REVERSE JENSEN INTEGRAL INEQUALITIES FOR OPERATOR CONVEX FUNCTIONS IN TERMS OF FRÉCHET DERIVATIVE

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Abstract. Let $f : I \to \mathbb{R}$ be an operator convex function of class $C^1(I)$. If $(A_t)_{t\in T}$ is a bounded continuous field of selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I defined on a locally compact Hausdorff space T with a bounded Radon measure μ and such that $\int_T \mathbf{1} d\mu(t) = \mathbf{1}$, then we have obtained among others the following reverse of Jensen's inequality

$$\begin{aligned}0 &\le \int_T f(A_t)\, d\mu(t) - f\left(\int_T A_s d\mu(s)\right)\\ &\le \int_T Df(A_t)(A_t)\, d\mu(t) - \int_T Df(A_t)\left(\int_T A_s d\mu(s)\right) d\mu(t)\end{aligned}$$

in terms of the *Fréchet derivative* $Df(\cdot)(\cdot)$. Some applications for the Hermite-Hadamard inequalities are also given.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \le (\ge)(1-\lambda) f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \le B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \le f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [9] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \le r \le 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \le r \le 2$ or $-1 \le r \le 0$ and is operator concave on $(0,\infty)$ if $0 \le r \le 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

For two distinct operators $A, B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$

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We observe that $A,\ B\in[A,B]$ and $[A,B]\subset\mathcal{SA}_I(H)$.

A continuous function $g:\mathcal{SA}_I(H)\to\mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A\in\mathcal{SA}_I(H)$ along the direction $B\in\mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B):=\lim_{s\to 0}\frac{g(A+sB)-g(A)}{s}\in\mathcal{B}(H). \tag{1.2}$$

If the limit (1.2) exists for all $B\in\mathcal{B}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $g\in\mathcal{G}(A)$. If this is true for any A in a subset $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g\in\mathcal{G}(\mathcal{S})$.

Let f be an operator convex function on I. For $A,\ B\in\mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\to\mathcal{SA}_I(H)$ defined by

$$\varphi_{(A,B)}(t):=f((1-t)A+tB). \tag{1.3}$$

For $x\in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle\varphi_{(A,B)}(t)x,x\right\rangle=\langle f((1-t)A+tB)x,x\rangle. \tag{1.4}$$

By employing the properties of convex functions of a real variable, we have the following basic facts, see for instance [8]:

Lemma 1. *Let f be an operator convex function on I. For any $A,\ B\in\mathcal{SA}_I(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $(A,B)\in\mathcal{SA}_I(H)$ and $x\in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$.*

Lemma 2. *Let f be an operator convex function on I and $A,\ B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t)=\nabla f_{(1-t)A+tB}(B-A). \tag{1.5}$$

Also we have for the lateral derivative that

$$\varphi'_{+(A,B)}(0)=\nabla f_A(B-A) \tag{1.6}$$

and

$$\varphi'_{-(A,B)}(1)=\nabla f_B(B-A). \tag{1.7}$$

and

Lemma 3. *Let f be an operator convex function on I and $A,\ B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then for $0<t_1<t_2<1$ we have*

$$\nabla g_{(1-t_1)A+t_1B}(B-A)\le\nabla g_{(1-t_2)A+t_2B}(B-A) \tag{1.8}$$

in the operator order.

We also have

$$\nabla f_A(B-A)\le\nabla g_{(1-t_1)A+t_1B}(B-A) \tag{1.9}$$

and

$$\nabla g_{(1-t_2)A+t_2B}(B-A)\le\nabla f_B(B-A). \tag{1.10}$$

Corollary 1. *Let f be an operator convex function on I and $A,\ B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then for all $t\in(0,1)$ we have*

$$\nabla f_A(B-A)\le\nabla f_{(1-t)A+tB}(B-A)\le\nabla f_B(B-A). \tag{1.11}$$

By making use of the gradient inequality for the convex function of a real variable $\varphi_{(A,B);x}$ with $x\in H$,

$$\varphi'_{+(A,B);x}(0)\le\varphi_{(A,B);x}(1)-\varphi_{(A,B);x}(0)\le\varphi'_{-(A,B)}(1),$$

namely

$$\langle\nabla f_A(B-A)x,x\rangle\le\langle f(B)x,x\rangle-\langle f(A)x,x\rangle\le\langle\nabla f_B(B-A)x,x\rangle$$

for any $x\in H$. This is equivalent in the operatorial order with the operator gradient inequality

$$\nabla f_A(B-A)\le f(B)-f(A)\le\nabla f_B(B-A).$$

It is well known that, if f is a C^1-function defined on an open interval, then the operator function $f(X)$ is *Fréchet differentiable* and the derivative $Df(A)(B)$ equals the Gâteaux derivative $\nabla f_A(B)$. So for operator convex functions f that are of class C^1 on I we have the *Fréchet gradient operator inequality*

$$Df(A)(B-A)\le f(B)-f(A)\le Df(B)(B-A) \tag{Gr}$$

for any $A,\ B\in\mathcal{SA}_I(H)$.

