are projections satisfying the condition $\sum_{j=1}^{k} P_j = 1_H$, then

$$\sum_{j=1}^{k} P_j f(A_j) P_j - f\left(\sum_{j=1}^{k} P_j A_j P_j\right) \tag{2.3}$$
$$\leq \frac{f'_-(M) - f'_+(m)}{M-m}\left(M 1_H - \sum_{j=1}^{k} P_j A_j P_j\right)\left(\sum_{j=1}^{k} P_j A_j P_j - m 1_H\right)$$
$$\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right] 1_H.$$

Proof. Utilising the continuous functional calculus for a selfadjoint operator T with $0 \leq T \leq 1_H$ and the convexity of f on $[m, M]$, we have

$$f(m(1_H - T) + MT) \leq f(m)(1_H - T) + f(M) T \tag{2.4}$$

in the operator order.

If we take in (2.4)

$$0 \leq T = \frac{A_j - m 1_H}{M-m} \leq 1_H,$$

then we get

$$f\left(m\left(1_H - \frac{A_j - m1_H}{M-m}\right) + M\frac{A_j - m1_H}{M-m}\right) \tag{2.5}$$
$$\leq f(m)\left(1_H - \frac{A_j - m1_H}{M-m}\right) + f(M)\frac{A_j - m1_H}{M-m}.$$

Observe that

$$m\left(1_H - \frac{A_j - m1_H}{M-m}\right) + M\frac{A_j - m1_H}{M-m}$$
$$= \frac{m(M1_H - A_j) + M(A_j - m1_H)}{M-m} = A_j$$

and

$$f(m)\left(1_H - \frac{A_j - m1_H}{M-m}\right) + f(M)\frac{A_j - m1_H}{M-m}$$
$$= \frac{f(m)(M1_H - A_j) + f(M)(A_j - m1_H)}{M-m}$$

and by (2.5) we get the following inequality of interest

$$f(A_j) \leq \frac{f(m)(M1_H - A_j) + f(M)(A_j - m1_H)}{M-m}. \tag{2.6}$$

If we multiply (2.6) to the left with C_j^* and to the right with C_j we get

$$
\begin{aligned}
&\sum_{j=1}^{k} C_j^* f(A_j) C_j^* \\
&\leq \sum_{j=1}^{k} C_j^* \left[\frac{f(m)(M1_H - A_j) + f(M)(A_j - m1_H)}{M-m} \right] C_j \\
&= \frac{f(m)\sum_{j=1}^{k} C_j^* (M1_H - A_j) C_j + f(M)\sum_{j=1}^{k} C_j^* (A_j - m1_H) C_j}{M-m} \\
&= \frac{f(m)\left(M\sum_{j=1}^{k} C_j^* C_j - \sum_{j=1}^{k} C_j^* A_j C_j\right) + f(M)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m\sum_{j=1}^{k} C_j^* C_j\right)}{M-m} \\
&= \frac{f(m)\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) + f(M)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right)}{M-m},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
&\leq \frac{f(m)\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) + f(M)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right)}{M-m} \\
&- f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right).
\end{aligned} \tag{2.7}
$$

Since $m1_H \leq \sum_{j=1}^{k} C_j^* A_j C_j \leq M1_H$, then by using (2.1) for $a = m$, $b = M$ and the continuous functional calculus, we have

$$
\begin{aligned}
&\frac{f(m)\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) + f(M)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right)}{M-m} \\
&- f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
&\leq \frac{f'_-(M) - f'_+(m)}{M-m}\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right) \\
&\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right] 1_H.
\end{aligned} \tag{2.8}
$$

By making use of (2.7) and (2.8) we get the desired result (2.2). □

Corollary 1. *Let $f : [m, M] \to \mathbb{R}$ be an operator convex function on $[m, M]$ and A_j selfadjoint operators with the spectrum $\operatorname{Sp}(A_j) \subset [m, M]$ for $j = 1, ..., k$. If*

$C_j \in \mathcal{B}(H)$ for $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$, then

$$
\begin{aligned}
(2.9) \qquad 0 &\le \sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
&\le \frac{f'_-(M) - f'_+(m)}{M - m} \left(M 1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m 1_H\right) \\
&\le \frac{1}{4}(M - m)\left[f'_-(M) - f'_+(m)\right] 1_H.
\end{aligned}
$$

