REVERSE OPERATOR INEQUALITIES FOR DAVIS
DIFFERENCE OF CONVEX FUNCTIONS IN HILBERT SPACES

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Abstract. In this paper we obtain several operator inequalities providing
upper bounds for the Davis difference

$$Pf(A)P - Pf(PAP)P$$

for any convex function $f : I \to \mathbb{R}$, any selfadjoint operator $A$ in $H$ with the
spectrum $\text{Sp}(A) \subset I$ and any orthogonal projection $P$. Some examples for
convex and operator convex functions are also provided.

1. Introduction

A real valued continuous function $f$ on an interval $I$ is said to be operator convex
(operator concave) on $I$ if

$$f((1 - \lambda)A + \lambda B) \leq (\geq) (1 - \lambda)f(A) + \lambda f(B)$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator $A$ and $B$
on a Hilbert space $H$ whose spectra are contained in $I$. Notice that a function $f$ is
operator concave if $-f$ is operator convex.

A real valued continuous function $f$ on an interval $I$ is said to be operator monotone if it is monotone with respect to the operator order, i.e., $A \leq B$ with
$\text{Sp}(A), \text{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and oper-
ator monotone functions, see [10] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$
if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$.
The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave
on $(0,\infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0,\infty)$. The
exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

For recent inequalities for operator convex functions see [1]-[9] and [11]-[20].

The following Davis-Jensen operator inequality is well know [4], see also [10, p.
10]:

**Theorem 1.** Let $H$ be a Hilbert space and $f$ be a real valued continuous function
on the interval $I$. Then $f$ is operator convex on the interval $I$ if and only if

$$Pf(PAP)P \leq Pf(A)P$$

for any selfadjoint operator $A$ in $H$ with the spectrum $\text{Sp}(A) \subset I$ and any orthogonal
projection $P$.

1991 Mathematics Subject Classification. 47A63; 47A99.

Key words and phrases. Selfadjoint bounded linear operators, Functions of operators, Operator
convex functions, Jensen’s operator inequality, Davis inequality.

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Note that the expression $P g(P A P) P$ can be interpreted to make sense even if $I$ does not contain 0. One way to do this is to extend $g$ arbitrarily to $I \cup \{0\}$, use the Borel functional calculus to define $g(P A P)$, and note that $P g(P A P) P$ depends only on $g|I$.

We observe that from (1.2) we get

$$
(1.3) \quad \sum_{j=1}^{n} P_j f(P_j A_j P_j) P_j \leq \sum_{j=1}^{n} P_j f(A_j) P_j
$$

for any selfadjoint operators $A_j$ in $H$ with the spectra $\text{Sp}(A_j) \subset I$ and any orthogonal projection $P_j$, $j \in \{1, \ldots, n\}$.

If $P_j$, with $j = 1, \ldots, k$ are orthogonal projections satisfying the condition $\sum_{j=1}^{k} P_j = 1_H$ and $B_j$ in $H$ with the spectra $\text{Sp}(B_j) \subset I$, $j \in \{1, \ldots, n\}$, then we have

$$
(1.4) \quad f\left(\sum_{j=1}^{k} P_j B_j P_j\right) \leq \sum_{j=1}^{k} P_j f(B_j) P_j.
$$

This inequality is also a sufficient condition for the function $f$ to be operator convex on $I$, see for instance [10, p. 10].

If we write the inequality (1.4) for $B_j = P_j A_j P_j$, $j \in \{1, \ldots, n\}$ then we have

$$
f\left(\sum_{j=1}^{k} P_j A_j P_j \right) \leq \sum_{j=1}^{k} P_j f(P_j A_j P_j) P_j.
$$

and since

$$
\sum_{j=1}^{k} P_j P_j A_j P_j P_j = \sum_{j=1}^{k} P_j^2 A_j P_j^2 = \sum_{j=1}^{k} P_j A_j P_j,
$$

hence

$$
(1.5) \quad f\left(\sum_{j=1}^{k} P_j A_j P_j\right) \leq \sum_{j=1}^{k} P_j f(P_j A_j P_j) P_j,
$$

provided $\sum_{j=1}^{k} P_j = 1_H$.

