REVERSE JENSEN INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS AND POSITIVE LINEAR MAPS IN C^*-ALGEBRAS

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. In this paper we obtain several operator inequalities providing upper bounds for the difference

$$\int_T \phi_t\left(f\left(x_t\right)\right) d\mu\left(t\right) - f\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right),$$

where $f : I \to \mathbb{R}$ is a convex function defined on an interval I , $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ and $(x_t)_{t\in T}$ is a bounded continuous field of selfadjoint elements in $\mathcal{A}$ with spectra contained in I. Several Hermite-Hadamard type inequalities are given. Some examples for convex and operator convex functions are also provided.

1. Introduction

Let T be a locally compact Hausdorff space and let $\mathcal{A}$ be a C^*-algebra. We say that a field $(x_t)_{t\in T}$ of operators in $\mathcal{A}$ is continuous if the function $t \mapsto x_t$ is norm continuous on T. If in addition μ is a Radon measure on T and the function $t \mapsto \|x_t\|$ is integrable, then we can form the Bochner integral $\int_T x_t d\mu\left(t\right)$, which is the unique element in $\mathcal{A}$ such that

$$\varphi\left(\int_T x_t d\mu\left(t\right)\right) = \int_T \varphi\left(x_t\right) d\mu\left(t\right)$$

for every linear functional φ in the norm dual $\mathcal{A}^*$, cf. [13, Section 4.1].

Assume furthermore that there is a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$. We say that such a field is continuous if the function $t \mapsto \phi_t\left(x\right)$ is continuous for every $x \in \mathcal{A}$. If the C^*-algebras are unital and the field $t \mapsto \phi_t\left(\mathbf{1}\right)$ is integrable with integral $\int_T \phi_t\left(\mathbf{1}\right) d\mu\left(t\right) = \mathbf{1}$, we say that $(\phi_t)_{t\in T}$ is unital.

A continuous function $I \to \mathbb{R}$ is said to be operator convex if

$$f\left(\left(1-\lambda\right)x + \lambda y\right) \le \left(1-\lambda\right) f\left(x\right) + \lambda f\left(y\right)$$

for any selfadjoint elements x, y in $\mathcal{A}$ with spectra $\operatorname{Sp}\left(x\right)$ and $\operatorname{Sp}\left(y\right)$ contained in I.

The following Jensen's integral inequality has been obtained in [12]:

Theorem 1. *Let $f : I \to \mathbb{R}$ be an operator convex function defined on an interval I , and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. If $(\phi_t)_{t\in T}$ is a unital field of positive*

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linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then the inequality

$$f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \le \int_T \phi_t(f(x_t))\,d\mu(t) \tag{1.1}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of selfadjoint elements in $\mathcal{A}$ with spectra contained in I.

The discrete case is as follows [15]:

$$f\left(\sum_{i=1}^n w_i\phi_i(x_i)\right) \le \sum_{i=1}^n w_i\phi_i(f(x_i))$$

for operator convex functions f defined on an interval I, where $\phi_i : \mathcal{A} \to \mathcal{B}$, $i \in \{1,...,n\}$ are unital positive linear maps, x_i, $i \in \{1,...,n\}$ are selfadjoint elements in $\mathcal{A}$ with spectra contained in I and $w_i \ge 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n w_i = 1$.

Also, if $f : I \to \mathbb{R}$ is operator convex on I and $a_i \in \mathcal{A}$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n a^* a_i = 1$, then [13]

$$f\left(\sum_{i=1}^n a^* x_i a_i\right) \le \sum_{i=1}^n a^* f(x_i)\,a_i$$

where x_i, $i \in \{1,...,n\}$ are selfadjoint elements in $\mathcal{A}$ with spectra contained in I.

For various reverse inequalities related to these results see [15], [13], [8] and [12]. For related inequalities for operator convex functions see [1]-[3], [9]-[11] and [16]-[20].

It is known that there are convex functions f for which the inequality (1.1) does not hold, however one can obtain several operator inequalities providing upper bounds for the difference

$$\int_T \phi_t(f(x_t))\,d\mu(t) - f\left(\int_T \phi_t(x_t)\,d\mu(t)\right)$$

for any convex function $f : I \to \mathbb{R}$, $(\phi_t)_{t\in T}$ and $(x_t)_{t\in T}$ as in Theorem 1. Several Hermite-Hadamard type inequalities are given. Some examples for convex and operator convex functions are also provided.

2. Some Hermite-Hadamard Type Inequalities

Let $T = [0,1]$ and μ be the Lebesgue measure on the interval $[0,1]$. Assume that the field $(\phi_t)_{t\in[0,1]}$ of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ is continuous and unital, i.e. $\int_0^1 \phi_t(\mathbf{1})\,dt = \mathbf{1}$ and x, y selfadjoint elements in $\mathcal{A}$ with the spectra in I. Then by taking $x_t := (1-t)x + ty$, $t \in [0,1]$ we get from (1.1) that

$$f\left(\int_0^1 \phi_t((1-t)x+ty)\,dt\right) \le \int_0^1 \phi_t(f((1-t)x+ty))\,dt. \tag{2.1}$$

We have

$$\int_0^1 \phi_t((1-t)x+ty)\,dt = \int_0^1 (1-t)\phi_t(x)\,dt + \int_0^1 t\phi_t(y)\,dt.$$

By the operator convexity of f we also have

$$\begin{aligned}(2.2)\qquad \int_0^1 \phi_t\left(f\left((1-t)x+ty\right)\right)dt &\le \int_0^1 \phi_t\left[(1-t)f(x)+tf(y)\right]dt\\ &= \int_0^1 \left[(1-t)\phi_t(f(x))+t\phi_t(f(y))\right]dt\\ &= \int_0^1 (1-t)\phi_t(f(x))\,dt+\int_0^1 t\phi_t(f(y))\,dt.\end{aligned}$$

Therefore by (2.1) we obtain the Hermite-Hadamard type inequality

$$\begin{aligned}(2.3)\qquad & f\left(\int_0^1 (1-t)\phi_t(x)\,dt+\int_0^1 t\phi_t(y)\,dt\right)\\ &\le \int_0^1 \phi_t\left(f\left((1-t)x+ty\right)\right)dt\\ &\le \int_0^1 (1-t)\phi_t(f(x))\,dt+\int_0^1 t\phi_t(f(y))\,dt\end{aligned}$$

for x, y selfadjoint elements in $\mathcal{A}$ with the spectra in I, the field $(\phi_t)_{t\in[0,1]}$ of positive linear continuous mappings $\phi_t:\mathcal{A}\to\mathcal{B}$ with $\int_0^1\phi_t(\mathbf{1})\,dt=\mathbf{1}$ and the operator convex function $f:I\to\mathbb{R}$.