For a C^1-function f defined on I we also have by Lemma 2 that

$$\varphi'_{(A,B)}(t)=Df((1-t)A+tB)(B-A),\ t\in(0,1) \tag{1.12}$$

and

$$\varphi'_{+(A,B)}(0)=Df(A)(B-A),\ \varphi'_{-(A,B)}(1)=Df(B)(B-A). \tag{1.13}$$

Moreover, we have

$$Df(A)(B-A)\le Df((1-t)A+tB)(B-A)\le Df(B)(B-A) \tag{1.14}$$

for all $t\in(0,1)$.

Let T be a locally compact Hausdorff space. We say that a field $(A_t)_{t\in T}$ of operators in $\mathcal{B}(H)$ is continuous if the function $t\mapsto A_t$ is norm continuous on T. If in addition μ is a Radon measure on T and the function $t\mapsto\|A_t\|$ is integrable, then we can form the Bochner integral $\int_T A_t d\mu(t)$, which is the unique element in $\mathcal{B}(H)$ such that

$$\varphi\left(\int_T A_t d\mu(t)\right)=\int_T\varphi(A_t)\,d\mu(t)$$

for every linear functional φ in the norm dual $\mathcal{B}(H)^*$, cf. [14, Section 4.1].

Assume furthermore that there is a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t:\mathcal{B}(H)\to\mathcal{B}(K)$ from $\mathcal{B}(H)$ to another C^*-algebra $\mathcal{B}(K)$, with K a Hilbert space. We say that such a field is continuous if the function $t\mapsto\phi_t(A)$ is continuous for every $A\in\mathcal{B}(H)$. If the field $t\mapsto\phi_t(\mathbf{1})$ is integrable with integral $\int_T\phi_t(\mathbf{1})\,d\mu(t)=\mathbf{1}$, we say that $(\phi_t)_{t\in T}$ is unital.

The following Jensen's integral inequality has been obtained in [13]:

Theorem 1. *Let $f:I\to\mathbb{R}$ be an operator convex function defined on an interval I. If $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t:\mathcal{B}(H)\to\mathcal{B}(K)$ defined* *on a locally compact Hausdorff space T with a bounded Radon measure μ, then the inequality*

$$f\left(\int_T \phi_t(A_t)\,d\mu(t)\right) \le \int_T \phi_t(f(A_t))\,d\mu(t) \tag{1.15}$$

holds for every bounded continuous field $(A_t)_{t\in T}$ of selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I.

The discrete case is as follows [15]:

$$f\left(\sum_{i=1}^{n} w_i\phi_i(A_i)\right) \le \sum_{i=1}^{n} w_i\phi_i(f(A_i))$$

for operator convex functions f defined on an interval I, where $\phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$, $i \in \{1,...,n\}$ are unital positive linear maps, A_i, $i \in \{1,...,n\}$ are selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I and $w_i \ge 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} w_i = 1$.

Also, if $f : I \to \mathbb{R}$ is operator convex on I and $U_i \in \mathcal{B}(H)$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} U^*U_i = 1$, then [14]

$$f\left(\sum_{i=1}^{n} U^*A_iU_i\right) \le \sum_{i=1}^{n} U^*f(A_i)\,U_i,$$

where A_i, $i \in \{1,...,n\}$ are selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I.

In this paper we establish some reverses of Jensen's integral inequality for operator convex functions of class $C^1(I)$, continuous fields $(A_t)_{t\in T}$ of selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I defined on a locally compact Hausdorff space T with a bounded Radon measure μ and such that $\int_T \mathbf{1}d\mu(t) = \mathbf{1}$. These reverses are given in terms of the Fréchet derivative $Df(\cdot)(\cdot)$. Some applications for the Hermite-Hadamard inequalities are also provided.

2. Main Results

We have the following inequalities in terms of the Fréchet derivative $Df(\cdot)(\cdot)$:

Theorem 2. *Let $f : I \to \mathbb{R}$ be an operator convex function of class $C^1(I)$. If $(A_t)_{t\in T}$ is a bounded continuous field $(A_t)_{t\in T}$ of selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I defined on a locally compact Hausdorff space T with a bounded Radon measure μ and such that $\int_T \mathbf{1}d\mu(t) = \mathbf{1}$, then we have the double inequality in terms of the Fréchet derivative $Df(\cdot)(\cdot)$*

$$\begin{aligned} &f(A) - Df(A)(A) + Df(A)\left(\int_T A_t d\mu(t)\right) \\ &\le \int_T f(A_t)\,d\mu(t) \\ &\le f(A) - \int_T Df(A_t)(A)\,d\mu(t) + \int_T Df(A_t)(A_t)\,d\mu(t) \end{aligned} \tag{2.1}$$

for all $A \in \mathcal{SA}_I(H)$.

We have the reverse Jensen's inequality

$$0 \leq \int_T f(A_t)\, d\mu(t) - f\left(\int_T A_s d\mu(s)\right) \leq \int_T Df(A_t)(A_t)\, d\mu(t) - \int_T Df(A_t)\left(\int_T A_s d\mu(s)\right) d\mu(t)\,. \tag{2.2}$$

If $S \in \mathcal{SA}_I(H)$ is an operator satisfying the equality

$$\int_T Df(A_t)(S)\, d\mu(t) = \int_T Df(A_t)(A_t)\, d\mu(t)\,, \tag{Sl}$$

then we have the Slater type inequality

$$0 \leq f(S) - \int_T f(A_t)\, d\mu(t) \leq Df(S)(S) - Df(S)\left(\int_T A_t d\mu(t)\right). \tag{2.3}$$

Proof. From (Gr) we have

$$Df(A)(A_t - A) \leq f(A_t) - f(A) \leq Df(A_t)(A_t - A) \tag{2.4}$$

for all $t \in T$.

