OPERATOR UPPER BOUNDS FOR DAVIS-CHOI-JENSEN'S DIFFERENCE IN HILBERT SPACES

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Abstract. In this paper we obtain several operator inequalities providing upper bounds for the Davis-Choi-Jensen's Difference

$$\Phi(f(A)) - f(\Phi(A))$$

for any convex function $f : I \to \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\mathrm{Sp}(A) \subset I$ and any linear, positive and normalized map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, where H and K are Hilbert spaces. Some examples for convex and operator convex functions are also provided.

1. Introduction

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [1] (see also [12, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

for any λ, $\mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e., $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \le B \text{ implies } \Phi(A) \le \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \le A \le \beta 1_H$, then $\alpha 1_K \le \Phi(A) \le \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \le (\ge)(1-\lambda) f(A) + \lambda f(B)$$

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for all $\lambda \in [0,1]$ and for every selfadjoint operators A, $B \in \mathcal{B}(H)$ whose spectra are contained in I.

The following Jensen's type result is well known [12, p. 22]:

Theorem 1 (Davis-Choi-Jensen's Inequality). *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$f(\Phi(A)) \leq \Phi(f(A)). \tag{1.1}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (1.1) we get

$$f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right) \leq \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps*

$$\Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H) \leq \Psi(f(A)). \tag{1.2}$$

Let $C_j \in \mathcal{B}(H)$, $j = 1, ..., k$ be contractions with

$$\sum_{j=1}^{k} C_j^* C_j = 1_H. \tag{1.3}$$

The map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ defined by [12, p. 19]

$$\Phi(A) := \sum_{j=1}^{k} C_j^* A C_j$$

is a normalized positive linear map on $\mathcal{B}(H)$.

In this paper we obtain several operator inequalities providing upper bounds for the Davis-Choi-Jensen's Difference

$$\Phi(f(A)) - f(\Phi(A))$$

for any convex function $f : I \to \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A) \subset I$ and any linear, positive and normalized map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, where H and K are Hilbert spaces. Some examples for convex and operator convex functions are also provided.

2. Main Results

We use the following result that was obtained in [5]:

Lemma 1. *If $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$, then*

$$\begin{aligned} 0 &\leq \frac{(b-t) f(a) + (t-a) f(b)}{b-a} - f(t) \\ &\leq (b-t)(t-a)\frac{f'_-(b) - f'_+(a)}{b-a} \leq \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right] \end{aligned} \tag{2.1}$$

for any $t \in [a,b]$.

If the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $1/4$ are sharp.

We have:

Theorem 2. *Let $f:[m,M]\to\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then*

$$
\begin{aligned}
&\Phi(f(A))-f(\Phi(A)) \\
&\le \frac{f'_-(M)-f'_+(m)}{M-m}(M1_K-\Phi(A))(\Phi(A)-m1_K) \\
&\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_K.
\end{aligned}
\tag{2.2}
$$

Proof. Utilising the continuous functional calculus for a selfadjoint operator T with $0\le T\le 1_H$ and the convexity of f on $[m,M]$, we have

$$
f(m(1_H-T)+MT)\le f(m)(1_H-T)+f(M)T \tag{2.3}
$$

in the operator order.

If we take in (2.3)

$$
0\le T=\frac{A-m1_H}{M-m}\le 1_H,
$$

then we get

$$
\begin{aligned}
&f\left(m\left(1_H-\frac{A-m1_H}{M-m}\right)+M\frac{A-m1_H}{M-m}\right) \\
&\le f(m)\left(1_H-\frac{A-m1_H}{M-m}\right)+f(M)\frac{A-m1_H}{M-m}.
\end{aligned}
\tag{2.4}
$$

Observe that

$$
\begin{aligned}
&m\left(1_H-\frac{A-m1_H}{M-m}\right)+M\frac{A-m1_H}{M-m} \\
&=\frac{m(M1_H-A)+M(A-m1_H)}{M-m}=A
\end{aligned}
$$

and

$$
\begin{aligned}
&f(m)\left(1_H-\frac{A-m1_H}{M-m}\right)+f(M)\frac{A-m1_H}{M-m} \\
&=\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}
\end{aligned}
$$

and by (2.4) we get the following inequality of interest

$$
f(A)\le\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}. \tag{2.5}
$$

If we take the map Φ in (2.5), then we get

$$
\begin{aligned}
\Phi(f(A)) &\le \Phi\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right] \\
&=\frac{f(m)\Phi(M1_H-A)+f(M)\Phi(A-m1_H)}{M-m} \\
&=\frac{f(m)(M\Phi(1_H)-\Phi(A))+f(M)(\Phi(A)-m\Phi(1_H))}{M-m} \\
&=\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m},
\end{aligned}
$$

which implies that

$$\begin{aligned}
&\Phi\left(f\left(A\right)\right)-f\left(\Phi\left(A\right)\right) \\
&\leq\frac{f\left(m\right)\left(M1_{K}-\Phi\left(A\right)\right)+f\left(M\right)\left(\Phi\left(A\right)-m1_{K}\right)}{M-m}-f\left(\Phi\left(A\right)\right).
\end{aligned}\tag{2.6}$$