If we take $\phi_t=\phi$, $t\in[0,1]$, a positive linear mapping with $\phi(\mathbf{1})=\mathbf{1}$ and since

$$\begin{aligned}\int_0^1 (1-t)\phi_t(x)\,dt+\int_0^1 t\phi_t(y)\,dt &= \left(\int_0^1(1-t)\,dt\right)\phi(x)+\left(\int_0^1 tdt\right)\phi(y)\\ &= \frac{\phi(x)+\phi(y)}{2}\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 (1-t)\phi_t(f(x))\,dt+\int_0^1 t\phi_t(f(y))\,dt\\ &= \left(\int_0^1(1-t)\,dt\right)\phi(f(x))+\left(\int_0^1 tdt\right)\phi(f(y))\\ &= \frac{\phi(f(x))+\phi(f(y))}{2},\end{aligned}$$

then by (2.3) we get

$$(2.4)\qquad f\left(\frac{\phi(x)+\phi(y)}{2}\right)\le\int_0^1\phi\left(f\left((1-t)x+ty\right)\right)dt\le\frac{\phi(f(x))+\phi(f(y))}{2}$$

for x, y selfadjoint elements in $\mathcal{A}$ with the spectra in I, the positive linear mapping $\phi:\mathcal{A}\to\mathcal{B}$ with $\phi(\mathbf{1})=\mathbf{1}$ and the operator convex function $f:I\to\mathbb{R}$.

However, this inequality is not as good as the following result obtained for Banach algebras of operators [7], which can be also stated, with a similar proof, for the unital

C^{*}-algebras $\mathcal{A}$ and $\mathcal{B}$

$$
\begin{aligned}
(2.5)\qquad & \left(f\left(\frac{\phi(x)+\phi(y)}{2}\right)\leq\right)\phi\left(f\left(\frac{x+y}{2}\right)\right) \\
& \leq(1-\lambda)\phi\left(f\left[\frac{(1-\lambda)x+(1+\lambda)}{2}\right]\right)+\lambda\phi\left(f\left[\frac{(2-\lambda)x+\lambda y}{2}\right]\right) \\
& \leq\int_0^1\phi\left(f\left((1-t)x+ty\right)\right)dt \\
& \leq\frac{1}{2}\left[\phi\left(f\left((1-\lambda)x+\lambda y\right)\right)+(1-\lambda)\phi\left(f(y)\right)+\lambda\phi\left(f(x)\right)\right] \\
& \leq\frac{\phi\left(f(x)\right)+\phi\left(f(y)\right)}{2}
\end{aligned}
$$

for x, y selfadjoint elements in $\mathcal{A}$ with the spectra in I, the positive linear mapping $\phi:\mathcal{A}\to\mathcal{B}$ with $\phi(\mathbf{1})=\mathbf{1}$ and the operator convex function $f:I\to\mathbb{R}$.

Let a, $b\in\mathcal{A}$ with $a^{*}a=b^{*}b=\mathbf{1}$ and define

$$\phi_t(x)=(1-t)a^{*}xa+tb^{*}xb,\ t\in[0,1]\ \text{ and } x\in\mathcal{A}.$$

This field $(\phi_t)_{t\in[0,1]}$ is of positive linear continuous mappings with

$$
\begin{aligned}
\int_0^1\phi_t(\mathbf{1})\,dt&=\int_0^1\left[(1-t)a^{*}\mathbf{1}a+tb^{*}\mathbf{1}b\right]dt \\
&=\int_0^1\left[(1-t)a^{*}a+tb^{*}b\right]dt=\frac{a^{*}a+b^{*}b}{2}=\mathbf{1}.
\end{aligned}
$$

If we use the inequality (1.1) for this filed of positive linear mappings, we get

$$f\left(\int_0^1\left[(1-t)a^{*}x_ta+tb^{*}x_tb\right]dt\right)\leq\int_0^1\left[(1-t)a^{*}f(x_t)a+tb^{*}f(x_t)b\right]dt$$

namely

$$
\begin{aligned}
(2.6)\qquad & f\left(a^{*}\left(\int_0^1(1-t)x_tdt\right)a+b^{*}\left(\int_0^1tx_tdt\right)b\right) \\
& \leq a^{*}\left(\int_0^1(1-t)f(x_t)\,dt\right)a+b^{*}\left(\int_0^1tf(x_t)\,dt\right)b
\end{aligned}
$$

for every bounded continuous field $(x_t)_{t\in[0,1]}$ of selfadjoint elements in $\mathcal{A}$ with spectra contained in I.

If we take $x_t:=(1-t)x+ty$, $t\in[0,1]$, then

$$\int_0^1(1-t)x_tdt=\int_0^1(1-t)\left[(1-t)x+ty\right]dt=\frac{1}{3}x+\frac{1}{6}y,$$

$$\int_0^1tx_tdt=\int_0^1t\left[(1-t)x+ty\right]dt=\frac{1}{6}x+\frac{1}{3}y.$$

Also, by the operator convexity of f we have

$$\begin{aligned}
a^*\left(\int_0^1 (1-t)\,f(x_t)\,dt\right)a &= a^*\left(\int_0^1 (1-t)\,f((1-t)\,x+ty)\,dt\right)a \\
&\le a^*\left(\int_0^1 \left((1-t)^2 f(x)+(1-t)\,tf(y)\right)dt\right)a \\
&= \frac{1}{3}a^*f(x)\,a+\frac{1}{6}a^*f(y)\,a
\end{aligned}$$

and

$$\begin{aligned}
b^*\left(\int_0^1 tf(x_t)\,dt\right)b &= b^*\left(\int_0^1 tf((1-t)\,x+ty)\,dt\right)b \\
&\le b^*\left(\int_0^1 \left((1-t)\,tf(x)+t^2 f(y)\right)dt\right)b \\
&= \frac{1}{6}b^*f(x)\,b+\frac{1}{3}b^*f(y)\,b.
\end{aligned}$$

From (2.6) we get

$$\begin{aligned}
& f\left(a^*\left(\frac{1}{3}x+\frac{1}{6}y\right)a+b^*\left(\frac{1}{6}x+\frac{1}{3}y\right)b\right) \\
&\le a^*\left(\int_0^1 (1-t)\,f((1-t)\,x+ty)\,dt\right)a+b^*\left(\int_0^1 tf((1-t)\,x+ty)\,dt\right)b \\
&\le \frac{1}{3}a^*f(x)\,a+\frac{1}{6}a^*f(y)\,a+\frac{1}{6}b^*f(x)\,b+\frac{1}{3}b^*f(y)\,b
\end{aligned}$$

namely

$$\begin{aligned}
\text{(2.7)}\quad & f\left(\frac{1}{2}\left[a^*\left(\frac{2x+y}{3}\right)a+b^*\left(\frac{x+2y}{3}\right)b\right]\right) \\
&\le a^*\left(\int_0^1 (1-t)\,f((1-t)\,x+ty)\,dt\right)a+b^*\left(\int_0^1 tf((1-t)\,x+ty)\,dt\right)b \\
&\le \frac{1}{2}\left[a^*\left(\frac{2f(x)+f(y)}{3}\right)a+b^*\left(\frac{f(x)+2f(y)}{3}\right)b\right],
\end{aligned}$$

where f is operator convex on I, $a, b \in \mathcal{A}$ with $a^*a = b^*b = \mathbf{1}$ and x, y are selfadjoint elements with spectra in I.

3. Main Results

The following result provides an operator inequality that generalizes the scalar version obtained in [6]. In the formulation below it was obtained in [14]:

Theorem 2. *Let $f : I \to \mathbb{R}$ be a continuous convex function defined on an interval I and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. If $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a* *bounded Radon measure μ, then*

$$\begin{aligned}(3.1)\qquad &\int_T \phi_t\left(f\left(x_t\right)\right) d\mu\left(t\right) - f\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right)\\ &\le \frac{f'_-\left(M\right) - f'_+\left(m\right)}{M-m}\left(M\mathbf{1} - \int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right)\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right) - m\mathbf{1}\right)\\ &\le \frac{1}{4}\left(M-m\right)\left[f'_-\left(M\right) - f'_+\left(m\right)\right]\mathbf{1}\end{aligned}$$

for every bounded continuous field $(x_t)_{t\in T}$ of selfadjoint elements in $\mathcal{A}$ with spectra contained in a closed subinterval $[m, M]$ of I.