If $P_j$, with $j = 1, \ldots, k$ are orthogonal projections satisfying the condition $\sum_{j=1}^{k} P_j = 1_H$ and $A_j$ in $H$ with the spectra $\text{Sp}(A_j) \subset I$, $j \in \{1, \ldots, n\}$, then we have the following refinement of Jensen’s discrete inequality

$$
(1.6) \quad f\left(\sum_{j=1}^{k} P_j A_j P_j\right) \leq \sum_{j=1}^{k} P_j f(P_j A_j P_j) P_j \leq \sum_{j=1}^{n} P_j f(A_j) P_j.
$$

It is known that there are convex functions $f$ for which the inequality (1.2) does not hold, however one can obtain several operator inequalities providing upper bounds for the difference

$$
P f(A) P - P f(P A P) P
$$

for any convex function $f : I \to \mathbb{R}$, any selfadjoint operator $A$ in $H$ with the spectrum $\text{Sp}(A) \subset I$ and any orthogonal projection $P$. Some examples for convex and operator convex functions are also provided.
2. Main Results

We use the following result that was obtained in [5]:

**Lemma 1.** If \( f : [a, b] \to \mathbb{R} \) is a convex function on \([a, b]\), then

\[
0 \leq \frac{(b-t) f(a) + (t-a) f(b)}{b-a} - f(t) \leq (b-t) (t-a) \frac{f'_-(b) - f'_+(a)}{b-a} \leq \frac{1}{4} (b-a) \left[ f'_-(b) - f'_+(a) \right]
\]

for any \( t \in [a, b] \).

If the lateral derivatives \( f'_- (b) \) and \( f'_+ (a) \) are finite, then the second inequality and the constant \( 1/4 \) are sharp.

We have:

**Theorem 2.** Let \( f : [m, M] \to \mathbb{R} \) be a convex function on \([m, M]\) and \( A \) a self adjoint operator with the spectrum \( \text{Sp}(A) \subset [m, M] \). If \( P \) is an orthogonal projection, then

\[
P f(A) P - P f(PAP) P \leq \frac{f'_-(M) - f'_+(m)}{M-m} (MP - PAP) (PAP - mP) \leq \frac{1}{4} (M-m) \left[ f'_-(M) - f'_+(m) \right] P.
\]

**Proof.** Utilising the continuous functional calculus for a selfadjoint operator \( T \) with \( 0 \leq T \leq 1_H \) and the convexity of \( f \) on \([m, M]\), we have

\[
f(m (1_H - T) + MT) \leq f(m) (1_H - T) + f(M) T
\]

in the operator order.

If we take in (2.3)

\[
0 \leq T = \frac{A - m 1_H}{M - m} \leq 1_H,
\]

then we get

\[
f \left( m \left(1_H - \frac{A - m 1_H}{M - m} \right) + M \frac{A - m 1_H}{M - m} \right) \leq f(m) \left(1_H - \frac{A - m 1_H}{M - m} \right) + f(M) \frac{A - m 1_H}{M - m}.
\]

Observe that

\[
m \left(1_H - \frac{A - m 1_H}{M - m} \right) + M \frac{A - m 1_H}{M - m} = m (M1_H - A) + M (A - m 1_H) = A
\]

and

\[
f(m) \left(1_H - \frac{A - m 1_H}{M - m} \right) + f(M) \frac{A - m 1_H}{M - m} = f(m) (M1_H - A) + f(M) (A - m 1_H)
\]
and by (2.4) we get the following inequality of interest

\[(2.5)\]
\[f(A) \leq \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}.\]