We also have the following scalar inequality of interest:

Lemma 2. *Let $f : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$ and $t \in [0, 1]$, then*

$$
\begin{aligned}
(2.10) \qquad & 2 \min\{t, 1 - t\}\left[\frac{f(a) + f(b)}{2} - f\left(\frac{a + b}{2}\right)\right] \\
&\le (1 - t) f(a) + t f(b) - f((1 - t) a + t b) \\
&\le 2 \max\{t, 1 - t\}\left[\frac{f(a) + f(b)}{2} - f\left(\frac{a + b}{2}\right)\right].
\end{aligned}
$$

The proof follows, for instance, by Corollary 1 from [5] for $n = 2$, $p_1 = 1 - t$, $p_2 = t$, $t \in [0, 1]$ and $x_1 = a$, $x_2 = b$.

Theorem 3. *Let $f : [m, M] \to \mathbb{R}$ be a convex function on $[m, M]$ and A_j selfadjoint operators with the spectrum $\operatorname{Sp}(A_j) \subset [m, M]$ for $j = 1, ..., k$. If $C_j \in \mathcal{B}(H)$ for $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$, then*

$$
\begin{aligned}
(2.11) \qquad & 2\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m + M}{2}\right)\right] \\
&\times \left(\frac{1}{2}(M - m) 1_H - \left|\sum_{j=1}^{k} C_j^* A_j C_j - \frac{1}{2}(m + M) 1_H\right|\right) \\
&\le \frac{f(m)\left(M 1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) + f(M)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m 1_H\right)}{M - m} \\
&- f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
&\le 2\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m + M}{2}\right)\right] \\
&\times \left(\frac{1}{2}(M - m) 1_H + \left|\sum_{j=1}^{k} C_j^* A_j C_j - \frac{1}{2}(m + M) 1_H\right|\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_H-\sum_{j=1}^{k}C_j^*\left|A_j-\frac{1}{2}(m+M)1_H\right|C_j\right) \\
&\leq\frac{f(m)\left(M1_H-\sum_{j=1}^{k}C_j^*A_jC_j\right)+f(M)\left(\sum_{j=1}^{k}C_j^*A_jC_j-m1_H\right)}{M-m} \\
&-\sum_{j=1}^{k}C_j^*f(A_j)C_j \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_H+\sum_{j=1}^{k}C_j^*\left|A_j-\frac{1}{2}(m+M)1_H\right|C_j\right).
\end{aligned}
\tag{2.12}
$$

Proof. We have from (2.10) that

$$
\begin{aligned}
&2\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\leq(1-t)f(m)+tf(M)-f((1-t)m+tM) \\
&\leq 2\left(\frac{1}{2}+\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],
\end{aligned}
\tag{2.13}
$$

for all $t\in[0,1]$.

Utilising the continuous functional calculus for a selfadjoint operator T with $0\leq T\leq 1_H$ we get from (2.13) that

$$
\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H-\left|T-\frac{1}{2}1_H\right|\right) \\
&\leq(1-T)f(m)+Tf(M)-f((1-T)m+TM) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H+\left|T-\frac{1}{2}1_H\right|\right),
\end{aligned}
\tag{2.14}
$$

in the operator order.

If we take in (2.14)

$$
0\leq T=\frac{A_j-m1_H}{M-m}\leq 1_H,
$$

then, like in the proof of Theorem 2, we get

$$
\begin{aligned}
(2.15) \qquad & 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
& \times\left(\frac{1}{2}(M-m)1_H-\left|A_j-\frac{1}{2}(m+M)1_H\right|\right) \\
& \leq \frac{f(m)(M1_H-A_j)+f(M)(A_j-m1_H)}{M-m}-f(A_j) \\
& \leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
& \times\left(\frac{1}{2}(M-m)1_H+\left|A_j-\frac{1}{2}(m+M)1_H\right|\right).
\end{aligned}
$$

Since $m1_H \leq \sum_{j=1}^{k} C_j^* A_j C_j \leq M1_H$, then by writing the inequality (2.15) for $\sum_{j=1}^{k} C_j^* A_j C_j$ instead of A_j we get (2.11).