By the linearity of the Fréchet derivative we have

$$f(A) - Df(A)(A) + Df(A)(A_t) \leq f(A_t) \leq f(A) - Df(A_t)(A) + Df(A_t)(A_t) \tag{2.5}$$

for all $t \in T$.

By taking the integral over $t \in T$, we have

$$f(A) - Df(A)(A) + \int_T Df(A)(A_t)\, d\mu(t) \leq \int_T f(A_t)\, d\mu(t) \leq f(A) - \int_T Df(A_t)(A)\, d\mu(t) + \int_T Df(A_t)(A_t)\, d\mu(t)\,. \tag{2.6}$$

Since $\mathrm{Sp}(A_t) \subset I$, $t \in T$, then there exists $m < M$ such that $\mathrm{Sp}(A_t) \subseteq [m, M] \subset I$, $t \in T$, namely $\mathbf{1}m \leq A_t \leq \mathbf{1}M$ which implies that $\mathbf{1}m \leq \int_T A_t d\mu(t) \leq \mathbf{1}M$. Namely, $\int_T A_t d\mu(t) \in \mathcal{SA}_I(H)$. By the linearity and continuity of the Fréchet derivative we then have

$$\int_T Df(A)(A_t)\, d\mu(t) = Df(A)\left(\int_T A_t d\mu(t)\right)$$

and by (2.6) we get (2.1).

By taking $A = \int_T A_t d\mu(t)$ in (2.1) we get (2.2). If we take $A = S$ in (2.1), then we also get (2.3). □

We assume that $\mathfrak{D}$ is a bounded linear operator that acts on $\mathcal{SA}_I(H)$ with values in $\mathcal{SA}_I(H)$. We denote this as $\mathfrak{D} \in \mathcal{B}(\mathcal{SA}_I(H))$.

Corollary 2. *Let $f : I \to \mathbb{R}$ be an operator convex function of class $C^1(I)$. If $(A_t)_{t\in T}$ is a bounded continuous field $(A_t)_{t\in T}$ of selfadjoint operators in $\mathcal{B}(H)$*

with spectra contained in I defined on a locally compact Hausdorff space T with a bounded Radon measure μ and such that $\int_T \mathbf{1} d\mu(t) = \mathbf{1}$, then

$$\left\| \int_T f(A_t)\, d\mu(t) - f\left(\int_T A_s d\mu(s) \right) \right\| \tag{2.7}$$
$$\leq \begin{cases} \inf_{\mathfrak{D} \in \mathcal{B}(\mathcal{SA}_I(H))} \left(\sup_{t\in T} \| Df(A_t) - \mathfrak{D} \| \right) \\ \times \int_T \left\| A_t - \int_T A_s d\mu(s) \right\| d\mu(t) \\ \\ \inf_{\mathfrak{D} \in \mathcal{B}(\mathcal{SA}_I(H))} \left(\int_T \| Df(A_t) - \mathfrak{D} \|^p d\mu(t) \right)^{1/p} \\ \times \left(\int_T \left\| A_t - \int_T A_s d\mu(s) \right\|^q d\mu(t) \right)^{1/q};\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \inf_{\mathfrak{D} \in \mathcal{B}(\mathcal{SA}_I(H))} \left(\int_T \| Df(A_t) - \mathfrak{D} \| d\mu(t) \right) \\ \times \sup_{t\in T} \left\| A_t - \int_T A_s d\mu(s) \right\|. \end{cases}$$

In particular,

$$\left\| \int_T f(A_t)\, d\mu(t) - f\left(\int_T A_s d\mu(s) \right) \right\| \tag{2.8}$$
$$\leq \begin{cases} \sup_{t\in T} \| Df(A_t) \| \int_T \left\| A_t - \int_T A_s d\mu(s) \right\| d\mu(t) \\ \\ \left(\int_T \| Df(A_t) \|^p d\mu(t) \right)^{1/p} \left(\int_T \left\| A_t - \int_T A_s d\mu(s) \right\|^q d\mu(t) \right)^{1/q}; \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \int_T \| Df(A_t) \| d\mu(t) \sup_{t\in T} \left\| A_t - \int_T A_s d\mu(s) \right\|. \end{cases}$$

Proof. We have for any operator $\mathfrak{D} \in \mathcal{B}(\mathcal{SA}_I(H))$ and the properties of Fréchet derivative and integral, that

$$\begin{aligned} &\int_T (Df(A_t) - \mathfrak{D}) \left(A_t - \int_T A_s d\mu(s) \right) d\mu(t) \\ &= \int_T Df(A_t) \left(A_t - \int_T A_s d\mu(s) \right) d\mu(t) - \int_T \mathfrak{D} \left(A_t - \int_T A_s d\mu(s) \right) d\mu(t) \\ &= \int_T Df(A_t)(A_t)\, d\mu(t) - \int_T Df(A_t) \left(\int_T A_s d\mu(s) \right) d\mu(t) \\ &- \mathfrak{D} \int_T \left(\int_T A_t d\mu(t) - \int_T \left(\int_T A_s d\mu(s) \right) d\mu(t) \right) \\ &= \int_T Df(A_t)(A_t)\, d\mu(t) - \int_T Df(A_t) \left(\int_T A_s d\mu(s) \right) d\mu(t) \\ &- \mathfrak{D} \int_T \left(\int_T A_t d\mu(t) - \left(\int_T A_s d\mu(s) \right) \right) \\ &= \int_T Df(A_t)(A_t)\, d\mu(t) - \int_T Df(A_t) \left(\int_T A_s d\mu(s) \right) d\mu(t). \end{aligned}$$