If we multiply (2.5) to the left with \(P\) and to the right with \(P\) we get

\[Pf(A)P \leq P \left[ \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right]P\]
\[= \frac{f(m)P(M1_H - A)P + f(M)P(A - m1_H)P}{M - m}\]
\[= \frac{f(m)(MP^2 - PAP) + f(M)(PAP - mP^2)}{M - m}\]
\[= \frac{f(m)(MP - PAP) + f(M)(PAP - mP)}{M - m},\]

which implies that

\[(2.6)\]
\[Pf(A)P - Pf(PAP)P \leq \frac{f(m)(MP - PAP) + f(M)(PAP - mP)}{M - m} - Pf(PAP)P.\]

By using (2.1) and the continuous functional calculus, we have

\[(2.7)\]
\[\frac{f(m)(M1_H - PAP) + f(M)(PAP - m1_H)}{M - m} - f(PAP)\]
\[\leq \frac{f'_-(M) - f'_+(m)}{M - m} (M1_H - PAP)(PAP - m1_H)\]
\[\leq \frac{1}{4} (M - m) [f'_-(M) - f'_+(m)] 1_H.\]

If we multiply (2.7) to the left with \(P\) and to the right with \(P\) we get

\[P \left[ \frac{f(m)(M1_H - PAP) + f(M)(PAP - m1_H)}{M - m} \right]P - Pf(PAP)P\]
\[\leq \frac{f'_-(M) - f'_+(m)}{M - m} P(M1_H - PAP)(PAP - m1_H)P\]
\[\leq \frac{1}{4} (M - m) [f'_-(M) - f'_+(m)] P^2,\]

namely, as above,

\[(2.8)\]
\[\frac{f(m)(MP - PAP) + f(M)(PAP - mP)}{M - m} - Pf(PAP)P\]
\[\leq \frac{f'_-(M) - f'_+(m)}{M - m} (MP - PAP)(PAP - mP)\]
\[\leq \frac{1}{4} (M - m) [f'_-(M) - f'_+(m)] P,\]

By making use of (2.6) and (2.8) we get the desired result (2.2). \(\square\)

**Corollary 1.** Let \(f : [m, M] \to \mathbb{R}\) be an operator convex function on \([m, M]\) and \(A\) a selfadjoint operator with the spectrum \(\text{Sp}(A) \subset [m, M]\). If \(P\) is an orthogonal
projection, then

\begin{align}
0 \leq Pf(A)P - Pf(PAP)P \\
\leq \frac{f'_-(M) - f'_+(m)}{M - m} (MP - PAP) (PAP - mP) \\
\leq \frac{1}{4} (M - m) [f'_-(M) - f'_+(m)] P.
\end{align}

We also have the following scalar inequality of interest:

**Lemma 2.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function on \([a, b]\) and \( t \in [0, 1] \), then

\begin{align}
2 \min \{ t, 1 - t \} \left[ \frac{f(a) + f(b)}{2} - f \left( \frac{a + b}{2} \right) \right] \\
\leq (1 - t) f(a) + tf(b) - f((1 - t) a + tb) \\
\leq 2 \max \{ t, 1 - t \} \left[ \frac{f(a) + f(b)}{2} - f \left( \frac{a + b}{2} \right) \right].
\end{align}

The proof follows, for instance, by Corollary 1 from [6] for \( n = 2, p_1 = 1 - t, p_2 = t, t \in [0, 1] \) and \( x_1 = a, x_2 = b \).

**Theorem 3.** Let \( f : [m, M] \to \mathbb{R} \) be a convex function on \([m, M]\) and \( A \) a self-adjoint operator with the spectrum \( \text{Sp}(A) \subset [m, M] \). If \( P \) is an orthogonal projection, then

\begin{align}
2 \left[ \frac{f(m) + f(M)}{2} - f \left( \frac{m + M}{2} \right) \right] \\
\times \left( \frac{1}{2} (M - m) P - \frac{1}{2} (m + M) 1_H \right) P \\
\leq \frac{f(m) (MP - PAP) + f(M) (PAP - mP)}{M - m} - Pf(A)P \\
\leq 2 \left[ \frac{f(m) + f(M)}{2} - f \left( \frac{m + M}{2} \right) \right] \\
\times \left( \frac{1}{2} (M - m) P + P \left| A - \frac{1}{2} (m + M) 1_H \right| P \right)
\end{align}