If we multiply (2.15) to the left with C_j^* and to the right with C_j we get

$$
\begin{aligned}
& 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
& \times\sum_{j=1}^{k} C_j^*\left(\frac{1}{2}(M-m)1_H-\left|A_j-\frac{1}{2}(m+M)1_H\right|\right)C_j \\
& \leq \sum_{j=1}^{k} C_j^*\left[\frac{f(m)(M1_H-A_j)+f(M)(A_j-m1_H)}{M-m}\right]C_j \\
& -\sum_{j=1}^{k} C_j^* f(A_j)\,C_j \\
& \leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
& \times\sum_{j=1}^{k} C_j^*\left(\frac{1}{2}(M-m)1_H+\left|A_j-\frac{1}{2}(m+M)1_H\right|\right)C_j,
\end{aligned}
$$

which is equivalent to (2.12). □

Corollary 2. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A_j selfadjoint operators with the spectrum $\mathrm{Sp}(A_j)\subset[m,M]$ for $j=1,...,k$. If*

$C_j \in \mathcal{B}(H)$ for $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$, then,

$$
\begin{aligned}
(2.16)\qquad & 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
& \times \left(\frac{1}{2}(M-m)1_H - \sum_{j=1}^{k} C_j^* \left|A_j - \frac{1}{2}(m+M)1_H\right| C_j\right) \\
& \leq \frac{f(m)\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) + f(M)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right)}{M-m} \\
& - \sum_{j=1}^{k} C_j^* f(A_j) C_j \\
& \leq \frac{f(m)\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) + f(M)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right)}{M-m} \\
& - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
& \leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
& \times \left(\frac{1}{2}(M-m)1_H + \left|\sum_{j=1}^{k} C_j^* A_j C_j - \frac{1}{2}(m+M)1_H\right|\right).
\end{aligned}
$$

We also have:

Corollary 3. *Let $f : [m, M] \to \mathbb{R}$ be a convex function on $[m, M]$ and A_j selfadjoint operators with the spectrum $\mathrm{Sp}(A_j) \subset [m, M]$ for $j = 1, ..., k$. If $C_j \in \mathcal{B}(H)$ for $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$, then*

$$
\begin{aligned}
(2.17)\qquad & \sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
& \leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
& \times \left(\frac{1}{2}(M-m)1_H + \left|\sum_{j=1}^{k} C_j^* A_j C_j - \frac{1}{2}(m+M)1_H\right|\right) \\
& \leq 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] 1_H.
\end{aligned}
$$

If P_j, with $j=1,...,k$ are projections satisfying the condition $\sum_{j=1}^{k} P_j = 1_H$, then

$$\begin{aligned}
&\sum_{j=1}^{k} P_j f(A_j) P_j - f\left(\sum_{j=1}^{k} P_j A_j P_j\right) \\
&\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&\times \left(\frac{1}{2}(M-m)1_H + \left|\sum_{j=1}^{k} P_j A_j P_j - \frac{1}{2}(m+M)1_H\right|\right) \\
&\leq 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] 1_H.
\end{aligned} \tag{2.18}$$

Proof. From (2.7) we have

$$\begin{aligned}
&\sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
&\leq \frac{f(m)\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) + f(M)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right)}{M-m} \\
&- f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right)
\end{aligned}$$

and from (2.12) we have

$$\begin{aligned}
&\frac{f(m)\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) + f(M)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right)}{M-m} \\
&- f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
&\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&\times \left(\frac{1}{2}(M-m)1_H + \left|\sum_{j=1}^{k} C_j^* A_j C_j - \frac{1}{2}(m+M)1_H\right|\right).
\end{aligned}$$

which produce the desired result (2.17). □

Remark 1. *If $f:[m,M]\to\mathbb{R}$ is an operator convex function on $[m,M]$, A_j selfadjoint operators with the spectrum $\operatorname{Sp}(A_j)\subset[m,M]$ for $j=1,...,k$. If $C_j \in$*

$\mathcal{B}(H)$ for $j=1,...,k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$, then

$$0 \le \sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \times \left(\frac{1}{2}(M-m)1_H + \left|\sum_{j=1}^{k} C_j^* A_j C_j - \frac{1}{2}(m+M)1_H\right|\right) \le 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]1_H. \tag{2.19}$$

We also have [4]:

Lemma 3. *Assume that $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. If f' is K-Lipschitzian on $[a,b]$, then*

$$|(1-t)f(a) + tf(b) - f((1-t)a+tb)| \le \frac{1}{2}K(b-t)(t-a) \le \frac{1}{8}K(b-a)^2 \tag{2.20}$$

for all $t\in[0,1]$.