Since $m1_{K}\leq\Phi\left(A\right)\leq M1_{K}$, then by using (2.1) for $a=m$, $b=M$ and the continuous functional calculus, we have

$$\begin{aligned}
&\frac{f\left(m\right)\left(M1_{K}-\Phi\left(A\right)\right)+f\left(M\right)\left(\Phi\left(A\right)-m1_{K}\right)}{M-m}-f\left(\Phi\left(A\right)\right) \\
&\leq\frac{f'_{-}\left(M\right)-f'_{+}\left(m\right)}{M-m}\left(M1_{K}-\Phi\left(A\right)\right)\left(\Phi\left(A\right)-m1_{K}\right) \\
&\leq\frac{1}{4}\left(M-m\right)\left[f'_{-}\left(M\right)-f'_{+}\left(m\right)\right]1_{K}.
\end{aligned}\tag{2.7}$$

By making use of (2.6) and (2.7) we get the desired result (2.2). □

Corollary 1. *Let $f:[m,M]\rightarrow\mathbb{R}$ be an operator convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}\left(A\right)\subset[m,M]$. If $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$, then*

$$\begin{aligned}
0&\leq\Phi\left(f\left(A\right)\right)-f\left(\Phi\left(A\right)\right) \\
&\leq\frac{f'_{-}\left(M\right)-f'_{+}\left(m\right)}{M-m}\left(M1_{K}-\Phi\left(A\right)\right)\left(\Phi\left(A\right)-m1_{K}\right) \\
&\leq\frac{1}{4}\left(M-m\right)\left[f'_{-}\left(M\right)-f'_{+}\left(m\right)\right]1_{K}.
\end{aligned}\tag{2.8}$$

We also have the following scalar inequality of interest:

Lemma 2. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a convex function on $[a,b]$ and $t\in[0,1]$, then*

$$\begin{aligned}
&2\min\left\{t,1-t\right\}\left[\frac{f\left(a\right)+f\left(b\right)}{2}-f\left(\frac{a+b}{2}\right)\right] \\
&\leq\left(1-t\right)f\left(a\right)+tf\left(b\right)-f\left(\left(1-t\right)a+tb\right) \\
&\leq2\max\left\{t,1-t\right\}\left[\frac{f\left(a\right)+f\left(b\right)}{2}-f\left(\frac{a+b}{2}\right)\right].
\end{aligned}\tag{2.9}$$

The proof follows, for instance, by Corollary 1 from [6] for $n=2$, $p_{1}=1-t$, $p_{2}=t$, $t\in[0,1]$ and $x_{1}=a$, $x_{2}=b$.

Theorem 3. *Let $f:[m,M]\rightarrow\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}\left(A\right)\subset[m,M]$. If $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$,* *then*

$$
\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_K-\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right) \\
&\leq \frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A)) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right)
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_K-\Phi\left(\left|A-\frac{1}{2}(m+M)1_K\right|\right)\right) \\
&\leq \frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-\Phi(f(A)) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_K+\Phi\left(\left|A-\frac{1}{2}(m+M)1_H\right|\right)\right).
\end{aligned}
\tag{2.11}
$$

Proof. We have from (2.9) that

$$
\begin{aligned}
&2\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\leq (1-t)f(m)+tf(M)-f((1-t)m+tM) \\
&\leq 2\left(\frac{1}{2}+\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],
\end{aligned}
\tag{2.12}
$$

for all $t\in[0,1]$.

Utilising the continuous functional calculus for a selfadjoint operator T with $0\leq T\leq 1_H$ we get from (2.12) that

$$
\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H-\left|T-\frac{1}{2}1_H\right|\right) \\
&\leq (1-T)f(m)+Tf(M)-f((1-T)m+TM) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H+\left|T-\frac{1}{2}1_H\right|\right),
\end{aligned}
\tag{2.13}
$$

in the operator order.