and

\begin{align}
2 \left[ \frac{f(m) + f(M)}{2} - f \left( \frac{m + M}{2} \right) \right] \\
\times \left( \frac{1}{2} (M - m) P - \frac{1}{2} (m + M) 1_H \right) P \\
\leq \frac{f(m) (MP - PAP) + f(M) (PAP - mP)}{M - m} - Pf(PAP)P \\
\leq 2 \left[ \frac{f(m) + f(M)}{2} - f \left( \frac{m + M}{2} \right) \right] \\
\times \left( \frac{1}{2} (M - m) P + P \left| PAP - \frac{1}{2} (m + M) 1_H \right| P \right).
\end{align}
Proof. We have from (2.10) that
\begin{equation}
2 \left( \frac{1}{2} - t^2 \right) \left[ f(m) + f(M) - f \left( \frac{m + M}{2} \right) \right]
\leq (1 - t) f(m) + tf(M) - f((1 - t)m + tM)
\leq 2 \left( \frac{1}{2} + t^2 \right) \left[ f(m) + f(M) - f \left( \frac{m + M}{2} \right) \right],
\end{equation}
for all \( t \in [0, 1] \).

Utilising the continuous functional calculus for a selfadjoint operator \( T \) with \( 0 \leq T \leq 1_H \) we get from (2.13) that
\begin{equation}
2 \left[ \frac{f(m) + f(M)}{2} - f \left( \frac{m + M}{2} \right) \right] \left( \frac{1}{2} \left| T - \frac{1}{2} 1_H \right| \right)
\leq (1 - T) f(m) + Tf(M) - f((1 - T)m + TM)
\leq 2 \left[ \frac{f(m) + f(M)}{2} - f \left( \frac{m + M}{2} \right) \right] \left( \frac{1}{2} \left| 1_H - \frac{1}{2} 1_H \right| \right),
\end{equation}
in the operator order.

If we take in (2.14) \( 0 \leq T = \frac{A - m 1_H}{M - m} \leq 1_H \), then, like in the proof of Theorem 2, we get
\begin{equation}
2 \left[ \frac{f(m) + f(M)}{2} - f \left( \frac{m + M}{2} \right) \right]
\times \left( \frac{1}{2} (M - m) 1_H - \left| A - \frac{1}{2} (m + M) 1_H \right| \right)
\leq \frac{f(m)(M 1_H - A) + f(M)(A - m 1_H)}{M - m} - f(A)
\leq 2 \left[ \frac{f(m) + f(M)}{2} - f \left( \frac{m + M}{2} \right) \right]
\times \left( \frac{1}{2} (M - m) 1_H + \left| A - \frac{1}{2} (m + M) 1_H \right| \right).
\end{equation}

If we multiply (2.15) to the left with \( P \) and to the right with \( P \) and by taking into account that \( P^2 = P \), then we get
\begin{equation}
2 \left[ \frac{f(m) + f(M)}{2} - f \left( \frac{m + M}{2} \right) \right]
\times \left( \frac{1}{2} (M - m) P - P \left| A - \frac{1}{2} (m + M) 1_H \right| P \right)
\leq \frac{f(m)(MP - PAP) + f(M)(PAP - mP)}{M - m} - Pf(A)P
\leq 2 \left[ \frac{f(m) + f(M)}{2} - f \left( \frac{m + M}{2} \right) \right]
\times \left( \frac{1}{2} (M - m) P + P \left| A - \frac{1}{2} (m + M) 1_H \right| P \right),
\end{equation}
which proves (2.11).
Like in (2.15) we get
\[ 2 \left[ \frac{f(m) + f(M)}{2} - f\left( \frac{m + M}{2} \right) \right] \]
\[ \times \left( \frac{1}{2} (M - m) 1_H - \left| PAP - \frac{1}{2} (m + M) 1_H \right| \right) \]
\[ \leq \frac{f(m) (M1_H - PAP) + f(M) (PAP - m1_H)}{M - m} - f(PAP) \]
\[ \leq 2 \left[ \frac{f(m) + f(M)}{2} - f\left( \frac{m + M}{2} \right) \right] \]
\[ \times \left( \frac{1}{2} (M - m) 1_H + \left| PAP - \frac{1}{2} (m + M) 1_H \right| \right). \]