The constants $1/2$ and $1/8$ are the best possible in (2.20).

Remark 2. *If $f:[a,b]\to\mathbb{R}$ is twice differentiable and $f''\in L_\infty[a,b]$, then*

$$|(1-t)f(a) + tf(b) - f((1-t)a+tb)| \le \frac{1}{2}\|f''\|_{[a,b],\infty}(b-t)(t-a) \le \frac{1}{8}\|f''\|_{[a,b],\infty}(b-a)^2, \tag{2.21}$$

where $\|f''\|_{[a,b],\infty} := \operatorname{essup}_{t\in[a,b]}|f''(t)| < \infty$. The constants $1/2$ and $1/8$ are the best possible in (2.21).

We have:

Theorem 4. *Let $f:[m,M]\to\mathbb{R}$ be a twice differentiable convex function on $[m,M]$ with $\|f''\|_{[m,M],\infty} := \operatorname{essup}_{t\in[m,M]} f''(t) < \infty$ and A_j selfadjoint operators with the spectrum $\operatorname{Sp}(A_j)\subset[m,M]$ for $j=1,...,k$. If $C_j\in\mathcal{B}(H)$ for $j=1,...,k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$, then*

$$\sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \le \frac{1}{2}\|f''\|_{[m,M],\infty}\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right) \le \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H. \tag{2.22}$$

If P_j, with $j = 1, ..., k$ are projections satisfying the condition $\sum_{j=1}^{k} P_j = 1_H$, then

$$\begin{aligned} &\sum_{j=1}^{k} P_j f(A_j) P_j - f\left(\sum_{j=1}^{k} P_j A_j P_j\right) \\ &\leq \frac{1}{2}\|f''\|_{[m,M],\infty}\left(M1_H - \sum_{j=1}^{k} P_j A_j P_j\right)\left(\sum_{j=1}^{k} P_j A_j P_j - m1_H\right) \\ &\leq \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H. \end{aligned} \tag{2.23}$$

Proof. From (2.21) and the continuous functional calculus, we get

$$\begin{aligned} 0 &\leq \frac{f(m)(M1_H - B) + f(M)(B - m1_H)}{M-m} - f(B) \\ &\leq \frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H - B)(B - m1_H) \\ &\leq \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H \end{aligned} \tag{2.24}$$

where B is a selfadjoint operator with the spectrum $\mathrm{Sp}(B) \subset [m, M]$.

If we take $B = \sum_{j=1}^{k} C_j^* A_j C_j$ in (2.24) we get

$$\begin{aligned} 0 &\leq \frac{f(m)\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) + f(M)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right)}{M-m} \\ &- f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\ &\leq \frac{1}{2}\|f''\|_{[m,M],\infty}\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right) \\ &\leq \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H. \end{aligned} \tag{2.25}$$

Since

$$\begin{aligned} &\sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\ &\leq \frac{f(m)\left(M1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) + f(M)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m1_H\right)}{M-m} \\ &- f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right), \end{aligned}$$

hence by (2.25) we get (2.22). □

Corollary 4. *Let $f : [m, M] \to \mathbb{R}$ be an operator convex function on $[m, M]$ and A_j selfadjoint operators with the spectrum $\mathrm{Sp}(A_j) \subset [m, M]$ for $j = 1, ..., k$. If*

$C_j \in \mathcal{B}(H)$ *for* $j = 1, ..., k$ *satisfying the condition* $\sum_{j=1}^{k} C_j^* C_j = 1_H$*, then*

$$0 \le \sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \tag{2.26}$$
$$\le \frac{1}{2} \|f''\|_{[m,M],\infty} \left(M 1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) \left(\sum_{j=1}^{k} C_j^* A_j C_j - m 1_H\right)$$
$$\le \frac{1}{8} \|f''\|_{[m,M],\infty} (M-m)^2 1_H.$$

