REVERSE OPERATOR INEQUALITIES FOR DAVIS DIFFERENCE OF CONVEX FUNCTIONS IN HILBERT SPACES

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Abstract. In this paper we obtain several operator inequalities providing upper bounds for the Davis difference

$$Pf(A)P - Pf(PAP)P$$

for any convex function $f: I \to \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A) \subset I$ and any orthogonal projection P. Some examples for convex and operator convex functions are also provided.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [10] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

For recent inequalities for operator convex functions see [1]-[9] and [11]-[20].

The following Davis-Jensen operator inequality is well know [4], see also [10, p. 10]:

Theorem 1. *Let H be a Hilbert space and f be a real valued continuous function on the interval I. Then f is operator convex on the interval I if and only if*

$$Pf(PAP)P \leq Pf(A)P \tag{1.2}$$

for any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A) \subset I$ and any orthogonal projection P.

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Note that the expression $Pg(PAP)P$ can be interpreted to make sense even if I does not contain 0. One way to do this is to extend g arbitrarily to $I\cup\{0\}$, use the Borel functional calculus to define $g(PAP)$, and note that $Pg(PAP)P$ depends only on $g_{|I}$.

We observe that from (1.2) we get

$$\sum_{j=1}^{n} P_j f\left(P_jA_jP_j\right)P_j \le \sum_{j=1}^{n} P_j f\left(A_j\right)P_j \tag{1.3}$$

for any selfadjoint operators A_j in H with the spectra $\operatorname{Sp}(A_j)\subset I$ and any orthogonal projection P_j, $j\in\{1,...,n\}$.

If P_j, with $j=1,...,k$ are orthogonal projections satisfying the condition $\sum_{j=1}^{k}P_j = 1_H$ and B_j in H with the spectra $\operatorname{Sp}(B_j)\subset I$, $j\in\{1,...,n\}$, then we have

$$f\left(\sum_{j=1}^{k} P_jB_jP_j\right) \le \sum_{j=1}^{k} P_j f\left(B_j\right)P_j. \tag{1.4}$$

This inequality is also a sufficient condition for the function f to be operator convex on I, see for instance [10, p. 10].

If we write the inequality (1.4) for $B_j = P_jA_jP_j$, $j\in\{1,...,n\}$ then we have

$$f\left(\sum_{j=1}^{k} P_jP_jA_jP_jP_j\right) \le \sum_{j=1}^{k} P_j f\left(P_jA_jP_j\right)P_j.$$

and since

$$\sum_{j=1}^{k} P_jP_jA_jP_jP_j = \sum_{j=1}^{k} P_j^2A_jP_j^2 = \sum_{j=1}^{k} P_jA_jP_j,$$

hence

$$f\left(\sum_{j=1}^{k} P_jA_jP_j\right) \le \sum_{j=1}^{k} P_j f\left(P_jA_jP_j\right)P_j, \tag{1.5}$$

provided $\sum_{j=1}^{k}P_j = 1_H$.

If P_j, with $j=1,...,k$ are orthogonal projections satisfying the condition $\sum_{j=1}^{k}P_j = 1_H$ and A_j in H with the spectra $\operatorname{Sp}(A_j)\subset I$, $j\in\{1,...,n\}$, then we have the following refinement of Jensen's discrete inequality

$$f\left(\sum_{j=1}^{k} P_jA_jP_j\right) \le \sum_{j=1}^{k} P_j f\left(P_jA_jP_j\right)P_j \le \sum_{j=1}^{n} P_j f\left(A_j\right)P_j. \tag{1.6}$$

It is known that there are convex functions f for which the inequality (1.2) does not hold, however one can obtain several operator inequalities providing upper bounds for the difference

$$Pf(A)P - Pf(PAP)P$$

for any convex function $f: I\to\mathbb{R}$, any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A)\subset I$ and any orthogonal projection P. Some examples for convex and operator convex functions are also provided.

2. Main Results

We use the following result that was obtained in [5]:

Lemma 1. *If $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$, then*

$$\begin{aligned} 0 &\le \frac{(b-t)\,f(a)+(t-a)\,f(b)}{b-a}-f(t) \\ &\le (b-t)(t-a)\frac{f'_-(b)-f'_+(a)}{b-a}\le\frac{1}{4}(b-a)\left[f'_-(b)-f'_+(a)\right] \end{aligned} \tag{2.1}$$

for any $t\in[a,b]$.