and if we multiply it to the left with \( P \) and to the right with \( P \) and by taking into account that \( P^2 = P \), then we get
\[ 2 \left[ \frac{f(m) + f(M)}{2} - f\left( \frac{m + M}{2} \right) \right] \]
\[ \times \left( \frac{1}{2} (M - m) P - P \left| PAP - \frac{1}{2} (m + M) 1_H \right| P \right) \]
\[ \leq \frac{f(m) (MP - PAP) + f(M) (PAP - mP)}{M - m} - Pf(PAP)P \]
\[ \leq 2 \left[ \frac{f(m) + f(M)}{2} - f\left( \frac{m + M}{2} \right) \right] \]
\[ \times \left( \frac{1}{2} (M - m) P + P \left| PAP - \frac{1}{2} (m + M) 1_H \right| P \right), \]

which proves (2.12).

**Corollary 2.** Let \( f : [m, M] \to \mathbb{R} \) be an operator convex function on \([m, M]\) and \( A \) a selfadjoint operator with the spectrum \( \text{Sp}(A) \subset [m, M] \). If \( P \) is an orthogonal projection, then

\[ 2 \left[ \frac{f(m) + f(M)}{2} - f\left( \frac{m + M}{2} \right) \right] \]
\[ \times \left( \frac{1}{2} (M - m) P - P \left| A - \frac{1}{2} (m + M) 1_H \right| P \right) \]
\[ \leq \frac{f(m) (MP - PAP) + f(M) (PAP - mP)}{M - m} - Pf(A)P \]
\[ \leq \frac{f(m) (MP - PAP) + f(M) (PAP - mP)}{M - m} - Pf(PAP)P \]
\[ \leq 2 \left[ \frac{f(m) + f(M)}{2} - f\left( \frac{m + M}{2} \right) \right] \]
\[ \times \left( \frac{1}{2} (M - m) P + P \left| PAP - \frac{1}{2} (m + M) 1_H \right| P \right). \]

We also have

**Corollary 3.** Let \( f : [m, M] \to \mathbb{R} \) be a convex function on \([m, M]\) and \( A \) a selfadjoint operator with the spectrum \( \text{Sp}(A) \subset [m, M] \). If \( P \) is an orthogonal projection,
then
\begin{equation}
Pf (A) P - Pf (PAP) P \leq 2 \left[ \frac{f (m) + f (M)}{2} - f \left( \frac{m + M}{2} \right) \right]
\end{equation}
\times \left( \frac{1}{2} (M - m) P + P \left| PAP - \frac{1}{2} (m + M) H \right| P \right)
\leq 2 (M - m) \left[ \frac{f (m) + f (M)}{2} - f \left( \frac{m + M}{2} \right) \right] P.

\textbf{Proof.} From (2.6) we have
\begin{equation}
Pf (A) P - Pf (PAP) P \leq \frac{f (m) (MP - PAP) + f (M) (PAP - mP)}{M - m} - Pf (PAP) P
\end{equation}
and from (2.12) we have
\begin{align*}
f (m) (MP - PAP) + f (M) (PAP - mP) & \leq 2 \left[ \frac{f (m) + f (M)}{2} - f \left( \frac{m + M}{2} \right) \right] \\
& \times \left( \frac{1}{2} (M - m) P + P \left| PAP - \frac{1}{2} (m + M) H \right| P \right).
\end{align*}
which produce the desired result (2.17). \hfill \Box