Proof. We use the following inequality for convex functions $f : [m, b] \to \mathbb{R}$ that was obtained in [4]:

$$\begin{aligned}(3.2)\qquad 0 &\le \frac{\left(M-t\right) f\left(m\right) + \left(t-m\right) f\left(M\right)}{M-m} - f\left(t\right)\\ &\le \left(M-t\right)\left(t-m\right)\frac{f'_-\left(M\right) - f'_+\left(m\right)}{M-m} \le \frac{1}{4}\left(M-m\right)\left[f'_-\left(M\right) - f'_+\left(m\right)\right]\end{aligned}$$

for any $t \in [m, M]$.

If the lateral derivatives $f'_-(M)$ and $f'_+(m)$ are finite, then the second inequality and the constant $1/4$ are sharp.

Utilising the continuous functional calculus for a selfadjoint element y with $0 \le y \le \mathbf{1}$ and the convexity of f on $[m, M]$, we have

$$(3.3)\qquad f\left(m\left(\mathbf{1}-y\right) + My\right) \le f\left(m\right)\left(\mathbf{1}-y\right) + f\left(M\right) y$$

in the operator order.

Let $t \in T$. If we take in (3.3)

$$0 \le y = \frac{x_t - m\mathbf{1}}{M-m} \le \mathbf{1},$$

then we get

$$\begin{aligned}(3.4)\qquad &f\left(m\left(\mathbf{1} - \frac{x_t - m\mathbf{1}}{M-m}\right) + M\frac{x_t - m\mathbf{1}}{M-m}\right)\\ &\le f\left(m\right)\left(\mathbf{1} - \frac{x_t - m\mathbf{1}}{M-m}\right) + f\left(M\right)\frac{x_t - m\mathbf{1}}{M-m}.\end{aligned}$$

Observe that

$$m\left(\mathbf{1} - \frac{x_t - m\mathbf{1}}{M-m}\right) + M\frac{x_t - m\mathbf{1}}{M-m} = \frac{m\left(M\mathbf{1} - x_t\right) + M\left(x_t - m\mathbf{1}\right)}{M-m} = x_t$$

and

$$f\left(m\right)\left(\mathbf{1} - \frac{x_t - m\mathbf{1}}{M-m}\right) + f\left(M\right)\frac{x_t - m\mathbf{1}}{M-m} = \frac{f\left(m\right)\left(M\mathbf{1} - x_t\right) + f\left(M\right)\left(x_t - m\mathbf{1}\right)}{M-m}$$

and by (3.4) we get the following inequality of interest

$$(3.5)\qquad f\left(x_t\right) \le \frac{f\left(m\right)\left(M\mathbf{1} - x_t\right) + f\left(M\right)\left(x_t - m\mathbf{1}\right)}{M-m}$$

for all $t \in T$.

If we take the functional ϕ_t in (3.5) we get

$$(3.6)\qquad \phi_t\left(f\left(x_t\right)\right) \le \frac{f\left(m\right)\left(M\phi_t\left(\mathbf{1}\right) - \phi_t\left(x_t\right)\right) + f\left(M\right)\left(\phi_t\left(x_t\right) - m\phi_t\left(\mathbf{1}\right)\right)}{M-m}$$

for all $t \in T$.

If we take the integral $\int_T$ in (3.6) we get

$$\begin{aligned}
(3.7)\qquad & \int_T \phi_t\left(f\left(x_t\right)\right) d\mu\left(t\right) \\
& \le \frac{1}{M-m}\left[f\left(m\right)\left(M\int_T \phi_t\left(\mathbf{1}\right) d\mu\left(t\right) - \int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right)\right. \\
& \left. + f\left(M\right)\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right) - m\int_T \phi_t\left(\mathbf{1}\right) d\mu\left(t\right)\right)\right] \\
& = \frac{1}{M-m}\left[f\left(m\right)\left(M\mathbf{1} - \int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right)\right. \\
& \left. + f\left(M\right)\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right) - m\mathbf{1}\right)\right].
\end{aligned}$$

Therefore, by (3.7) we obtain

$$\begin{aligned}
(3.8)\qquad & \int_T \phi_t\left(f\left(x_t\right)\right) d\mu\left(t\right) - f\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right) \\
& \le \frac{f\left(m\right)\left(M\mathbf{1} - \int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right) + f\left(M\right)\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right) - m\mathbf{1}\right)}{M-m} \\
& - f\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right).
\end{aligned}$$

Using the inequality (3.2) and the functional calculus, we get

$$\begin{aligned}
(3.9)\qquad & \frac{f\left(m\right)\left(M\mathbf{1} - \int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right) + f\left(M\right)\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right) - m\mathbf{1}\right)}{M-m} \\
& - f\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right) \\
& \le \frac{f'_-\left(M\right) - f'_+\left(m\right)}{M-m}\left(M\mathbf{1} - \int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right)\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right) - m\mathbf{1}\right) \\
& \le \frac{1}{4}\left(M-m\right)\left[f'_-\left(M\right) - f'_+\left(m\right)\right]\mathbf{1}.
\end{aligned}$$

By utilising (3.8) and (3.9) we derive (3.1). □

Corollary 1. *With the assumptions of Theorem 2 and if f is operator convex on I, then we have the following reverse of (1.1)*

$$\begin{aligned}
(3.10)\qquad 0 & \le \int_T \phi_t\left(f\left(x_t\right)\right) d\mu\left(t\right) - f\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right) \\
& \le \frac{f'_-\left(M\right) - f'_+\left(m\right)}{M-m}\left(M\mathbf{1} - \int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right)\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right) - m\mathbf{1}\right) \\
& \le \frac{1}{4}\left(M-m\right)\left[f'_-\left(M\right) - f'_+\left(m\right)\right]\mathbf{1}.
\end{aligned}$$

We also have the norm inequalities

$$\begin{aligned}\left\|\int_T \phi_t(f(x_t))\,d\mu(t) - f\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\right\| &\le \frac{f'_-(M)-f'_+(m)}{M-m}\left\|\left(M\mathbf{1}-\int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t)-m\mathbf{1}\right)\right\| \\ &\le \frac{f'_-(M)-f'_+(m)}{M-m}\left\|M\mathbf{1}-\int_T \phi_t(x_t)\,d\mu(t)\right\|\left\|\int_T \phi_t(x_t)\,d\mu(t)-m\mathbf{1}\right\| \\ &\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]\mathbf{1}.\end{aligned} \tag{3.11}$$

Remark 1. *Let $f : I \to \mathbb{R}$ be a continuous convex function defined on an interval I and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Assume that the field $(\phi_t)_{t\in[0,1]}$ of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ is continuous and unital, i.e. $\int_0^1 \phi_t(\mathbf{1})\,dt = \mathbf{1}$ and x, y selfadjoint elements in $\mathcal{A}$ with the spectra in $[m, M] \subset I$. Then by taking $x_t := (1-t)x + ty$, $t \in [0,1]$ we get from (3.1) that*

$$\begin{aligned}&\int_0^1 \phi_t(f((1-t)x+ty))\,dt - f\left(\int_0^1 (1-t)\phi_t(x)\,dt + \int_0^1 t\phi_t(y)\,dt\right) \\ &\le \frac{f'_-(M)-f'_+(m)}{M-m}\left(M\mathbf{1}-\int_0^1 (1-t)\phi_t(x)\,dt - \int_0^1 t\phi_t(y)\,dt\right) \\ &\times\left(\int_0^1 (1-t)\phi_t(x)\,dt + \int_0^1 t\phi_t(y)\,dt - m\mathbf{1}\right) \\ &\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]\mathbf{1}.\end{aligned} \tag{3.12}$$

If f is operator convex, then the first term in (3.12) is also nonnegative in the operator order.