If the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $1/4$ are sharp.

We have:

Theorem 2. *Let $f:[m,M]\to\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If P is an orthogonal projection, then*

$$\begin{aligned} &Pf(A)P-Pf(PAP)P \\ &\le\frac{f'_-(M)-f'_+(m)}{M-m}(MP-PAP)(PAP-mP) \\ &\le\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]P. \end{aligned} \tag{2.2}$$

Proof. Utilising the continuous functional calculus for a selfadjoint operator T with $0\le T\le 1_H$ and the convexity of f on $[m,M]$, we have

$$f(m(1_H-T)+MT)\le f(m)(1_H-T)+f(M)T \tag{2.3}$$

in the operator order.

If we take in (2.3)

$$0\le T=\frac{A-m1_H}{M-m}\le 1_H,$$

then we get

$$\begin{aligned} &f\left(m\left(1_H-\frac{A-m1_H}{M-m}\right)+M\frac{A-m1_H}{M-m}\right) \\ &\le f(m)\left(1_H-\frac{A-m1_H}{M-m}\right)+f(M)\frac{A-m1_H}{M-m}. \end{aligned} \tag{2.4}$$

Observe that

$$\begin{aligned} &m\left(1_H-\frac{A-m1_H}{M-m}\right)+M\frac{A-m1_H}{M-m} \\ &=\frac{m(M1_H-A)+M(A-m1_H)}{M-m}=A \end{aligned}$$

and

$$\begin{aligned} &f(m)\left(1_H-\frac{A-m1_H}{M-m}\right)+f(M)\frac{A-m1_H}{M-m} \\ &=\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m} \end{aligned}$$

and by (2.4) we get the following inequality of interest

$$f(A) \leq \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}. \tag{2.5}$$

If we multiply (2.5) to the left with P and to the right with P we get

$$\begin{aligned} Pf(A)P &\leq P\left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}\right]P \\ &= \frac{f(m)P(M1_H - A)P + f(M)P(A - m1_H)P}{M - m} \\ &= \frac{f(m)\left(MP^2 - PAP\right) + f(M)\left(PAP - mP^2\right)}{M - m} \\ &= \frac{f(m)(MP - PAP) + f(M)(PAP - mP)}{M - m}, \end{aligned}$$

which implies that

$$\begin{aligned} &Pf(A)P - Pf(PAP)P \\ &\leq \frac{f(m)(MP - PAP) + f(M)(PAP - mP)}{M - m} - Pf(PAP)P. \end{aligned} \tag{2.6}$$

By using (2.1) and the continuous functional calculus, we have

$$\begin{aligned} &\frac{f(m)(M1_H - PAP) + f(M)(PAP - m1_H)}{M - m} - f(PAP) \\ &\leq \frac{f'_-(M) - f'_+(m)}{M - m}(M1_H - PAP)(PAP - m1_H) \\ &\leq \frac{1}{4}(M - m)\left[f'_-(M) - f'_+(m)\right]1_H. \end{aligned} \tag{2.7}$$

If we multiply (2.7) to the left with P and to the right with P we get

$$\begin{aligned} &P\left[\frac{f(m)(M1_H - PAP) + f(M)(PAP - m1_H)}{M - m}\right]P - Pf(PAP)P \\ &\leq \frac{f'_-(M) - f'_+(m)}{M - m}P(M1_H - PAP)(PAP - m1_H)P \\ &\leq \frac{1}{4}(M - m)\left[f'_-(M) - f'_+(m)\right]P^2, \end{aligned}$$

namely, as above,

$$\begin{aligned} &\frac{f(m)(MP - PAP) + f(M)(PAP - mP)}{M - m} - Pf(PAP)P \\ &\leq \frac{f'_-(M) - f'_+(m)}{M - m}(MP - PAP)(PAP - mP) \\ &\leq \frac{1}{4}(M - m)\left[f'_-(M) - f'_+(m)\right]P, \end{aligned} \tag{2.8}$$