3. Some Examples

We consider the exponential function $f(x) = \exp(\alpha x)$ with $\alpha \in \mathbb{R} \setminus \{0\}$. This function is convex but not operator convex on $\mathbb{R}$. If A_j selfadjoint operators with the spectrum $\operatorname{Sp}(A_j) \subset [m, M]$ for $j = 1, ..., k$. If $C_j \in \mathcal{B}(H)$ for $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$, then, then by (2.2), (2.17) and (2.22) we have

$$\sum_{j=1}^{k} C_j^* \exp(\alpha A_j) C_j - \exp\left(\alpha \sum_{j=1}^{k} C_j^* A_j C_j\right) \tag{3.1}$$
$$\le \alpha \frac{\exp(\alpha M) - \exp(\alpha m)}{M-m} \left(M 1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) \left(\sum_{j=1}^{k} C_j^* A_j C_j - m 1_H\right)$$
$$\le \frac{1}{4} \alpha (M-m) [\exp(\alpha M) - \exp(\alpha m)] 1_H,$$

$$C_j^* \exp(\alpha A_j) C_j - \exp\left(\alpha C_j^* A_j C_j\right) \tag{3.2}$$
$$\le 2\left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha \frac{m+M}{2}\right)\right]$$
$$\times \left(\frac{1}{2}(M-m) 1_H + \left|\sum_{j=1}^{k} C_j^* A_j C_j - \frac{1}{2}(m+M) 1_H\right|\right)$$
$$\le 2(M-m)\left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha \frac{m+M}{2}\right)\right] 1_H$$

and

$$\begin{aligned}
(3.3)\qquad &\sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
&\le \frac{1}{2}\alpha^2 \begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \\
&\times \left(M 1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m 1_H\right) \\
&\le \frac{1}{8}\alpha^2 (M-m)^2 \begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \times 1_H.
\end{aligned}$$

The function $f(x) = -\ln x$, $x > 0$ is operator convex on $(0, \infty)$. If A_j selfadjoint operators with the spectrum $\mathrm{Sp}(A_j) \subset [m, M]$ for $j = 1, ..., k$, $C_j \in \mathcal{B}(H)$ for $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$, then by (2.9), (2.19) and (2.26) we have

$$\begin{aligned}
(3.4)\qquad 0 &\le \ln\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) - \sum_{j=1}^{k} C_j^* \ln(A_j) C_j \\
&\le \frac{1}{mM}\left(M 1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m 1_H\right) \\
&\le \frac{1}{4mM}(M-m)^2 1_H,
\end{aligned}$$

$$\begin{aligned}
(3.5)\qquad 0 &\le \ln\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) - \sum_{j=1}^{k} C_j^* \ln(A_j) C_j \\
&\le 2\ln\left(\frac{m+M}{2\sqrt{mM}}\right)\left(\frac{1}{2}(M-m) 1_H + \left|\sum_{j=1}^{k} C_j^* A_j C_j - \frac{1}{2}(m+M) 1_H\right|\right) \\
&\le 2(M-m)\ln\left(\frac{m+M}{2\sqrt{mM}}\right) 1_H
\end{aligned}$$

and

$$\begin{aligned}
(3.6)\qquad 0 &\le \ln\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) - \sum_{j=1}^{k} C_j^* \ln(A_j) C_j \\
&\le \frac{1}{2m^2}\left(M 1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m 1_H\right) \\
&\le \frac{1}{8m^2}(M-m)^2 1_H.
\end{aligned}$$

We observe that if $M > 2m$ then the bound in (3.4) is better than the one from (3.6). If $M < 2m$, then the conclusion is the other way around.

The function $f(x) = x \ln x$, $x > 0$ is operator convex on $(0, \infty)$. If A_j selfadjoint operators with the spectrum $\operatorname{Sp}(A_j) \subset [m, M]$ for $j = 1, ..., k$, $C_j \in \mathcal{B}(H)$ for $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$, then by (2.9), (2.19) and (2.26) we have