For x, y selfadjoint elements in $\mathcal{A}$ with the spectra in I, the positive linear mapping $\phi : \mathcal{A} \to \mathcal{B}$ with $\phi(\mathbf{1}) = \mathbf{1}$ and the convex function $f : I \to \mathbb{R}$, we have

$$\begin{aligned}&\int_0^1 \phi(f((1-t)x+ty))\,dt - f\left(\frac{\phi(x)+\phi(y)}{2}\right) \\ &\le \frac{f'_-(M)-f'_+(m)}{M-m}\left(M\mathbf{1}-\frac{\phi(x)+\phi(y)}{2}\right)\left(\frac{\phi(x)+\phi(y)}{2}-m\mathbf{1}\right) \\ &\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]\mathbf{1}.\end{aligned} \tag{3.13}$$

If f is operator convex, then the first term in (3.12) is also nonnegative in the operator order.

For $\mathcal{A} = \mathcal{B}$ and $\phi(x) = x$, then by (3.13) we get

$$\begin{aligned}&\int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right) \\ &\le \frac{f'_-(M)-f'_+(m)}{M-m}\left(M\mathbf{1}-\frac{x+y}{2}\right)\left(\frac{x+y}{2}-m\mathbf{1}\right) \\ &\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]\mathbf{1}.\end{aligned} \tag{3.14}$$

If f is operator convex, then the first term in (3.14) is also nonnegative in the operator order.

Let a, $b \in \mathcal{A}$ with $a^*a = b^*b = \mathbf{1}$ and a bounded continuous field $(x_t)_{t\in[0,1]}$ of selfadjoint elements in $\mathcal{A}$ with spectra contained in $[m, M] \subset I$. If f is convex on I, then by (3.1) we get

$$\begin{aligned}(3.15)\quad & a^*\left(\int_0^1 (1-t) f(x_t)\, dt\right) a + b^*\left(\int_0^1 t f(x_t)\, dt\right) b \\ & - f\left(a^*\left(\int_0^1 (1-t) x_t dt\right) a + b^*\left(\int_0^1 t x_t dt\right) b\right) \\ & \le \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - a^*\left(\int_0^1 (1-t) x_t dt\right) a - b^*\left(\int_0^1 t x_t dt\right) b\right) \\ & \times \left(a^*\left(\int_0^1 (1-t) x_t dt\right) a + b^*\left(\int_0^1 t x_t dt\right) b - m\mathbf{1}\right) \\ & \le \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]\mathbf{1}.\end{aligned}$$

If f is operator convex, then the first term in (3.15) is also nonnegative in the operator order.

If f is convex on I, $a, b \in \mathcal{A}$ with $a^*a = b^*b = \mathbf{1}$ and x, y are selfadjoint elements with spectra in $[m, M] \subset I$, then

$$\begin{aligned}(3.16)\quad & a^*\left(\int_0^1 (1-t) f((1-t)x + ty)\, dt\right) a + b^*\left(\int_0^1 t f((1-t)x + ty)\, dt\right) b \\ & - f\left(\frac{1}{2}\left[a^*\left(\frac{2x+y}{3}\right) a + b^*\left(\frac{x+2y}{3}\right) b\right]\right) \\ & \le \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \frac{1}{2}\left[a^*\left(\frac{2x+y}{3}\right) a + b^*\left(\frac{x+2y}{3}\right) b\right]\right) \\ & \times \left(\frac{1}{2}\left[a^*\left(\frac{2x+y}{3}\right) a + b^*\left(\frac{x+2y}{3}\right) b\right] - m\mathbf{1}\right) \\ & \le \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]\mathbf{1}.\end{aligned}$$

If f is operator convex, then the first term in (3.16) is also nonnegative in the operator order.

Further, we also have the following result that provides an operator inequality that generalizes the scalar version obtained in [6].

Theorem 3. *Let $f : I \to \mathbb{R}$ be a continuous convex function defined on an interval I and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. If $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a* *bounded Radon measure μ, then*

$$\begin{aligned} &\int_T \phi_t\left(f\left(x_t\right)\right) d\mu\left(t\right) - f\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right) \\ &\leq 2\left[\frac{f\left(m\right)+f\left(M\right)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times\left(\frac{1}{2}\left(M-m\right)\mathbf{1} + \left|\int_T \phi_t\left(x_t\right) d\mu\left(t\right) - \frac{1}{2}\left(m+M\right)\mathbf{1}\right|\right) \\ &\leq 2\left(M-m\right)\left[\frac{f\left(m\right)+f\left(M\right)}{2} - f\left(\frac{m+M}{2}\right)\right] \end{aligned} \tag{3.17}$$

for every bounded continuous field $(x_t)_{t\in T}$ of selfadjoint elements in $\mathcal{A}$ with spectra contained in a closed subinterval $[m, M]$ of I.

Proof. We also have the following scalar inequality of interest:

$$\begin{aligned} &2\min\{t, 1-t\}\left[\frac{f\left(m\right)+f\left(M\right)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\leq \left(1-t\right) f\left(m\right) + t f\left(M\right) - f\left(\left(1-t\right) m + tM\right) \\ &\leq 2\max\{t, 1-t\}\left[\frac{f\left(m\right)+f\left(M\right)}{2} - f\left(\frac{m+M}{2}\right)\right], \end{aligned} \tag{3.18}$$

where $f : [m, M] \to \mathbb{R}$ be a convex function on $[m, M]$ and $t \in [0, 1]$.

The proof follows, for instance, by Corollary 1 from [5] for $n = 2$, $p_1 = 1 - t$, $p_2 = t$, $t \in [0, 1]$ and $x_1 = m$, $x_2 = M$.

We have from (3.18) that

$$\begin{aligned} &2\left(\frac{1}{2} - \left|t - \frac{1}{2}\right|\right)\left[\frac{f\left(m\right)+f\left(M\right)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\leq \left(1-t\right) f\left(m\right) + t f\left(M\right) - f\left(\left(1-t\right) m + tM\right) \\ &\leq 2\left(\frac{1}{2} + \left|t - \frac{1}{2}\right|\right)\left[\frac{f\left(m\right)+f\left(M\right)}{2} - f\left(\frac{m+M}{2}\right)\right], \end{aligned} \tag{3.19}$$

for all $t \in [0, 1]$.

Utilising the continuous functional calculus for a selfadjoint element y with $0 \leq y \leq \mathbf{1}$ we get from (3.19) that

$$\begin{aligned} &2\left[\frac{f\left(m\right)+f\left(M\right)}{2} - f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}\mathbf{1} - \left|y - \frac{1}{2}\mathbf{1}\right|\right) \\ &\leq \left(\mathbf{1}-y\right) f\left(m\right) + y f\left(M\right) - f\left(\left(\mathbf{1}-y\right) m + yM\right) \\ &\leq 2\left[\frac{f\left(m\right)+f\left(M\right)}{2} - f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}\mathbf{1} + \left|y - \frac{1}{2}\mathbf{1}\right|\right), \end{aligned} \tag{3.20}$$

in the operator order.