By making use of (2.6) and (2.8) we get the desired result (2.2). □

Corollary 1. *Let $f : [m, M] \to \mathbb{R}$ be an operator convex function on $[m, M]$ and A a selfadjoint operator with the spectrum* $\operatorname{Sp}(A) \subset [m, M]$. *If P is an orthogonal* *projection, then*

$$0 \le Pf(A)P - Pf(PAP)P \le \frac{f'_-(M) - f'_+(m)}{M-m}(MP - PAP)(PAP - mP) \le \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]P. \tag{2.9}$$

We also have the following scalar inequality of interest:

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and $t\in[0,1]$, then*

$$\begin{aligned} &2\min\{t,1-t\}\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right] \\ &\le (1-t)f(a) + tf(b) - f((1-t)a + tb) \\ &\le 2\max\{t,1-t\}\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right]. \end{aligned} \tag{2.10}$$

The proof follows, for instance, by Corollary 1 from [6] for $n=2$, $p_1 = 1-t$, $p_2 = t$, $t\in[0,1]$ and $x_1 = a$, $x_2 = b$.

Theorem 3. *Let $f:[m,M]\to\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If P is an orthogonal projection, then*

$$\begin{aligned} &2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times\left(\frac{1}{2}(M-m)P - P\left|A - \frac{1}{2}(m+M)1_H\right|P\right) \\ &\le \frac{f(m)(MP-PAP) + f(M)(PAP-mP)}{M-m} - Pf(A)P \\ &\le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times\left(\frac{1}{2}(M-m)P + P\left|A - \frac{1}{2}(m+M)1_H\right|P\right) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times\left(\frac{1}{2}(M-m)P - P\left|PAP - \frac{1}{2}(m+M)1_H\right|P\right) \\ &\le \frac{f(m)(MP-PAP) + f(M)(PAP-mP)}{M-m} - Pf(PAP)P \\ &\le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times\left(\frac{1}{2}(M-m)P + P\left|PAP - \frac{1}{2}(m+M)1_H\right|P\right). \end{aligned} \tag{2.12}$$

Proof. We have from (2.10) that

$$2\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \tag{2.13}$$
$$\leq(1-t)f(m)+tf(M)-f((1-t)m+tM)$$
$$\leq 2\left(\frac{1}{2}+\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],$$

for all $t\in[0,1]$.

Utilising the continuous functional calculus for a selfadjoint operator T with $0\leq T\leq 1_H$ we get from (2.13) that

$$2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H-\left|T-\frac{1}{2}1_H\right|\right) \tag{2.14}$$
$$\leq(1-T)f(m)+Tf(M)-f((1-T)m+TM)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H+\left|T-\frac{1}{2}1_H\right|\right),$$

in the operator order.

If we take in (2.14)

$$0\leq T=\frac{A-m1_H}{M-m}\leq 1_H,$$

then, like in the proof of Theorem 2, we get

$$2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \tag{2.15}$$
$$\times\left(\frac{1}{2}(M-m)1_H-\left|A-\frac{1}{2}(m+M)1_H\right|\right)$$
$$\leq\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)1_H+\left|A-\frac{1}{2}(m+M)1_H\right|\right).$$

If we multiply (2.15) to the left with P and to the right with P and by taking into account that $P^2=P$, then we get

$$2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)P-P\left|A-\frac{1}{2}(m+M)1_H\right|P\right)$$
$$\leq\frac{f(m)(MP-PAP)+f(M)(PAP-mP)}{M-m}-Pf(A)P$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)P+P\left|A-\frac{1}{2}(m+M)1_H\right|P\right),$$

which proves (2.11).

Like in (2.15) we get

$$\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\frac{1}{2}(M-m)1_H-\left|PAP-\frac{1}{2}(m+M)1_H\right|\right)\\
&\leq\frac{f(m)(M1_H-PAP)+f(M)(PAP-m1_H)}{M-m}-f(PAP)\\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\frac{1}{2}(M-m)1_H+\left|PAP-\frac{1}{2}(m+M)1_H\right|\right).
\end{aligned}$$

and if we multiply it to the left with P and to the right with P and by taking into account that $P^2=P$, then we get

$$\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\frac{1}{2}(M-m)P-P\left|PAP-\frac{1}{2}(m+M)1_H\right|P\right)\\
&\leq\frac{f(m)(MP-PAP)+f(M)(PAP-mP)}{M-m}-Pf(PAP)P\\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\frac{1}{2}(M-m)P+P\left|PAP-\frac{1}{2}(m+M)1_H\right|P\right),
\end{aligned}$$