From (2.2) we have

$$\begin{aligned} 0 &\leq \int_T f(A_t)\, d\mu(t) - f\left(\int_T A_s d\mu(s) \right) \\ &\leq \int_T (Df(A_t) - \mathfrak{D}) \left(A_t - \int_T A_s d\mu(s) \right) d\mu(t) \end{aligned} \tag{2.9}$$

for any operator $\mathfrak{D} \in \mathcal{B}\left(\mathcal{SA}_I\left(H\right)\right)$.

By taking the norm in (2.9) we get

$$\begin{aligned}
&\left\| \int_T f\left(A_t\right) d\mu\left(t\right) - f\left(\int_T A_s d\mu\left(s\right)\right) \right\| \\
&\leq \int_T \left\| \left(Df(A_t) - \mathfrak{D}\right)\left(A_t - \int_T A_s d\mu\left(s\right)\right) \right\| d\mu\left(t\right) \\
&\leq \int_T \left\| Df(A_t) - \mathfrak{D} \right\| \left\| A_t - \int_T A_s d\mu\left(s\right) \right\| d\mu\left(t\right) \\
&\leq \begin{cases} \sup_{t\in T} \left\| Df(A_t) - \mathfrak{D} \right\| \int_T \left\| A_t - \int_T A_s d\mu\left(s\right) \right\| d\mu\left(t\right) \\ \\ \left(\int_T \left\| Df(A_t) - \mathfrak{D} \right\|^p\right)^{1/p} \left(\int_T \left\| A_t - \int_T A_s d\mu\left(s\right) \right\|^q d\mu\left(t\right)\right)^{1/q} \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \int_T \left\| Df(A_t) - \mathfrak{D} \right\| d\mu\left(t\right) \sup_{t\in T} \left\| A_t - \int_T A_s d\mu\left(s\right) \right\| \end{cases}
\end{aligned} \tag{2.10}$$

for any operator $\mathfrak{D} \in \mathcal{B}\left(\mathcal{SA}_I\left(H\right)\right)$.

By taking the infimum over $\mathfrak{D} \in \mathcal{B}\left(\mathcal{SA}_I\left(H\right)\right)$ in (2.10) we obtain the desired result. □

Corollary 3. *With the assumptions of Corollary 2 and if there exists* $\mathfrak{D}_1, \mathfrak{D}_2 \in \mathcal{B}\left(\mathcal{SA}_I\left(H\right)\right)$ *such that*

$$\left\| Df(A_t) - \frac{\mathfrak{D}_1 + \mathfrak{D}_2}{2} \right\| \leq \frac{1}{2} \left\| \mathfrak{D}_2 - \mathfrak{D}_1 \right\| \tag{2.11}$$

then

$$\begin{aligned}
&\left\| \int_T f\left(A_t\right) d\mu\left(t\right) - f\left(\int_T A_s d\mu\left(s\right)\right) \right\| \\
&\leq \frac{1}{2} \left\| \mathfrak{D}_2 - \mathfrak{D}_1 \right\| \int_T \left\| A_t - \int_T A_s d\mu\left(s\right) \right\| d\mu\left(t\right).
\end{aligned} \tag{2.12}$$

The proof follows by the first inequality in (2.7) and the condition (2.11).

Corollary 4. *With the assumptions of Corollary 2 and if* $S \in \mathcal{SA}_I\left(H\right)$ *is an operator satisfying the equality (Sl), then*

$$\begin{aligned}
\left\| f\left(S\right) - \int_T f\left(A_t\right) d\mu\left(t\right) \right\| &\leq \left\| Df(S) \right\| \left\| S - \int_T A_t d\mu\left(t\right) \right\| \\
&\leq \left\| Df(S) \right\| \int_T \left\| S - A_t \right\| d\mu\left(t\right).
\end{aligned} \tag{2.13}$$

Proof. By taking the norm in (2.3) we get

$$\begin{aligned}
\left\| f\left(S\right) - \int_T f\left(A_t\right) d\mu\left(t\right) \right\| &\leq \left\| Df(S)\left(S - \int_T A_t d\mu\left(t\right)\right) \right\| \\
&\leq \left\| S - \int_T A_t d\mu\left(t\right) \right\| \\
&= \left\| \int_T \left(S - A_t\right) d\mu\left(t\right) \right\| \leq \int_T \left\| S - A_t \right\| d\mu\left(t\right)
\end{aligned}$$

and the inequality (2.13) is obtained. □

We assume that $f : I \to \mathbb{R}$ is an operator convex function of class $C^1(I)$ and $Df(\cdot)$ is Lipschitzian with constant $L > 0$ on $\mathcal{SA}_I(H)$, namely

$$\|Df(A) - Df(B)\| \leq L \|A - B\| \tag{2.14}$$

for any A, $B \in \mathcal{SA}_I(H)$.