Now, if x is selfadjoint with $\operatorname{Sp}(x) \subset [m, M]$, then $m\mathbf{1} \leq x \leq M\mathbf{1}$. If we take in (3.20)

$$0 \leq y = \frac{x - m\mathbf{1}}{M - m} \leq \mathbf{1},$$

then, we get

$$
\begin{aligned}
(3.21)\qquad & 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
& \times\left(\frac{1}{2}(M-m)\mathbf{1}-\left|x-\frac{1}{2}(m+M)\mathbf{1}\right|\right) \\
& \leq \frac{f(m)(M\mathbf{1}-x)+f(M)(x-m\mathbf{1})}{M-m}-f(x) \\
& \leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
& \times\left(\frac{1}{2}(M-m)\mathbf{1}+\left|x-\frac{1}{2}(m+M)\mathbf{1}\right|\right).
\end{aligned}
$$

From (3.8) we get

$$
\begin{aligned}
(3.22)\qquad & \int_T \phi_t(f(x_t))\,d\mu(t)-f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \\
& \leq \frac{f(m)\left(M\mathbf{1}-\int_T \phi_t(x_t)\,d\mu(t)\right)+f(M)\left(\int_T \phi_t(x_t)\,d\mu(t)-m\mathbf{1}\right)}{M-m} \\
& -f\left(\int_T \phi_t(x_t)\,d\mu(t)\right).
\end{aligned}
$$

Using the inequality (3.21), we also have

$$
\begin{aligned}
(3.23)\qquad & \frac{f(m)\left(M\mathbf{1}-\int_T \phi_t(x_t)\,d\mu(t)\right)+f(M)\left(\int_T \phi_t(x_t)\,d\mu(t)-m\mathbf{1}\right)}{M-m} \\
& -f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \\
& \leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
& \times\left(\frac{1}{2}(M-m)\mathbf{1}+\left|\int_T \phi_t(x_t)\,d\mu(t)-\frac{1}{2}(m+M)\mathbf{1}\right|\right).
\end{aligned}
$$

By utilising (3.22) and (3.23) we obtain the first part of (3.17).

If $u\in[m,M]$, then $\left|u-\frac{m+M}{2}\right|\leq\frac{1}{2}(M-m)$ and by the continuous functional calculus we have $\left|x-\frac{1}{2}(m+M)\mathbf{1}\right|\leq\frac{1}{2}(M-m)\mathbf{1}$ if x is a selfadjoint element with $\operatorname{Sp}(x)\subset[m,M]$.

Since $m\mathbf{1}\leq\phi_t(x_t)\leq M\mathbf{1}$ then $m\mathbf{1}\leq\int_T\phi_t(x_t)\,d\mu(t)\leq M\mathbf{1}$, which proves the last part of (3.17). $\square$

Corollary 2. *With the assumptions of Theorem 2 and if f is operator convex on I, then we have the following reverse of (1.1)*

$$
\begin{aligned}
0 &\leq \int_T \phi_t(f(x_t))\,d\mu(t) - f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \\
&\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&\times \left(\frac{1}{2}(M-m)\mathbf{1} + \left|\int_T \phi_t(x_t)\,d\mu(t) - \frac{1}{2}(m+M)\mathbf{1}\right|\right) \\
&\leq 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]\mathbf{1}.
\end{aligned} \tag{3.24}
$$

We have the norm inequalities

$$
\begin{aligned}
&\left\|\int_T \phi_t(f(x_t))\,d\mu(t) - f\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\right\| \\
&\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&\times \left\|\frac{1}{2}(M-m)\mathbf{1} + \left|\int_T \phi_t(x_t)\,d\mu(t) - \frac{1}{2}(m+M)\mathbf{1}\right|\right\| \\
&\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&\times \left(\frac{1}{2}(M-m)\mathbf{1} + \left\|\int_T \phi_t(x_t)\,d\mu(t) - \frac{1}{2}(m+M)\mathbf{1}\right\|\right) \\
&\leq 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right].
\end{aligned} \tag{3.25}
$$

Remark 2. *Let $f : I \to \mathbb{R}$ be a continuous convex function defined on an interval I and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Assume that the field $(\phi_t)_{t\in[0,1]}$ of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ is continuous and unital, i.e. $\int_0^1 \phi_t(\mathbf{1})\,dt = \mathbf{1}$ and x, y selfadjoint elements in $\mathcal{A}$ with the spectra in $[m, M] \subset I$. Then by taking $x_t := (1-t)x + ty$, $t \in [0,1]$ we get from (3.17) that*

$$
\begin{aligned}
&\int_0^1 \phi_t(f((1-t)x+ty))\,dt - f\left(\int_0^1 (1-t)\phi_t(x)\,dt + \int_0^1 t\phi_t(y)\,dt\right) \\
&\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&\times \left(\frac{1}{2}(M-m)\mathbf{1} + \left|\int_0^1 (1-t)\phi_t(x)\,dt + \int_0^1 t\phi_t(y)\,dt - \frac{1}{2}(m+M)\mathbf{1}\right|\right) \\
&\leq 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right].
\end{aligned} \tag{3.26}
$$

If f is operator convex, then the first term in (3.26) is also nonnegative in the operator order.

For x, y selfadjoint elements in $\mathcal{A}$ with the spectra in I, the positive linear mapping $\phi : \mathcal{A} \to \mathcal{B}$ with $\phi(\mathbf{1}) = \mathbf{1}$ and the convex function $f : I \to \mathbb{R}$, we have

$$\begin{aligned}(3.27)\qquad & \int_0^1 \phi\left(f\left((1-t)x+ty\right)\right)dt - f\left(\frac{\phi(x)+\phi(y)}{2}\right) \\ & \leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ & \times\left(\frac{1}{2}(M-m)\mathbf{1} + \left|\frac{\phi(x)+\phi(y)}{2} - \frac{1}{2}(m+M)\mathbf{1}\right|\right) \\ & \leq 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]\mathbf{1}.\end{aligned}$$

If f is operator convex, then the first term in (3.27) is also nonnegative in the operator order.

For $\mathcal{A} = \mathcal{B}$ and $\phi(x) = x$, then by (3.27) we get

$$\begin{aligned}(3.28)\qquad & \int_0^1 f\left((1-t)x+ty\right)dt - f\left(\frac{x+y}{2}\right) \\ & \leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ & \times\left(\frac{1}{2}(M-m)\mathbf{1} + \left|\frac{x+y}{2} - \frac{1}{2}(m+M)\mathbf{1}\right|\right) \\ & \leq 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]\mathbf{1}.\end{aligned}$$

If f is operator convex, then the first term in (3.28) is also nonnegative in the operator order.

We also have [4]:

Lemma 1. *Assume that $f : [m, M] \to \mathbb{R}$ is absolutely continuous on $[m, M]$. If f' is K-Lipschitzian on $[m, M]$, then*

$$\begin{aligned}(3.29)\qquad & \left|(1-t)f(m) + tf(M) - f\left((1-t)m+tM\right)\right| \\ & \leq \frac{1}{2}K(M-t)(t-m) \leq \frac{1}{8}K(M-m)^2\end{aligned}$$

for all $t \in [0, 1]$.