\textbf{Remark 1.} If \( f : [m, M] \rightarrow \mathbb{R} \) is an operator convex function on \([m, M]\), \( A \) a selfadjoint operator with the spectrum \( \text{Sp} (A) \subset [m, M] \) and \( P \) is an orthogonal projection, then
\begin{equation}
0 \leq Pf (A) P - Pf (PAP) P \leq 2 \left[ \frac{f (m) + f (M)}{2} - f \left( \frac{m + M}{2} \right) \right] \\
\times \left( \frac{1}{2} (M - m) P + P \left| PAP - \frac{1}{2} (m + M) H \right| P \right)
\leq 2 (M - m) \left[ \frac{f (m) + f (M)}{2} - f \left( \frac{m + M}{2} \right) \right] P.
\end{equation}

We also have [5]:

\textbf{Lemma 3.} Assume that \( f : [a, b] \rightarrow \mathbb{R} \) is absolutely continuous on \([a, b]\). If \( f' \) is \( K \)-Lipschitzian on \([a, b]\), then
\begin{equation}
|(1 - t) f (a) + t f (b) - f ((1 - t) a + t b)| \leq \frac{1}{2} K (b - t) (t - a) \leq \frac{1}{8} K (b - a)^2
\end{equation}
for all \( t \in [0, 1] \).

The constants \( 1/2 \) and \( 1/8 \) are the best possible in (2.20).
Remark 2. If $f : [a, b] \to \mathbb{R}$ is twice differentiable and $f'' \in L_\infty [a, b]$, then
\begin{equation}
(2.21)
| (1 - t) f (a) + t f (b) - f ((1 - t) a + t b) | \\
\leq \frac{1}{2} \| f'' \|_{[a, b], \infty} (b - t) (t - a) \leq \frac{1}{8} \| f'' \|_{[a, b], \infty} (b - a)^2,
\end{equation}
where $\| f'' \|_{[a, b], \infty} := \text{essup}_{t \in [a, b]} | f'' (t) | < \infty$. The constants $1/2$ and $1/8$ are the best possible in (2.21).

We have:

**Theorem 4.** Let $f : [m, M] \to \mathbb{R}$ be a twice differentiable convex function on $[m, M]$ with $\| f'' \|_{[m, M], \infty} := \text{essup}_{t \in [m, M]} f'' (t) < \infty$ and $A$ a selfadjoint operator with the spectrum $\text{Sp} (A) \subset [m, M]$. If $P$ is an orthogonal projection, then
\begin{equation}
(2.22)
P f (A) P - P f (PAP) P \\
\leq \frac{1}{2} \| f'' \|_{m, M}, \infty (MP - PAP) (PAP - mP) \\
\leq \frac{1}{8} \| f'' \|_{m, M}, \infty (M - m)^2 P.
\end{equation}

**Proof.** From (2.21) and the continuous functional calculus, we get
\begin{equation}
(2.23)
0 \leq \frac{f (m) (M 1_H - B) + f (M) (B - m 1_H)}{M - m} - f (B) \\
\leq \frac{1}{2} \| f'' \|_{m, M}, \infty (M 1_H - B) (B - m 1_H) \\
\leq \frac{1}{8} \| f'' \|_{m, M}, \infty (M - m)^2 1_H
\end{equation}
where $B$ is a selfadjoint operator with the spectrum $\text{Sp} (B) \subset [m, M]$.

Therefore
\[ 0 \leq \frac{f (m) (M 1_H - PAP) + f (M) (PAP - m 1_H)}{M - m} - f (PAP) \]
\[ \leq \frac{1}{2} \| f'' \|_{m, M}, \infty (M 1_H - PAP) (PAP - m 1_H) \]
\[ \leq \frac{1}{8} \| f'' \|_{m, M}, \infty (M - m)^2 1_H \]
if we multiply it to the left with $P$ and to the right with $P$ and by taking into account that $P^2 = P$, then we get
\[ 0 \leq \frac{f (m) (MP - PAP) + f (M) (PAP - mP)}{M - m} - P f (PAP) P \]
\[ \leq \frac{1}{2} \| f'' \|_{m, M}, \infty (MP - PAP) (PAP - mP) \]
\[ \leq \frac{1}{8} \| f'' \|_{m, M}, \infty (M - m)^2 P \]
and by (2.18) we get (2.22). \qed