Corollary 5. *With the assumptions of Corollary 1 and if $Df(\cdot)$ is Lipschitzian with constant $L > 0$ on $\mathcal{SA}_I(H)$,*

$$\begin{aligned}
&\left\| \int_T f(A_t)\, d\mu(t) - f\left(\int_T A_s d\mu(s)\right) \right\| \\
&\leq L \begin{cases} \inf_{B\in\mathcal{SA}_I(H)} \left(\sup_{t\in T} \|A_t - B\|\right) \int_T \left\|A_t - \int_T A_s d\mu(s)\right\| d\mu(t) \\ \\ \inf_{B\in\mathcal{SA}_I(H)} \left(\int_T \|A_t - B\|^p d\mu(t)\right)^{1/p} \\ \times \left(\int_T \left\|A_t - \int_T A_s d\mu(s)\right\|^q d\mu(t)\right)^{1/q},\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \inf_{B\in\mathcal{SA}_I(H)} \int_T \|A_t - B\| d\mu(t) \sup_{t\in T} \left\|A_t - \int_T A_s d\mu(s)\right\|. \end{cases}
\end{aligned} \tag{2.15}$$

In particular,

$$\begin{aligned}
&\left\| \int_T f(A_t)\, d\mu(t) - f\left(\int_T A_s d\mu(s)\right) \right\| \\
&\leq L \begin{cases} \sup_{t\in T} \left\|A_t - \int_T A_s d\mu(s)\right\| \int_T \left\|A_t - \int_T A_s d\mu(s)\right\| d\mu(t) \\ \\ \left(\int_T \left\|A_t - \int_T A_s d\mu(s)\right\|^p d\mu(t)\right)^{1/p} \\ \times \left(\int_T \left\|A_t - \int_T A_s d\mu(s)\right\|^q d\mu(t)\right)^{1/q},\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1. \end{cases}
\end{aligned} \tag{2.16}$$

For $p = q = 2$ we also get

$$\left\| \int_T f(A_t)\, d\mu(t) - f\left(\int_T A_s d\mu(s)\right) \right\| \leq L \int_T \left\| A_t - \int_T A_s d\mu(s) \right\|^2 d\mu(t). \tag{2.17}$$

Proof. Let $B \in \mathcal{SA}_I(H)$. From (2.9) we have

$$\begin{aligned}
0 &\leq \int_T f(A_t)\, d\mu(t) - f\left(\int_T A_s d\mu(s)\right) \\
&\leq \int_T (Df(A_t) - Df(B)) \left(A_t - \int_T A_s d\mu(s)\right) d\mu(t).
\end{aligned} \tag{2.18}$$

By taking the norm in (2.18) we get

$$\begin{aligned}
&\left\|\int_T f(A_t)\,d\mu(t) - f\left(\int_T A_s d\mu(s)\right)\right\| \\
&\le \int_T \left\|(Df(A_t)-Df(B))\left(A_t - \int_T A_s d\mu(s)\right)\right\| d\mu(t) \\
&\le \int_T \|Df(A_t)-Df(B)\| \left\|A_t - \int_T A_s d\mu(s)\right\| d\mu(t) \\
&\le L\int_T \|A_t - B\| \left\|A_t - \int_T A_s d\mu(s)\right\| d\mu(t) \\
&\le L\begin{cases} \sup_{t\in T}\|A_t - B\| \int_T \left\|A_t - \int_T A_s d\mu(s)\right\| d\mu(t) \\ \\ \left(\int_T \|A_t - B\|^p d\mu(t)\right)^{1/p} \left(\int_T \left\|A_t - \int_T A_s d\mu(s)\right\|^q d\mu(t)\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_T \|A_t - B\|\,d\mu(t) \sup_{t\in T}\left\|A_t - \int_T A_s d\mu(s)\right\| \end{cases}
\end{aligned} \tag{2.19}$$

for any $B\in\mathcal{SA}_I(H)$.

By taking the infimum over $B\in\mathcal{SA}_I(H)$ in (2.19) we obtain the desired result. □

Remark 1. *For the sake of completeness, we give here the discrete case as well. Let $f: I\to\mathbb{R}$ be an operator convex function of class $C^1(I)$. If $(A_i)_{i\in\{1,...,n\}}$ is a sequence of selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I and $p_i\ge 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$. Then for all $A\in\mathcal{SA}_I(H)$ we have*

$$\begin{aligned}
&f(A) - Df(A)(A) + Df(A)\left(\sum_{i=1}^n p_i A_i\right) \\
&\le \sum_{i=1}^n p_i f(A_i) \\
&\le f(A) - \sum_{i=1}^n p_i Df(A_i)(A) + \sum_{i=1}^n p_i Df(A_i)(A_i).
\end{aligned} \tag{2.20}$$

We have

$$\begin{aligned}
0 &\le \sum_{i=1}^n p_i f(A_i) - f\left(\sum_{i=1}^n p_i A_i\right) \\
&\le \sum_{i=1}^n p_i Df(A_i)(A_i) - \sum_{i=1}^n p_i Df(A_i)\left(\sum_{j=1}^n p_j A_j\right).
\end{aligned} \tag{2.21}$$