The constants $1/2$ and $1/8$ are the best possible in (3.19).

Remark 3. *If $f : [m, M] \to \mathbb{R}$ is twice differentiable and $f'' \in L_\infty[m, M]$, then*

$$\begin{aligned}(3.30)\qquad & \left|(1-t)f(m) + tf(M) - f\left((1-t)m+tM\right)\right| \\ & \leq \frac{1}{2}\left\|f''\right\|_{[m,M],\infty}(M-t)(t-m) \leq \frac{1}{8}\left\|f''\right\|_{[m,M],\infty}(M-m)^2,\end{aligned}$$

where $\left\|f''\right\|_{[m,M],\infty} := \operatorname{essup}_{t\in[m,M]}\left|f''(t)\right| < \infty$. The constants $1/2$ and $1/8$ are the best possible in (3.20).

The following result also holds:

Theorem 4. *Let $f : I \to \mathbb{R}$ be a twice differentiable convex function defined on an interval I and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. If $(\phi_t)_{t\in T}$ is a unital field of* *positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then*

$$\begin{aligned}
(3.31)\qquad & \int_T \phi_t(f(x_t))\,d\mu(t) - f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \\
&\leq \frac{1}{2}\|f''\|_{[m,M],\infty}\left(M\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2\,\mathbf{1}
\end{aligned}$$

for every bounded continuous field $(x_t)_{t\in T}$ of selfadjoint elements in $\mathcal{A}$ with spectra contained in a closed subinterval $[m,M]$ of I.

Proof. From (3.30), the convexity of f and the continuous functional calculus, we get

$$\begin{aligned}
(3.32)\qquad 0 &\leq \frac{f(m)(M\mathbf{1}-x) + f(M)(x-m\mathbf{1})}{M-m} - f(x) \\
&\leq \frac{1}{2}\|f''\|_{[m,M],\infty}(M\mathbf{1}-x)(x-m\mathbf{1}) \leq \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2\,\mathbf{1},
\end{aligned}$$

where x is a selfadjoint element with the spectrum $\mathrm{Sp}(x) \subset [m,M]$.

Since

$$\begin{aligned}
(3.33)\qquad & \int_T \phi_t(f(x_t))\,d\mu(t) - f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \\
&\leq \frac{f(m)\left(M\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right) + f(M)\left(\int_T \phi_t(x_t)\,d\mu(t) - m\mathbf{1}\right)}{M-m} \\
&- f\left(\int_T \phi_t(x_t)\,d\mu(t)\right)
\end{aligned}$$

and, by (3.32) for $x = \int_T \phi_t(x_t)\,d\mu(t)$,

$$\begin{aligned}
(3.34)\qquad 0 &\leq \frac{f(m)\left(M\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right) + f(M)\left(\int_T \phi_t(x_t)\,d\mu(t) - m\mathbf{1}\right)}{M-m} \\
&- f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \\
&\leq \frac{1}{2}\|f''\|_{[m,M],\infty}\left(M\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2\,\mathbf{1},
\end{aligned}$$

hence by (3.33) and (3.34) we derive (3.31). □

Corollary 3. *With the assumptions of Theorem 4 and if f is operator convex on I, then we have the following reverse of (1.1)*

$$\begin{aligned}
(3.35)\qquad 0 &\leq \int_T \phi_t(f(x_t))\,d\mu(t) - f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \\
&\leq \frac{1}{2}\|f''\|_{[m,M],\infty}\left(M\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2\,\mathbf{1}.
\end{aligned}$$

We also have the norm inequalities

$$\begin{aligned}
(3.36)\qquad & \left\| \int_T \phi_t\left(f\left(x_t\right)\right) d\mu\left(t\right) - f\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right)\right\| \\
& \le \frac{1}{2}\left\|f''\right\|_{[m,M],\infty}\left\|\left(M\mathbf{1} - \int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right)\left(\int_T \phi_t\left(x_t\right) d\mu\left(t\right) - m\mathbf{1}\right)\right\| \\
& \le \left\|M\mathbf{1} - \int_T \phi_t\left(x_t\right) d\mu\left(t\right)\right\|\left\|\int_T \phi_t\left(x_t\right) d\mu\left(t\right) - m\mathbf{1}\right\| \\
& \le \frac{1}{8}\left\|f''\right\|_{[m,M],\infty}\left(M-m\right)^2.
\end{aligned}$$

Remark 4. *Let $f : I \to \mathbb{R}$ be a twice differentiable convex function defined on an interval I and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Assume that the field $(\phi_t)_{t\in[0,1]}$ of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ is continuous and unital, i.e. $\int_0^1 \phi_t\left(\mathbf{1}\right) dt = \mathbf{1}$ and x, y selfadjoint elements in $\mathcal{A}$ with the spectra in $[m, M] \subset I$. Then by taking $x_t := (1-t)\,x + ty$, $t \in [0,1]$ we get from (3.1) that*

$$\begin{aligned}
(3.37)\qquad & \int_0^1 \phi_t\left(f\left(\left(1-t\right)x + ty\right)\right) dt - f\left(\int_0^1 \left(1-t\right)\phi_t\left(x\right) dt + \int_0^1 t\phi_t\left(y\right) dt\right) \\
& \le \frac{1}{2}\left\|f''\right\|_{[m,M],\infty}\left(M\mathbf{1} - \int_0^1 \left(1-t\right)\phi_t\left(x\right) dt - \int_0^1 t\phi_t\left(y\right) dt\right) \\
& \times \left(\int_0^1 \left(1-t\right)\phi_t\left(x\right) dt + \int_0^1 t\phi_t\left(y\right) dt - m\mathbf{1}\right) \\
& \le \frac{1}{8}\left\|f''\right\|_{[m,M],\infty}\left(M-m\right)^2\mathbf{1}.
\end{aligned}$$

If f is operator convex, then the first term in (3.37) is also nonnegative in the operator order.

For x, y selfadjoint elements in $\mathcal{A}$ with the spectra in $[m, M] \subset I$, the positive linear mapping $\phi : \mathcal{A} \to \mathcal{B}$ with $\phi\left(\mathbf{1}\right) = \mathbf{1}$ and the convex function $f : I \to \mathbb{R}$, we have

$$\begin{aligned}
(3.38)\qquad & \int_0^1 \phi\left(f\left(\left(1-t\right)x + ty\right)\right) dt - f\left(\frac{\phi\left(x\right) + \phi\left(y\right)}{2}\right) \\
& \le \frac{1}{2}\left\|f''\right\|_{[m,M],\infty}\left(M\mathbf{1} - \frac{\phi\left(x\right) + \phi\left(y\right)}{2}\right)\left(\frac{\phi\left(x\right) + \phi\left(y\right)}{2} - m\mathbf{1}\right) \\
& \le \frac{1}{8}\left\|f''\right\|_{[m,M],\infty}\left(M-m\right)^2\mathbf{1}.
\end{aligned}$$

If f is operator convex, then the first term in (3.38) is also nonnegative in the operator order.