**Corollary 4.** Let $f : [m, M] \to \mathbb{R}$ be an operator convex function on $[m, M]$ and $A$ a selfadjoint operator with the spectrum $\text{Sp} (A) \subset [m, M]$. If $P$ is an orthogonal
projection, then

\begin{equation}
0 \leq Pf(A)P - Pf(PAP)P
\leq \frac{1}{2} \|f''\|_{[m,M],\infty} (MP - PAP)(PAP - mP)
\leq \frac{1}{8} \|f''\|_{[m,M],\infty} (M - m)^2 P.
\end{equation}

3. Some Examples

We consider the exponential function \(f(x) = \exp(\alpha x)\) with \(\alpha \in \mathbb{R} \setminus \{0\}\). This function is convex but not operator convex on \(\mathbb{R}\). If \(A\) is selfadjoint with \(\text{Sp}(A) \subset [m,M]\) for some \(m < M\) and \(P\) is an orthogonal projection, then by (2.2), (2.17) and (2.22) we have

\begin{equation}
P\exp(\alpha A)P - P\exp(\alpha PAP)P
\leq \frac{\alpha}{M - m} (MP - PAP)(PAP - mP)
\leq \frac{1}{4} \alpha (M - m) \exp(\alpha M) - \exp(\alpha m) P,
\end{equation}

\begin{equation}
P\exp(\alpha A)P - P\exp(\alpha PAP)P
\leq 2 \left[ \frac{\exp(\alpha M) + f(\alpha M)}{2} - \exp\left(\frac{\alpha M}{2}\right) \right]
\times \left( \frac{1}{2} (M - m) P + P \left\lvert PAP - \frac{1}{2} (m + M) 1_P \right\rvert P \right)
\leq 2 (M - m) \left[ \frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\frac{\alpha m + M}{2}\right) \right] P
\end{equation}

and

\begin{equation}
P\exp(\alpha A)P - P\exp(\alpha PAP)P
\leq \frac{1}{2} \alpha^2 \begin{cases} 
\exp(\alpha M) & \text{if } \alpha > 0 \\
\exp(\alpha M) & \text{if } \alpha < 0
\end{cases}
\times (MP - PAP)(PAP - mP)
\leq \frac{1}{8} \alpha^2 (M - m)^2 \begin{cases} 
\exp(\alpha M) & \text{if } \alpha > 0 \\
\exp(\alpha m) & \text{if } \alpha < 0
\end{cases}
\times P.
\end{equation}

The function \(f(x) = -\ln x, x > 0\) is operator convex on \((0, \infty)\). If \(A\) is selfadjoint with \(\text{Sp}(A) \subset [m,M]\) for some \(0 < m < M\) and \(P\) is an orthogonal projection, then by (2.9), (2.19) and (2.24) we have

\begin{equation}
0 \leq Pf(A)P - Pf(PAP)P
\leq \frac{1}{mM} (MP - PAP)(PAP - mP)
\leq \frac{1}{4mM} (M - m)^2 P,
\end{equation}
(3.5) \[ 0 \leq Pf(A)P - Pf(PAP)P \]
\[ \leq 2 \ln \left( \frac{m + M}{2\sqrt{mM}} \right) \]
\[ \times \left( \frac{1}{2} (M - m) P + P \left| PAP - \frac{1}{2} (m + M) 1_H \right| P \right) \]
\[ \leq 2 (M - m) \ln \left( \frac{m + M}{2\sqrt{mM}} \right) P \]

and

(3.6) \[ 0 \leq Pf(A)P - Pf(PAP)P \]
\[ \leq \frac{1}{2m^2} (MP - PAP) (PAP - mP) \leq \frac{1}{8m^2} (M - m)^2 P. \]

We observe that if \( M > 2m \) then the bound in (3.4) is better than the one from (3.6). If \( M < 2m \), then the conclusion is the other way around.