If $S\in\mathcal{SA}_I(H)$ is an operator satisfying the equality

$$\sum_{i=1}^n p_i Df(A_i)(S)\,d\mu(t) = \sum_{i=1}^n p_i Df(A_i)(A_i), \tag{Sld}$$

then we have the Slater type discrete inequality

$$0 \le f(S) - \sum_{i=1}^{n} p_i f(A_i) \le Df(S)(S) - Df(S)\left(\sum_{i=1}^{n} p_i A_i\right). \tag{2.22}$$

We have the norm inequalities

$$\left\| \sum_{i=1}^{n} p_i f(A_i) - f\left(\sum_{i=1}^{n} p_i A_i\right) \right\| \tag{2.23}$$
$$\le \begin{cases} \inf_{\mathfrak{D}\in\mathcal{B}(\mathcal{SA}_I(H))} \left(\max_{i\in\{1,\dots,n\}} \|Df(A_i) - \mathfrak{D}\|\right) \\ \times \sum_{i=1}^{n} p_i \left\| A_i - \sum_{j=1}^{n} p_j A_j \right\| \\ \\ \inf_{\mathfrak{D}\in\mathcal{B}(\mathcal{SA}_I(H))} \left(\sum_{i=1}^{n} p_i \|Df(A_i) - \mathfrak{D}\|^p\right)^{1/p} \\ \times \left(\sum_{i=1}^{n} p_i \left\| A_i - \sum_{j=1}^{n} p_j A_j \right\|^q\right)^{1/q}, \ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \inf_{\mathfrak{D}\in\mathcal{B}(\mathcal{SA}_I(H))} \left(\sum_{i=1}^{n} p_i \|Df(A_i) - \mathfrak{D}\|\right) \\ \times \max_{i\in\{1,\dots,n\}} \left\| A_i - \sum_{j=1}^{n} p_j A_j \right\|. \end{cases}$$

In particular,

$$\left\| \sum_{i=1}^{n} p_i f(A_i) - f\left(\sum_{i=1}^{n} p_i A_i\right) \right\| \tag{2.24}$$
$$\le \begin{cases} \max_{i\in\{1,\dots,n\}} \|Df(A_i)\| \sum_{i=1}^{n} p_i \left\| A_i - \sum_{j=1}^{n} p_j A_j \right\| \\ \\ \left(\sum_{i=1}^{n} p_i \|Df(A_i)\|^p\right)^{1/p} \\ \times \left(\sum_{i=1}^{n} p_i \left\| A_i - \sum_{j=1}^{n} p_j A_j \right\|^q\right)^{1/q}, \ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \sum_{i=1}^{n} p_i \|Df(A_i)\| \max_{i\in\{1,\dots,n\}} \left\| A_i - \sum_{j=1}^{n} p_j A_j \right\|. \end{cases}$$

If there exists $\mathfrak{D}_1, \mathfrak{D}_2 \in \mathcal{B}(\mathcal{SA}_I(H))$ such that

$$\left\| Df(A_i) - \frac{\mathfrak{D}_1 + \mathfrak{D}_2}{2} \right\| \le \frac{1}{2} \|\mathfrak{D}_2 - \mathfrak{D}_1\|, \tag{2.25}$$

then

$$\left\| \sum_{i=1}^{n} p_i f(A_i) - f\left(\sum_{i=1}^{n} p_i A_i\right) \right\| \le \frac{1}{2} \|\mathfrak{D}_2 - \mathfrak{D}_1\| \sum_{i=1}^{n} p_i \left\| A_i - \sum_{j=1}^{n} p_j A_j \right\|. \tag{2.26}$$

If $S \in \mathcal{SA}_I(H)$ is an operator satisfying the equality (Sld), then

$$\left\| f(S) - \sum_{i=1}^{n} p_i f(A_i) \right\| \le \|Df(S)\| \left\| S - \sum_{i=1}^{n} p_i A_i \right\| \tag{2.27}$$
$$\le \|Df(S)\| \sum_{i=1}^{n} p_i \|S - A_i\|.$$

Moreover, if $Df(\cdot)$ is Lipschitzian with constant $L>0$ on $\mathcal{SA}_I(H)$, then

$$\left\|\sum_{i=1}^{n} p_i f(A_i) - f\left(\sum_{i=1}^{n} p_i A_i\right)\right\| \tag{2.28}$$
$$\leq L \begin{cases} \inf_{B\in\mathcal{SA}_I(H)}\left(\sup_{i\in\{1,\dots,n\}}\|A_i-B\|\right)\sum_{i=1}^{n} p_i\left\|A_i-\sum_{j=1}^{n} p_j A_j\right\| \\ \\ \inf_{B\in\mathcal{SA}_I(H)}\left(\sum_{i=1}^{n} p_i\|A_i-B\|^p\right)^{1/p} \\ \times\left(\sum_{i=1}^{n} p_i\left\|A_i-\sum_{j=1}^{n} p_j A_j\right\|^q\right)^{1/q},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \\ \inf_{B\in\mathcal{SA}_I(H)}\sum_{i=1}^{n} p_i\|A_i-B\|\sup_{i\in\{1,\dots,n\}}\left\|A_i-\sum_{j=1}^{n} p_j A_j\right\|. \end{cases}$$