For $\mathcal{A} = \mathcal{B}$ and $\phi\left(x\right) = x$, then by (3.38) we get

$$\begin{aligned}
(3.39)\qquad & \int_0^1 f\left(\left(1-t\right)x + ty\right) dt - f\left(\frac{x+y}{2}\right) \\
& \le \frac{1}{2}\left\|f''\right\|_{[m,M],\infty}\left(M\mathbf{1} - \frac{x+y}{2}\right)\left(\frac{x+y}{2} - m\mathbf{1}\right) \\
& \le \frac{1}{8}\left\|f''\right\|_{[m,M],\infty}\left(M-m\right)^2\mathbf{1}.
\end{aligned}$$

If f is operator convex, then the first term in (3.39) is also nonnegative in the operator order.

4. Some Examples

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^{*}-algebras and $(\phi_t)_{t\in T}$ be a unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ.

We consider the exponential function $f(x) = \exp(\alpha x)$ with $\alpha \in \mathbb{R} \setminus \{0\}$. This function is convex but not operator convex on $\mathbb{R}$. Then by (3.1), (3.17) and (3.31) we get

$$\begin{aligned}
&\int_T \phi_t(\exp(\alpha x_t))\,d\mu(t) - \exp\left(\alpha \int_T \phi_t(x_t)\,d\mu(t)\right) \\
&\leq \alpha \frac{\exp(\alpha M) - \exp(\alpha m)}{M-m} \\
&\times \left(M\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{4}\alpha(M-m)\left[\exp(\alpha M) - \exp(\alpha m)\right]\mathbf{1},
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&\int_T \phi_t(\exp(\alpha x_t))\,d\mu(t) - \exp\left(\alpha \int_T \phi_t(x_t)\,d\mu(t)\right) \\
&\leq 2\left[\frac{\exp(m) + \exp(M)}{2} - \exp\left(\frac{m+M}{2}\right)\right] \\
&\times \left(\frac{1}{2}(M-m)\mathbf{1} + \left|\int_T \phi_t(x_t)\,d\mu(t) - \frac{1}{2}(m+M)\mathbf{1}\right|\right) \\
&\leq 2(M-m)\left[\frac{\exp(m) + \exp(M)}{2} - \exp\left(\frac{m+M}{2}\right)\right]
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\int_T \phi_t(\exp(\alpha x_t))\,d\mu(t) - \exp\left(\alpha \int_T \phi_t(x_t)\,d\mu(t)\right) \\
&\leq \frac{1}{2}\alpha^2 \begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \\
&\times \left(M\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{8}\alpha^2(M-m)^2 \begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \times \mathbf{1}
\end{aligned}$$

for every bounded continuous field $(x_t)_{t\in T}$ of selfadjoint elements in $\mathcal{A}$ with spectra contained in a closed subinterval $[m, M]$ of $\mathbb{R}$.

The function $f(x) = -\ln x$, $x > 0$ is operator convex on $(0,\infty)$. Then by (3.10), (3.24) and (3.35) we get

$$
\begin{aligned}
0 &\le \ln\left(\int_T \phi_t(x_t)\,d\mu(t)\right) - \int_T \phi_t(\ln(x_t))\,d\mu(t) \\
&\le \frac{1}{mM}\left(M\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - m\mathbf{1}\right) \\
&\le \frac{1}{4mM}(M-m)^2\,\mathbf{1},
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
0 &\le \ln\left(\int_T \phi_t(x_t)\,d\mu(t)\right) - \int_T \phi_t(\ln(x_t))\,d\mu(t) \\
&\le 2\ln\left(\frac{m+M}{2\sqrt{mM}}\right) \\
&\times\left(\frac{1}{2}(M-m)\,\mathbf{1} + \left|\int_T \phi_t(x_t)\,d\mu(t) - \frac{1}{2}(m+M)\,\mathbf{1}\right|\right) \\
&\le 2(M-m)\ln\left(\frac{m+M}{2\sqrt{mM}}\right)\mathbf{1}
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
0 &\le \ln\left(\int_T \phi_t(x_t)\,d\mu(t)\right) - \int_T \phi_t(\ln(x_t))\,d\mu(t) \\
&\le \frac{1}{2m^2}\left(M\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - m\mathbf{1}\right) \\
&\le \frac{1}{8m^2}(M-m)^2\,\mathbf{1}
\end{aligned} \tag{4.5}
$$

for every bounded continuous field $(x_t)_{t\in T}$ of selfadjoint elements in $\mathcal{A}$ with spectra contained in a closed subinterval $[m, M]$ of $(0,\infty)$.

We observe that if $M > 2m$ then the bound in (4.3) is better than the one from (4.5). If $M < 2m$, then the conclusion is the other way around.

For x, y selfadjoint elements in $\mathcal{A}$ with the spectra in $[m, M] \subset (0,\infty)$, we have the Hermite-Hadamard type inequalities

$$
\begin{aligned}
0 &\le \ln\left(\frac{x+y}{2}\right) - \int_0^1 \ln((1-t)x + ty)\,dt \\
&\le \frac{1}{mM}\left(M\mathbf{1} - \frac{x+y}{2}\right)\left(\frac{x+y}{2} - m\mathbf{1}\right) \le \frac{1}{4mM}(M-m)^2\,\mathbf{1},
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
0 &\le \ln\left(\frac{x+y}{2}\right) - \int_0^1 \ln((1-t)x + ty)\,dt \\
&\le 2\ln\left(\frac{m+M}{2\sqrt{mM}}\right)\left(\frac{1}{2}(M-m)\,\mathbf{1} + \left|\frac{x+y}{2} - \frac{1}{2}(m+M)\,\mathbf{1}\right|\right) \\
&\le 2(M-m)\ln\left(\frac{m+M}{2\sqrt{mM}}\right)\mathbf{1}
\end{aligned} \tag{4.7}
$$

and

$$0 \le \ln\left(\frac{x+y}{2}\right) - \int_0^1 \ln\left((1-t)x + ty\right) dt \tag{4.8}$$
$$\le \frac{1}{2m^2}\left(M\mathbf{1} - \frac{x+y}{2}\right)\left(\frac{x+y}{2} - m\mathbf{1}\right) \le \frac{1}{8m^2}(M-m)^2 \mathbf{1}.$$

The function $f(x) = x\ln x$, $x > 0$ is operator convex on $(0,\infty)$. Then by (3.10), (3.24) and (3.35) we get

$$0 \le \int_T \phi_t\left(x_t \ln(x_t)\right) d\mu(t) - \left(\int_T \phi_t(x_t)\, d\mu(t)\right) \ln\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \tag{4.9}$$
$$\le \frac{\ln(M) - \ln(m)}{M-m}$$
$$\times \left(M\mathbf{1} - \int_T \phi_t(x_t)\, d\mu(t)\right)\left(\int_T \phi_t(x_t)\, d\mu(t) - m\mathbf{1}\right)$$
$$\le \frac{1}{4}(M-m)\left[\ln(M) - \ln(m)\right]\mathbf{1},$$

$$0 \le \int_T \phi_t\left(x_t \ln(x_t)\right) d\mu(t) - \left(\int_T \phi_t(x_t)\, d\mu(t)\right) \ln\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \tag{4.10}$$
$$\le 2\left[\frac{m\ln(m) + M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]$$
$$\times \left(\frac{1}{2}(M-m)\mathbf{1} + \left|\int_T \phi_t(x_t)\, d\mu(t) - \frac{1}{2}(m+M)\mathbf{1}\right|\right)$$
$$\le 2(M-m)\left[\frac{m\ln(m) + M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]\mathbf{1}$$

and

$$0 \le \int_T \phi_t\left(x_t \ln(x_t)\right) d\mu(t) - \left(\int_T \phi_t(x_t)\, d\mu(t)\right) \ln\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \tag{4.11}$$
$$\le \frac{1}{2m}\left(M\mathbf{1} - \int_T \phi_t(x_t)\, d\mu(t)\right)\left(\int_T \phi_t(x_t)\, d\mu(t) - m\mathbf{1}\right)$$
$$\le \frac{1}{8m}(M-m)^2\mathbf{1}$$

for every bounded continuous field $(x_t)_{t\in T}$ of selfadjoint elements in $\mathcal{A}$ with spectra contained in a closed subinterval $[m, M]$ of $(0,\infty)$.