$$
\begin{aligned}
0 &\le \sum_{j=1}^{k} C_j^* A_j \ln(A_j) C_j - \sum_{j=1}^{k} C_j^* A_j C_j \ln\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
&\le \frac{\ln(M) - \ln(m)}{M - m} \left(M 1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m 1_H\right) \\
&\le \frac{1}{4}(M - m)[\ln(M) - \ln(m)] 1_H,
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
0 &\le \sum_{j=1}^{k} C_j^* A_j \ln(A_j) C_j - \sum_{j=1}^{k} C_j^* A_j C_j \ln\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
&\le 2\left[\frac{m \ln(m) + M \ln(M)}{2} - \left(\frac{m + M}{2}\right)\ln\left(\frac{m + M}{2}\right)\right] \\
&\times \left(\frac{1}{2}(M - m) 1_H + \left|\sum_{j=1}^{k} C_j^* A_j C_j - \frac{1}{2}(m + M) 1_H\right|\right) \\
&\le 2(M - m)\left[\frac{m \ln(m) + M \ln(M)}{2} - \left(\frac{m + M}{2}\right)\ln\left(\frac{m + M}{2}\right)\right] 1_H
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
0 &\le \sum_{j=1}^{k} C_j^* A_j \ln(A_j) C_j - \sum_{j=1}^{k} C_j^* A_j C_j \ln\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \\
&\le \frac{1}{2m}\left(M 1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right)\left(\sum_{j=1}^{k} C_j^* A_j C_j - m 1_H\right) \\
&\le \frac{1}{8m}(M - m)^2 1_H.
\end{aligned}
\tag{3.9}
$$

Consider the power function $f(x) = x^r$, $x \in (0, \infty)$ and r a real number. If $r \in (-\infty, 0] \cup [1, \infty)$, then f is convex and for $r \in [-1, 0] \cup [1, 2]$ is operator convex. If we use the inequalities (2.2), (2.17) and (2.22) we have for $r \in (-\infty, 0] \cup [1, \infty)$

that

$$\begin{aligned}(3.10)\qquad & \sum_{j=1}^{k} C_j^{*} A_j^{r} C_j-\left(\sum_{j=1}^{k} C_j^{*} A_j C_j\right)^{r} \\ & \leq r \frac{M^{r-1}-m^{r-1}}{M-m}\left(M 1_H-\sum_{j=1}^{k} C_j^{*} A_j C_j\right)\left(\sum_{j=1}^{k} C_j^{*} A_j C_j-m 1_H\right) \\ & \leq \frac{1}{4} r\left(M-m\right)\left[M^{r-1}-m^{r-1}\right] 1_H,\end{aligned}$$

$$\begin{aligned}(3.11)\qquad & \sum_{j=1}^{k} C_j^{*} A_j^{r} C_j-\left(\sum_{j=1}^{k} C_j^{*} A_j C_j\right)^{r} \\ & \leq 2\left[\frac{m^{r}+M^{r}}{2}-\left(\frac{m+M}{2}\right)^{r}\right] \\ & \times\left(\frac{1}{2}\left(M-m\right) 1_H+\left|\sum_{j=1}^{k} C_j^{*} A_j C_j-\frac{1}{2}\left(m+M\right) 1_H\right|\right) \\ & \leq 2\left(M-m\right)\left[\frac{m^{r}+M^{r}}{2}-\left(\frac{m+M}{2}\right)^{r}\right] 1_H\end{aligned}$$

and

$$\begin{aligned}(3.12)\qquad & \sum_{j=1}^{k} C_j^{*} A_j^{r} C_j-\left(\sum_{j=1}^{k} C_j^{*} A_j C_j\right)^{r} \\ & \leq \frac{1}{2} r\left(r-1\right)\begin{cases} M^{r-2} \text{ for } r \geq 2 \\ \\ m^{r-2} \text{ for } r \in(-\infty, 0] \cup[1,2)\end{cases} \\ & \times\left(M 1_H-\sum_{j=1}^{k} C_j^{*} A_j C_j\right)\left(\sum_{j=1}^{k} C_j^{*} A_j C_j-m 1_H\right) \\ & \leq \frac{1}{8} r\left(r-1\right)\left(M-m\right)^{2}\begin{cases} M^{r-2} \text{ for } r \geq 2 \\ \\ m^{r-2} \text{ for } r \in(-\infty, 0] \cup[1,2)\end{cases} \times 1_H,\end{aligned}$$

where A_j are selfadjoint operators with the spectrum $\operatorname{Sp}(A_j) \subset [m, M]$ for $j = 1, ..., k$ and $C_j \in \mathcal{B}(H)$ for $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^{*} C_j = 1_H$.

If $r \in [-1, 0] \cup [1, 2]$, then we also have $0 \leq \sum_{j=1}^{k} C_j^{*} A_j^{r} C_j - \left(\sum_{j=1}^{k} C_j^{*} A_j C_j\right)^{r}$ in the inequalities (3.10)-(3.12).

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