In particular,

$$\left\|\sum_{i=1}^{n} p_i f(A_i) - f\left(\sum_{i=1}^{n} p_i A_i\right)\right\| \tag{2.29}$$
$$\leq L \begin{cases} \sup_{i\in\{1,\dots,n\}}\left\|A_i-\sum_{j=1}^{n} p_j A_j\right\|\sum_{i=1}^{n} p_i\left\|A_i-\sum_{j=1}^{n} p_j A_j\right\| \\ \\ \left(\sum_{i=1}^{n} p_i\left\|A_i-\sum_{j=1}^{n} p_j A_j\right\|^p\right)^{1/p} \\ \times\left(\sum_{i=1}^{n} p_i\left\|A_i-\sum_{j=1}^{n} p_j A_j\right\|^q\right)^{1/q},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases}$$

For $p=q=2$ we also get

$$\left\|\sum_{i=1}^{n} p_i f(A_i) - f\left(\sum_{i=1}^{n} p_i A_i\right)\right\| \leq L\sum_{i=1}^{n} p_i\left\|A_i-\sum_{j=1}^{n} p_j A_j\right\|^q. \tag{2.30}$$

3. Hermite-Hadamard Type Inequalities

Let $f: I\to\mathbb{R}$ be an operator convex function of class $C^1(I)$. If A, B are selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I, then by taking $A_t := (1-t)A+tB$, $t\in[0,1]$ and the Lebesgue measure on $[0,1]$, we have by (2.1) the double inequality in terms of the Fréchet derivative $Df(\cdot)(\cdot)$

$$\begin{aligned} & f(A) - Df(A)(A) + Df(A)\left(\int_0^1 [(1-t)A+tB]\,dt\right) \\ & \leq \int_0^1 f((1-t)A+tB)\,dt \\ & \leq f(A) - \int_0^1 Df((1-t)A+tB)(A)\,dt \\ & + \int_0^1 Df((1-t)A+tB)((1-t)A+tB)\,dt \end{aligned} \tag{3.1}$$

for all A, $B\in\mathcal{SA}_I(H)$.

Observe that

$$\int_0^1 [(1-t)A+tB]\,dt = \frac{A+B}{2}$$

and

$$\int_0^1 Df((1-t)A+tB)((1-t)A+tB)\,dt - \int_0^1 Df((1-t)A+tB)(A)\,dt$$
$$= \int_0^1 tDf((1-t)A+tB)(B-A).$$

By utilising (3.1) we get the following inequality of interest

$$\begin{aligned} f(A) - Df(A)(A) + Df(A)\left(\frac{A+B}{2}\right) \\ \leq \int_0^1 f((1-t)A+tB)\,dt \\ \leq f(A) + \int_0^1 tDf((1-t)A+tB)(B-A) \end{aligned} \tag{3.2}$$

for all $A,\ B \in \mathcal{SA}_I(H)$.

From (2.2) we have the reverse of the first Hermite-Hadamard inequality

$$\begin{aligned} 0 &\leq \int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) \\ &\leq \int_0^1 Df((1-t)A+tB)((1-t)A+tB)\,dt \\ &- \int_0^1 Df((1-t)A+tB)\left(\frac{A+B}{2}\right)dt \\ &= \int_0^1 \left(t-\frac{1}{2}\right) Df((1-t)A+tB)(B-A)\,dt \end{aligned} \tag{3.3}$$

for all $A,\ B \in \mathcal{SA}_I(H)$.

If $S \in \mathcal{SA}_I(H)$ is an operator satisfying the equality

$$\int_0^1 Df((1-t)A+tB)(S)\,dt = \int_T Df((1-t)A+tB)((1-t)A+tB)\,dt, \tag{Sl}$$

then we have the Slater type inequality

$$0 \leq f(S) - \int_T f((1-t)A+tB)\,dt \leq Df(S)(S) - Df(S)\left(\frac{A+B}{2}\right). \tag{3.4}$$

Now, observe that, by (1.12) and integrating by parts, we have

$$\begin{aligned} &\int_0^1 \left(t-\frac{1}{2}\right) Df((1-t)A+tB)(B-A) \\ &= \int_0^1 \left(t-\frac{1}{2}\right) \varphi'_{(A,B)}(t) \\ &= \left(t-\frac{1}{2}\right) \varphi_{(A,B)}(t)\Big|_0^1 - \int_0^1 \varphi_{(A,B)}(t)\,dt \\ &= \frac{f(B)+f(A)}{2} - \int_0^1 f((1-t)A+tB)\,dt. \end{aligned}$$