For x, y selfadjoint elements in $\mathcal{A}$ with the spectra in $[m, M] \subset (0,\infty)$, we have the Hermite-Hadamard type inequalities

$$0 \le \int_0^1 \left((1-t)x + ty\right)\ln\left((1-t)x + ty\right) dt - \left(\frac{x+y}{2}\right)\ln\left(\frac{x+y}{2}\right) \tag{4.12}$$
$$\le \frac{\ln(M) - \ln(m)}{M-m}\left(M\mathbf{1} - \frac{x+y}{2}\right)\left(\frac{x+y}{2} - m\mathbf{1}\right)$$
$$\le \frac{1}{4}(M-m)\left[\ln(M) - \ln(m)\right]\mathbf{1},$$

$$
\begin{aligned}
(4.13)\qquad 0 &\leq \int_0^1 ((1-t)x+ty)\ln((1-t)x+ty)\,dt - \left(\frac{x+y}{2}\right)\ln\left(\frac{x+y}{2}\right) \\
&\leq 2\left[\frac{m\ln(m)+M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right] \\
&\times \left(\frac{1}{2}(M-m)\mathbf{1} + \left|\frac{x+y}{2} - \frac{1}{2}(m+M)\mathbf{1}\right|\right) \\
&\leq 2(M-m)\left[\frac{m\ln(m)+M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]\mathbf{1}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.14)\qquad 0 &\leq \int_0^1 ((1-t)x+ty)\ln((1-t)x+ty)\,dt - \left(\frac{x+y}{2}\right)\ln\left(\frac{x+y}{2}\right) \\
&\leq \frac{1}{2m}\left(M\mathbf{1} - \frac{x+y}{2}\right)\left(\frac{x+y}{2} - m\mathbf{1}\right) \leq \frac{1}{8m}(M-m)^2\mathbf{1}.
\end{aligned}
$$

Consider the power function $f(x) = x^r$, $x \in (0,\infty)$ and r a real number. If $r \in (-\infty, 0] \cup [1,\infty)$, then f is convex and for $r \in [-1,0] \cup [1,2]$ is operator convex. If we use the inequalities (3.1), (3.17) and (3.31) we have for $r \in (-\infty, 0] \cup [1,\infty)$ that

$$
\begin{aligned}
(4.15)\qquad &\int_T \phi_t(x_t^r)\,d\mu(t) - \left(\int_T \phi_t(x_t)\,d\mu(t)\right)^r \\
&\leq r\frac{M^{r-1}-m^{r-1}}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{4}r(M-m)\left(M^{r-1}-m^{r-1}\right)\mathbf{1},
\end{aligned}
$$

$$
\begin{aligned}
(4.16)\qquad &\int_T \phi_t(x_t^r)\,d\mu(t) - \left(\int_T \phi_t(x_t)\,d\mu(t)\right)^r \\
&\leq 2\left[\frac{m^r+M^r}{2} - \left(\frac{m+M}{2}\right)^r\right] \\
&\times\left(\frac{1}{2}(M-m)\mathbf{1} + \left|\int_T \phi_t(x_t)\,d\mu(t) - \frac{1}{2}(m+M)\mathbf{1}\right|\right) \\
&\leq 2(M-m)\left[\frac{m^r+M^r}{2} - \left(\frac{m+M}{2}\right)^r\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(4.17)\qquad &\int_T \phi_t(x_t^r)\,d\mu(t) - \left(\int_T \phi_t(x_t)\,d\mu(t)\right)^r \\
&\leq \frac{1}{2}r(r-1)\begin{cases} M^{r-2} \text{ for } r \geq 2 \\ \\ m^{r-2} \text{ for } r \in (-\infty,0] \cup [1,2) \end{cases} \\
&\times\left(M\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{8}r(r-1)(M-m)^2\begin{cases} M^{r-2} \text{ for } r \geq 2 \\ \\ m^{r-2} \text{ for } r \in (-\infty,0] \cup [1,2) \end{cases} \times\mathbf{1}
\end{aligned}
$$

for every bounded continuous field $(x_t)_{t\in T}$ of selfadjoint elements in $\mathcal{A}$ with spectra contained in a closed subinterval $[m,M]$ of $(0,\infty)$.

If $r\in[-1,0]\cup[1,2]$, then we also have

$$0\leq\int_T\phi_t\left(x_t^r\right)d\mu\left(t\right)-\left(\int_T\phi_t\left(x_t\right)d\mu\left(t\right)\right)^r$$

in the inequalities (4.15)-(4.17).

For x, y selfadjoint elements in $\mathcal{A}$ with the spectra in $[m,M]\subset(0,\infty)$, and if $r\in[-1,0]\cup[1,2]$, then we have the Hermite-Hadamard type inequalities

$$\begin{aligned}0&\leq\int_0^1\left((1-t)\,x+ty\right)^r dt-\left(\frac{x+y}{2}\right)^r\\&\leq r\frac{M^{r-1}-m^{r-1}}{M-m}\left(M\mathbf{1}-\frac{x+y}{2}\right)\left(\frac{x+y}{2}-m\mathbf{1}\right)\\&\leq\frac{1}{4}r\left(M-m\right)\left(M^{r-1}-m^{r-1}\right)\mathbf{1},\end{aligned}\tag{4.18}$$

$$\begin{aligned}0&\leq\int_0^1\left((1-t)\,x+ty\right)^r dt-\left(\frac{x+y}{2}\right)^r\\&\leq 2\left[\frac{m^r+M^r}{2}-\left(\frac{m+M}{2}\right)^r\right]\\&\times\left(\frac{1}{2}\left(M-m\right)\mathbf{1}+\left|\frac{x+y}{2}-\frac{1}{2}\left(m+M\right)\mathbf{1}\right|\right)\\&\leq 2\left(M-m\right)\left[\frac{m^r+M^r}{2}-\left(\frac{m+M}{2}\right)^r\right]\end{aligned}\tag{4.19}$$

and

$$\begin{aligned}0&\leq\int_0^1\left((1-t)\,x+ty\right)^r dt-\left(\frac{x+y}{2}\right)^r\\&\leq\frac{1}{2}r\left(r-1\right)\begin{cases}M^{r-2}\text{ for }r\geq2\\ \\ m^{r-2}\text{ for }r\in(-\infty,0]\cup[1,2)\end{cases}\\&\times\left(M\mathbf{1}-\frac{x+y}{2}\right)\left(\frac{x+y}{2}-m\mathbf{1}\right)\\&\leq\frac{1}{8}r\left(r-1\right)\left(M-m\right)^2\begin{cases}M^{r-2}\text{ for }r\geq2\\ \\ m^{r-2}\text{ for }r\in(-\infty,0]\cup[1,2)\end{cases}\times\mathbf{1}.\end{aligned}\tag{4.20}$$

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