If we take in (2.13)

$$0\leq T=\frac{A-m1_H}{M-m}\leq 1_H,$$

then, like in the proof of Theorem 2, we get

$$
\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_H-\left|A-\frac{1}{2}(m+M)1_H\right|\right) \\
&\leq \frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_H+\left|A-\frac{1}{2}(m+M)1_H\right|\right).
\end{aligned} \tag{2.14}
$$

Since $m1_K \leq \Phi(A) \leq M1_K$, then by writing the inequality (2.14) for $\Phi(A)$ instead of A we get (2.10).

If we take Φ in (2.14), then we get

$$
\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\Phi\left(\frac{1}{2}(M-m)1_H-\left|A-\frac{1}{2}(m+M)1_H\right|\right) \\
&\leq \Phi\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]-\Phi f(A) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\Phi\left(\frac{1}{2}(M-m)1_H+\left|A-\frac{1}{2}(m+M)1_H\right|\right),
\end{aligned}
$$

which is equivalent to (2.11). □

Corollary 2. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then*

$$
\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_K-\Phi\left(\left|A-\frac{1}{2}(m+M)1_H\right|\right)\right) \\
&\leq \frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-\Phi(f(A)) \\
&\leq \frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A)) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right).
\end{aligned} \tag{2.15}
$$

We also have:

Corollary 3. *Let $f:[m,M]\to\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then*

$$\begin{aligned}\Phi(f(A))-f(\Phi(A))&\\ \le 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]&\\ \times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right)&\\ \le 2(M-m)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_K.&\end{aligned}\tag{2.16}$$

Proof. From (2.6) we have

$$\begin{aligned}&\Phi(f(A))-f(\Phi(A))\\ &\le\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A))\end{aligned}$$

and from (2.11) we have

$$\begin{aligned}&\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A))\\ &\le 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\ &\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right),\end{aligned}$$

which produce the desired result (2.16). □

Remark 1. *If $f:[m,M]\to\mathbb{R}$ is an operator convex function on $[m,M]$, A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$ and $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then*

$$\begin{aligned}0\le\Phi(f(A))-f(\Phi(A))&\\ \le 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]&\\ \times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right)&\\ \le 2(M-m)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_K.&\end{aligned}\tag{2.17}$$

We also have [5]:

Lemma 3. *Assume that $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. If f' is K-Lipschitzian on $[a,b]$, then*

$$\begin{aligned}&|(1-t)f(a)+tf(b)-f((1-t)a+tb)|\\ &\le\frac{1}{2}K(b-t)(t-a)\le\frac{1}{8}K(b-a)^2\end{aligned}\tag{2.18}$$

for all $t\in[0,1]$.

The constants $1/2$ and $1/8$ are the best possible in (2.18).

Remark 2. *If $f:[a,b]\to\mathbb{R}$ is twice differentiable and $f''\in L_\infty[a,b]$, then*

$$|(1-t)f(a)+tf(b)-f((1-t)a+tb)| \le \frac{1}{2}\|f''\|_{[a,b],\infty}(b-t)(t-a)\le\frac{1}{8}\|f''\|_{[a,b],\infty}(b-a)^2, \tag{2.19}$$

where $\|f''\|_{[a,b],\infty}:=\operatorname{essup}_{t\in[a,b]}|f''(t)|<\infty$. The constants $1/2$ and $1/8$ are the best possible in (2.19).

We have:

Theorem 4. *Let $f:[m,M]\to\mathbb{R}$ be a twice differentiable convex function on $[m,M]$ with $\|f''\|_{[m,M],\infty}:=\operatorname{essup}_{t\in[m,M]}f''(t)<\infty$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then*

$$\begin{aligned}&\Phi(f(A))-f(\Phi(A))\\&\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_K-\Phi(A))(\Phi(A)-m1_K)\\&\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_K.\end{aligned}\tag{2.20}$$

Proof. From (2.19) and the continuous functional calculus, we get

$$\begin{aligned}0&\le\frac{f(m)(M1_H-B)+f(M)(B-m1_H)}{M-m}-f(B)\\&\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H-B)(B-m1_H)\\&\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H\end{aligned}\tag{2.21}$$

where B is a selfadjoint operator with the spectrum $\operatorname{Sp}(B)\subset[m,M]$.