which proves (2.12). □

Corollary 2. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If P is an orthogonal projection, then*

$$\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\frac{1}{2}(M-m)P-P\left|A-\frac{1}{2}(m+M)1_H\right|P\right)\\
&\leq\frac{f(m)(MP-PAP)+f(M)(PAP-mP)}{M-m}-Pf(A)P\\
&\leq\frac{f(m)(MP-PAP)+f(M)(PAP-mP)}{M-m}-Pf(PAP)P\\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\frac{1}{2}(M-m)P+P\left|PAP-\frac{1}{2}(m+M)1_H\right|P\right).
\end{aligned}\tag{2.16}$$

We also have

Corollary 3. *Let $f:[m,M]\to\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If P is an orthogonal projection,* *then*

$$\begin{aligned} & Pf(A)P - Pf(PAP)P \\ & \leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ & \times\left(\frac{1}{2}(M-m)P + P\left|PAP - \frac{1}{2}(m+M)1_H\right|P\right) \\ & \leq 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]P. \end{aligned} \tag{2.17}$$

Proof. From (2.6) we have

$$\begin{aligned} & Pf(A)P - Pf(PAP)P \\ & \leq \frac{f(m)(MP-PAP)+f(M)(PAP-mP)}{M-m} - Pf(PAP)P \end{aligned} \tag{2.18}$$

and from (2.12) we have

$$\begin{aligned} & \frac{f(m)(MP-PAP)+f(M)(PAP-mP)}{M-m} - Pf(PAP)P \\ & \leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ & \times\left(\frac{1}{2}(M-m)P + P\left|PAP - \frac{1}{2}(m+M)1_H\right|P\right). \end{aligned}$$

which produce the desired result (2.17). □

Remark 1. *If $f:[m,M]\to\mathbb{R}$ is an operator convex function on $[m,M]$, A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$ and P is an orthogonal projection, then*

$$\begin{aligned} 0 & \leq Pf(A)P - Pf(PAP)P \\ & \leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ & \times\left(\frac{1}{2}(M-m)P + P\left|PAP - \frac{1}{2}(m+M)1_H\right|P\right) \\ & \leq 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]P. \end{aligned} \tag{2.19}$$

We also have [5]:

Lemma 3. *Assume that $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. If f' is K-Lipschitzian on $[a,b]$, then*

$$\begin{aligned} & |(1-t)f(a)+tf(b)-f((1-t)a+tb)| \\ & \leq \frac{1}{2}K(b-t)(t-a) \leq \frac{1}{8}K(b-a)^2 \end{aligned} \tag{2.20}$$

for all $t\in[0,1]$.

The constants $1/2$ and $1/8$ are the best possible in (2.20).

Remark 2. *If* $f:[a,b]\to\mathbb{R}$ *is twice differentiable and* $f''\in L_\infty[a,b]$, *then*

$$
\begin{aligned}
&|(1-t)f(a)+tf(b)-f((1-t)a+tb)| \\
&\le \frac{1}{2}\|f''\|_{[a,b],\infty}(b-t)(t-a)\le \frac{1}{8}\|f''\|_{[a,b],\infty}(b-a)^2,
\end{aligned} \tag{2.21}
$$

where $\|f''\|_{[a,b],\infty}:=\operatorname{essup}_{t\in[a,b]}|f''(t)|<\infty$. *The constants* $1/2$ *and* $1/8$ *are the best possible in (2.21).*

We have:

Theorem 4. *Let* $f:[m,M]\to\mathbb{R}$ *be a twice differentiable convex function on* $[m,M]$ *with* $\|f''\|_{[m,M],\infty}:=\operatorname{essup}_{t\in[m,M]}f''(t)<\infty$ *and* A *a selfadjoint operator with the spectrum* $\operatorname{Sp}(A)\subset[m,M]$. *If* P *is an orthogonal projection, then*

$$
\begin{aligned}
&Pf(A)P-Pf(PAP)P \\
&\le \frac{1}{2}\|f''\|_{[m,M],\infty}(MP-PAP)(PAP-mP) \\
&\le \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2P.
\end{aligned} \tag{2.22}
$$

Proof. From (2.21) and the continuous functional calculus, we get

$$
\begin{aligned}
0&\le \frac{f(m)(M1_H-B)+f(M)(B-m1_H)}{M-m}-f(B) \\
&\le \frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H-B)(B-m1_H) \\
&\le \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H
\end{aligned} \tag{2.23}
$$

where B is a selfadjoint operator with the spectrum $\operatorname{Sp}(B)\subset[m,M]$.