The function \( f(x) = x \ln x, x > 0 \) is operator convex on \((0, \infty)\). If \( A \) is selfadjoint with \( \text{Sp}(A) \subseteq [m, M] \) for some \( 0 < m < M \) and \( P \) is an orthogonal projection, then by (2.9), (2.19) and (2.24) we have

(3.7) \[ 0 \leq PA \ln(A)P - PAP \ln(PAP)P \]
\[ \leq \frac{\ln(M) - \ln(m)}{M - m} (MP - PAP) (PAP - mP) \]
\[ \leq \frac{1}{4} (M - m) [\ln(M) - \ln(m)] P, \]

(3.8) \[ 0 \leq PA \ln(A)P - PAP \ln(PAP)P \]
\[ \leq 2 \left[ \frac{m \ln(m) + M \ln(M)}{2} - \left( \frac{m + M}{2} \right) \ln \left( \frac{m + M}{2} \right) \right] \]
\[ \times \left( \frac{1}{2} (M - m) P + P \left| PAP - \frac{1}{2} (m + M) 1_H \right| P \right) \]
\[ \leq 2 (M - m) \left[ \frac{m \ln(m) + M \ln(M)}{2} - \left( \frac{m + M}{2} \right) \ln \left( \frac{m + M}{2} \right) \right] P \]

and

(3.9) \[ 0 \leq PA \ln(A)P - PAP \ln(PAP)P \]
\[ \leq \frac{1}{2m} (MP - PAP) (PAP - mP) \leq \frac{1}{8m} (M - m)^2 P. \]

Consider the power function \( f(x) = x^r, x \in (0, \infty) \) and \( r \) a real number. If \( r \in (-\infty, 0] \cup [1, \infty), \) then \( f \) is convex and for \( r \in [-1, 0] \cup [1, 2] \) is operator convex. If we use the inequalities (2.2), (2.17) and (2.22) we have for \( r \in (-\infty, 0] \cup [1, \infty) \) that

(3.10) \[ PA^r P - P (PAP)^r P \]
\[ \leq r \frac{M^{r-1} - m^{r-1}}{M - m} (MP - PAP) (PAP - mP) \]
\[ \leq \frac{1}{4} r (M - m) [M^{r-1} - m^{r-1}] P, \]
\begin{align}
PA^r P - P (PAP)^r P & \leq 2 \left[ \frac{m^r + M^r}{2} - \left( \frac{m + M}{2} \right)^r \right] \\
& \times \left( \frac{1}{2} (M - m) P + P \right) \left( PAP - \frac{1}{2} (m + M) 1_H \right) P \\
& \leq 2 (M - m) \left[ \frac{m^r + M^r}{2} - \left( \frac{m + M}{2} \right)^r \right] P
\end{align}

and
\begin{align}
PA^r P - P (PAP)^r P & \leq \frac{1}{2} r (r - 1) \left\{ \begin{array}{ll}
M^{r-2} & \text{for } r \geq 2 \\
m^{r-2} & \text{for } r \in (-\infty, 0] \cup [1, 2)
\end{array} \right. \\
& \times (MP - PAP) (PAP - mP) \\
& \leq \frac{1}{8} r (r - 1) (M - m)^2 \left\{ \begin{array}{ll}
M^{r-2} & \text{for } r \geq 2 \\
m^{r-2} & \text{for } r \in (-\infty, 0] \cup [1, 2)
\end{array} \right. \times P,
\end{align}

where A is selfadjoint with \( \text{Sp}(A) \subset [m, M] \) for some \( 0 < m < M \) and \( P \) is an orthogonal projection.

If \( r \in [-1, 0] \cup [1, 2] \), then we also have \( 0 \leq PA^r P - P (PAP)^r P \) in the inequalities (3.10)-(3.12).

References


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