If we use (2.21) for $\Phi(A)$ we get

$$\begin{aligned}0&\le\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A))\\&\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_K-\Phi(A))(\Phi(A)-m1_K)\\&\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_K.\end{aligned}\tag{2.22}$$

Since

$$\begin{aligned}&\Phi(f(A))-f(\Phi(A))\\&\le\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A)),\end{aligned}$$

hence by (2.22) we get (2.20). □

Corollary 4. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then*

$$\begin{aligned}0&\le\Phi(f(A))-f(\Phi(A))\\&\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_K-\Phi(A))(\Phi(A)-m1_K)\\&\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_K.\end{aligned}\tag{2.23}$$

3. Some Examples

We consider the exponential function $f(x) = \exp(\alpha x)$ with $\alpha \in \mathbb{R} \setminus \{0\}$. This function is convex but not operator convex on $\mathbb{R}$. If A is selfadjoint with $\operatorname{Sp}(A) \subset [m, M]$ for some $m < M$ and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by (2.2), (2.16) and (2.20) we have

$$\begin{aligned}(3.1)\qquad &\Phi(\exp(\alpha A)) - \exp(\alpha\Phi(A))\\ &\leq \alpha\frac{\exp(\alpha M) - \exp(\alpha m)}{M-m}(M1_K - \Phi(A))(\Phi(A) - m1_K)\\ &\leq \frac{1}{4}\alpha(M-m)[\exp(\alpha M) - \exp(\alpha m)]1_K,\end{aligned}$$

$$\begin{aligned}(3.2)\qquad &\Phi(\exp(\alpha A)) - \exp(\alpha\Phi(A))\\ &\leq 2\left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha\frac{m+M}{2}\right)\right]\\ &\times\left(\frac{1}{2}(M-m)1_K + \left|\Phi(A) - \frac{1}{2}(m+M)1_K\right|\right)\\ &\leq 2(M-m)\left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha\frac{m+M}{2}\right)\right]1_K\end{aligned}$$

and

$$\begin{aligned}(3.3)\qquad &\Phi(\exp(\alpha A)) - \exp(\alpha\Phi(A))\\ &\leq \frac{1}{2}\alpha^2\begin{cases}\exp(\alpha M) & \text{if } \alpha > 0\\ \exp(\alpha m) & \text{if } \alpha < 0\end{cases} \times (M1_K - \Phi(A))(\Phi(A) - m1_K)\\ &\leq \frac{1}{8}\alpha^2(M-m)^2\begin{cases}\exp(\alpha M) & \text{if } \alpha > 0\\ \exp(\alpha m) & \text{if } \alpha < 0\end{cases} \times 1_K.\end{aligned}$$

The function $f(x) = -\ln x$, $x > 0$ is operator convex on $(0, \infty)$. If A is selfadjoint with $\operatorname{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by (2.8), (2.17) and (2.23) we have

$$\begin{aligned}(3.4)\qquad &0 \leq \ln(\Phi(A)) - \Phi(\ln(A))\\ &\leq \frac{1}{mM}(M1_V - \Phi(A))(\Phi(A) - m1_K) \leq \frac{1}{4mM}(M-m)^2 1_K,\end{aligned}$$

$$\begin{aligned}(3.5)\qquad &0 \leq \ln(\Phi(A)) - \Phi(\ln(A))\\ &\leq 2\ln\left(\frac{m+M}{2\sqrt{mM}}\right)\left(\frac{1}{2}(M-m)1_K + \left|\Phi(A) - \frac{1}{2}(m+M)1_K\right|\right)\\ &\leq 2(M-m)\ln\left(\frac{m+M}{2\sqrt{mM}}\right)1_K\end{aligned}$$

and

$$\begin{aligned}(3.6)\qquad &0 \leq \ln(\Phi(A)) - \Phi(\ln(A))\\ &\leq \frac{1}{2m^2}(M1_K - \Phi(A))(\Phi(A) - m1_K) \leq \frac{1}{8m^2}(M-m)^2 1_K.\end{aligned}$$

We observe that if $M>2m$ then the bound in (3.4) is better than the one from (3.6). If $M<2m$, then the conclusion is the other way around.

The function $f(x)=x\ln x$, $x>0$ is operator convex on $(0,\infty)$. If A is selfadjoint with $\operatorname{Sp}(A)\subset[m,M]$ for some $0<m<M$ and $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then by (2.8), (2.17) and (2.23) we have

$$\begin{aligned}0&\le\Phi(A\ln(A))-\Phi(A)\ln(\Phi(A))\\&\le\frac{\ln(M)-\ln(m)}{M-m}(M1_K-\Phi(A))(\Phi(A)-m1_K)\\&\le\frac{1}{4}(M-m)[\ln(M)-\ln(m)]1_K,\end{aligned}\tag{3.7}$$

$$\begin{aligned}0&\le\Phi(A\ln(A))-\Phi(A)\ln(\Phi(A))\\&\le2\left[\frac{m\ln(m)+M\ln(M)}{2}-\left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]\\&\quad\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right)\\&\le2(M-m)\left[\frac{m\ln(m)+M\ln(M)}{2}-\left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]1_K\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}0&\le\Phi(A\ln(A))-\Phi(A)\ln(\Phi(A))\\&\le\frac{1}{2m}(M1_K-\Phi(A))(\Phi(A)-m1_K)\le\frac{1}{8m}(M-m)^2 1_K.\end{aligned}\tag{3.9}$$