Therefore

$$
\begin{aligned}
0&\le \frac{f(m)(M1_H-PAP)+f(M)(PAP-m1_H)}{M-m}-f(PAP) \\
&\le \frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H-PAP)(PAP-m1_H) \\
&\le \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H
\end{aligned}
$$

if we multiply it to the left with P and to the right with P and by taking into account that $P^2=P$, then we get

$$
\begin{aligned}
0&\le \frac{f(m)(MP-PAP)+f(M)(PAP-mP)}{M-m}-Pf(PAP)P \\
&\le \frac{1}{2}\|f''\|_{[m,M],\infty}(MP-PAP)(PAP-mP) \\
&\le \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 P
\end{aligned}
$$

and by (2.18) we get (2.22). □

Corollary 4. *Let* $f:[m,M]\to\mathbb{R}$ *be an operator convex function on* $[m,M]$ *and* A *a selfadjoint operator with the spectrum* $\operatorname{Sp}(A)\subset[m,M]$. *If* P *is an orthogonal* *projection, then*

$$0 \le Pf(A)P - Pf(PAP)P \le \frac{1}{2}\|f''\|_{[m,M],\infty}(MP - PAP)(PAP - mP) \le \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 P. \tag{2.24}$$

3. Some Examples

We consider the exponential function $f(x) = \exp(\alpha x)$ with $\alpha \in \mathbb{R} \setminus \{0\}$. This function is convex but not operator convex on $\mathbb{R}$. If A is selfadjoint with $\mathrm{Sp}(A) \subset [m, M]$ for some $m < M$ and P is an orthogonal projection, then by (2.2), (2.17) and (2.22) we have

$$\begin{aligned} &P\exp(\alpha A)P - P\exp(\alpha PAP)P \\ &\le \alpha\frac{\exp(\alpha M) - \exp(\alpha m)}{M-m}(MP - PAP)(PAP - mP) \\ &\le \frac{1}{4}\alpha(M-m)[\exp(\alpha M) - \exp(\alpha m)]P, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &P\exp(\alpha A)P - P\exp(\alpha PAP)P \\ &\le 2\left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha\frac{m+M}{2}\right)\right] \\ &\times\left(\frac{1}{2}(M-m)P + P\left|PAP - \frac{1}{2}(m+M)1_H\right|P\right) \\ &\le 2(M-m)\left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha\frac{m+M}{2}\right)\right]P \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &P\exp(\alpha A)P - P\exp(\alpha PAP)P \\ &\le \frac{1}{2}\alpha^2\begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \times (MP - PAP)(PAP - mP) \\ &\le \frac{1}{8}\alpha^2(M-m)^2\begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \times P. \end{aligned} \tag{3.3}$$

The function $f(x) = -\ln x$, $x > 0$ is operator convex on $(0,\infty)$. If A is selfadjoint with $\mathrm{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and P is an orthogonal projection, then by (2.9), (2.19) and (2.24) we have

$$\begin{aligned} &0 \le Pf(A)P - Pf(PAP)P \\ &\le \frac{1}{mM}(MP - PAP)(PAP - mP) \le \frac{1}{4mM}(M-m)^2 P, \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
(3.5)\qquad 0 &\leq Pf(A)P - Pf(PAP)P \\
&\leq 2\ln\left(\frac{m+M}{2\sqrt{mM}}\right) \\
&\quad \times \left(\frac{1}{2}(M-m)P + P\left|PAP - \frac{1}{2}(m+M)1_H\right|P\right) \\
&\leq 2(M-m)\ln\left(\frac{m+M}{2\sqrt{mM}}\right)P
\end{aligned}
$$

and

$$
\begin{aligned}
(3.6)\qquad 0 &\leq Pf(A)P - Pf(PAP)P \\
&\leq \frac{1}{2m^2}(MP - PAP)(PAP - mP) \leq \frac{1}{8m^2}(M-m)^2 P.
\end{aligned}
$$

We observe that if $M > 2m$ then the bound in (3.4) is better than the one from (3.6). If $M < 2m$, then the conclusion is the other way around.

The function $f(x) = x\ln x$, $x > 0$ is operator convex on $(0,\infty)$. If A is selfadjoint with $\mathrm{Sp}(A) \subset [m,M]$ for some $0 < m < M$ and P is an orthogonal projection, then by (2.9), (2.19) and (2.24) we have

$$
\begin{aligned}
(3.7)\qquad 0 &\leq PA\ln(A)P - PAP\ln(PAP)P \\
&\leq \frac{\ln(M) - \ln(m)}{M-m}(MP - PAP)(PAP - mP) \\
&\leq \frac{1}{4}(M-m)[\ln(M) - \ln(m)]P,
\end{aligned}
$$

$$
\begin{aligned}
(3.8)\qquad 0 &\leq PA\ln(A)P - PAP\ln(PAP)P \\
&\leq 2\left[\frac{m\ln(m) + M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right] \\
&\quad \times \left(\frac{1}{2}(M-m)P + P\left|PAP - \frac{1}{2}(m+M)1_H\right|P\right) \\
&\leq 2(M-m)\left[\frac{m\ln(m) + M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]P
\end{aligned}
$$

and

$$
\begin{aligned}
(3.9)\qquad 0 &\leq PA\ln(A)P - PAP\ln(PAP)P \\
&\leq \frac{1}{2m}(MP - PAP)(PAP - mP) \leq \frac{1}{8m}(M-m)^2 P.
\end{aligned}
$$

Consider the power function $f(x) = x^r$, $x \in (0,\infty)$ and r a real number. If $r \in (-\infty, 0] \cup [1,\infty)$, then f is convex and for $r \in [-1,0] \cup [1,2]$ is operator convex. If we use the inequalities (2.2), (2.17) and (2.22) we have for $r \in (-\infty,0] \cup [1,\infty)$ that

$$
\begin{aligned}
(3.10)\qquad & PA^rP - P(PAP)^rP \\
&\leq r\frac{M^{r-1} - m^{r-1}}{M-m}(MP - PAP)(PAP - mP) \\
&\leq \frac{1}{4}r(M-m)\left[M^{r-1} - m^{r-1}\right]P,
\end{aligned}
$$

$$PA^{r}P-P\left(PAP\right) ^{r}P \tag{3.11}$$
$$\leq 2\left[\frac{m^{r}+M^{r}}{2}-\left(\frac{m+M}{2}\right) ^{r}\right]$$
$$\times \left(\frac{1}{2}\left(M-m\right) P+P\left\vert PAP-\frac{1}{2}\left(m+M\right) 1_{H}\right\vert P\right)$$
$$\leq 2\left(M-m\right) \left[\frac{m^{r}+M^{r}}{2}-\left(\frac{m+M}{2}\right) ^{r}\right] P$$

and

$$PA^{r}P-P\left(PAP\right) ^{r}P \tag{3.12}$$
$$\leq \frac{1}{2}r\left(r-1\right) \left\{ \begin{array}{l} M^{r-2}\text{ for }r\geq 2 \\ \\ m^{r-2}\text{ for }r\in (-\infty ,0]\cup \lbrack 1,2) \end{array} \right.$$
$$\times \left(MP-PAP\right) \left(PAP-mP\right)$$
$$\leq \frac{1}{8}r\left(r-1\right) \left(M-m\right) ^{2}\left\{ \begin{array}{l} M^{r-2}\text{ for }r\geq 2 \\ \\ m^{r-2}\text{ for }r\in (-\infty ,0]\cup \lbrack 1,2) \end{array} \right. \times P,$$

where A is selfadjoint with $\mathrm{Sp}\left(A\right) \subset \left[m,M\right]$ for some $0<m<M$ and P is an orthogonal projection.

If $r\in \left[-1,0\right] \cup \left[1,2\right]$, then we also have $0\leq PA^{r}P-P\left(PAP\right) ^{r}P$ in the inequalities (3.10)-(3.12).

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