Consider the power function $f(x)=x^r$, $x\in(0,\infty)$ and r a real number. If $r\in(-\infty,0]\cup[1,\infty)$, then f is convex and for $r\in[-1,0]\cup[1,2]$ is operator convex. If we use the inequalities (2.2), (2.16) and (2.20) we have for $r\in(-\infty,0]\cup[1,\infty)$ that

$$\begin{aligned}&\Phi(A^r)-(\Phi(A))^r\\&\le r\frac{M^{r-1}-m^{r-1}}{M-m}(M1_K-\Phi(A))(\Phi(A)-m1_K)\\&\le\frac{1}{4}r(M-m)\left(M^{r-1}-m^{r-1}\right)1_K,\end{aligned}\tag{3.10}$$

$$\begin{aligned}&\Phi(A^r)-(\Phi(A))^r\\&\le2\left[\frac{m^r+M^r}{2}-\left(\frac{m+M}{2}\right)^r\right]\\&\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right)\\&\le2(M-m)\left[\frac{m^r+M^r}{2}-\left(\frac{m+M}{2}\right)^r\right]1_K\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}(3.12)\qquad &\Phi\left(A^{r}\right)-\left(\Phi\left(A\right)\right)^{r}\\ &\leq \frac{1}{2}r\left(r-1\right)\begin{cases}M^{r-2} \text{ for } r\geq 2\\ \\ m^{r-2} \text{ for } r\in\left(-\infty,0\right]\cup\left[1,2\right)\end{cases}\\ &\times\left(M1_{K}-\Phi\left(A\right)\right)\left(\Phi\left(A\right)-m1_{K}\right)\\ &\leq \frac{1}{8}r\left(r-1\right)\left(M-m\right)^{2}\begin{cases}M^{r-2} \text{ for } r\geq 2\\ \\ m^{r-2} \text{ for } r\in\left(-\infty,0\right]\cup\left[1,2\right)\end{cases}\times 1_{K},\end{aligned}$$

where A is selfadjoint with $\operatorname{Sp}(A)\subset[m,M]$ for some $0<m<M$ and $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$.

If $r\in\left[-1,0\right]\cup\left[1,2\right]$, then we also have $0\leq\Phi\left(A^{r}\right)-\left(\Phi\left(A\right)\right)^{r}$ in the inequalities (3.10)-(3.12).

For $r=-1$ we have the inequalities

$$\begin{aligned}(3.13)\qquad &\Phi\left(A^{-1}\right)-\left(\Phi\left(A\right)\right)^{-1}\\ &\leq\frac{M+m}{M^{2}m^{2}}\left(M1_{K}-\Phi\left(A\right)\right)\left(\Phi\left(A\right)-m1_{K}\right)\\ &\leq\frac{1}{4}\left(M-m\right)^{2}\frac{M+m}{M^{2}m^{2}}1_{K},\end{aligned}$$

$$\begin{aligned}(3.14)\qquad &\Phi\left(A^{-1}\right)-\left(\Phi\left(A\right)\right)^{-1}\\ &\leq\frac{\left(M-m\right)^{2}}{mM\left(m+M\right)}\left(\frac{1}{2}\left(M-m\right)1_{K}+\left|\Phi\left(A\right)-\frac{1}{2}\left(m+M\right)1_{K}\right|\right)\\ &\leq\frac{\left(M-m\right)^{3}}{mM\left(m+M\right)}1_{K}\end{aligned}$$

and

$$\begin{aligned}(3.15)\qquad &\Phi\left(A^{-1}\right)-\left(\Phi\left(A\right)\right)^{-1}\\ &\leq\frac{1}{m^{3}}\left(M1_{K}-\Phi\left(A\right)\right)\left(\Phi\left(A\right)-m1_{K}\right)\leq\frac{1}{4m^{3}}\left(M-m\right)^{2}1_{K},\end{aligned}$$

where A is selfadjoint with $\operatorname{Sp}(A)\subset[m,M]$ for some $0<m<M$ and $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$.

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