By the inequality (3.3) we then have (see also [6])

$$0 \le \int_0^1 f\left(\left(1-t\right)A + tB\right)dt - f\left(\frac{A+B}{2}\right) \le \frac{f\left(B\right)+f\left(A\right)}{2} - \int_T f\left(\left(1-t\right)A + tB\right)dt \tag{3.5}$$

for all $A,\ B \in \mathcal{SA}_I\left(H\right).$

Observe that

$$\begin{aligned}&\int_0^1 \left(t-\frac{1}{2}\right) Df(\left(1-t\right)A + tB)\left(B-A\right)dt\\ &= \int_{1/2}^1 \left(t-\frac{1}{2}\right) Df(\left(1-t\right)A + tB)\left(B-A\right)dt\\ &- \int_0^{1/2} \left(\frac{1}{2}-t\right) Df(\left(1-t\right)A + tB)\left(B-A\right)dt\end{aligned}$$

for all $A,\ B \in \mathcal{SA}_I\left(H\right).$

From (1.14) we get

$$\begin{aligned}\left(t-\frac{1}{2}\right) Df\left(A\right)\left(B-A\right) &\le \left(t-\frac{1}{2}\right) Df(\left(1-t\right)A + tB)\left(B-A\right)\\ &\le \left(t-\frac{1}{2}\right) Df\left(B\right)\left(B-A\right),\ t\in[1/2,1)\end{aligned}$$

and

$$\begin{aligned}\left(\frac{1}{2}-t\right) Df\left(A\right)\left(B-A\right) &\le \left(\frac{1}{2}-t\right) Df(\left(1-t\right)A + tB)\left(B-A\right)\\ &\le \left(\frac{1}{2}-t\right) Df\left(B\right)\left(B-A\right),\ t\in(0,1/2].\end{aligned}$$

The second inequality can be written as

$$\begin{aligned}-\left(\frac{1}{2}-t\right) Df\left(B\right)\left(B-A\right) &\le -\left(\frac{1}{2}-t\right) Df(\left(1-t\right)A + tB)\left(B-A\right)\\ &\le -\left(\frac{1}{2}-t\right) Df\left(A\right)\left(B-A\right),\ t\in(0,1/2].\end{aligned}$$

By integration, we have

$$\begin{aligned}\int_{1/2}^1 \left(t-\frac{1}{2}\right) dt Df\left(A\right)\left(B-A\right) &\le \int_{1/2}^1 \left(t-\frac{1}{2}\right) Df(\left(1-t\right)A + tB)\left(B-A\right)dt\\ &\le \int_{1/2}^1 \left(t-\frac{1}{2}\right) dt Df\left(B\right)\left(B-A\right),\end{aligned}$$

namely

$$\begin{aligned}\frac{1}{8} Df\left(A\right)\left(B-A\right) &\le \int_{1/2}^1 \left(t-\frac{1}{2}\right) Df(\left(1-t\right)A + tB)\left(B-A\right)dt\\ &\le \frac{1}{8} Df\left(B\right)\left(B-A\right).\end{aligned} \tag{3.6}$$

Also

$$-\int_0^{1/2}\left(\frac{1}{2}-t\right)dtDf(B)(B-A)\le-\int_0^{1/2}\left(\frac{1}{2}-t\right)Df((1-t)A+tB)(B-A)\,dt$$
$$\le-\int_0^{1/2}\left(\frac{1}{2}-t\right)dtDf(A)(B-A),$$

namely

$$-\frac{1}{8}Df(B)(B-A)\le-\int_0^{1/2}\left(\frac{1}{2}-t\right)Df((1-t)A+tB)(B-A)\,dt \tag{3.7}$$
$$\le-\frac{1}{8}Df(A)(B-A).$$

By adding the right sides of the inequalities (3.6) and (3.7) we get

$$\int_{1/2}^1\left(t-\frac{1}{2}\right)Df((1-t)A+tB)(B-A)\,dt$$
$$-\int_0^{1/2}\left(\frac{1}{2}-t\right)Df((1-t)A+tB)(B-A)\,dt$$
$$\le\frac{1}{8}Df(B)(B-A)-\frac{1}{8}Df(A)(B-A).$$

By (3.5) we then get the following reverse of Hermite-Hadamard inequalities

$$0\le\int_0^1 f((1-t)A+tB)\,dt-f\left(\frac{A+B}{2}\right) \tag{3.8}$$
$$\le\frac{f(B)+f(A)}{2}-\int_0^1 f((1-t)A+tB)\,dt$$
$$\le\frac{1}{8}\left[Df(B)-Df(A)\right](B-A)$$

for all $A,\,B\in\mathcal{SA}_I(H)$.

From (3.8) we also have the norm inequalities

$$\left\|\int_0^1 f((1-t)A+tB)\,dt-f\left(\frac{A+B}{2}\right)\right\| \tag{3.9}$$
$$\le\left\|\frac{f(B)+f(A)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\|$$
$$\le\frac{1}{8}\left\|Df(B)-Df(A)\right\|\left\|B-A\right\|$$

for all $A,\,B\in\mathcal{SA}_I(H)$.

4. Some Examples

The function $f(x)=x^{-1}$ is operator convex on $(0,\infty)$, operator Fréchet differentiable and the Fréchet derivative $Df(\cdot)(\cdot)$ is given by

$$Df(T)(S)=-T^{-1}ST^{-1}$$

for $T,\,S>0$.

If $(A_t)_{t\in T}$ is a bounded continuous field of positive operators in $\mathcal{B}(H)$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ and such that $\int_T \mathbf{1}d\mu(t)=\mathbf{1}$, then by (2.1) we have the double inequality

$$\begin{aligned} 2A^{-1}&-A^{-1}\left(\int_T A_t d\mu(t)\right)A^{-1} \\ &\leq \int_T A_t^{-1}d\mu(t) \\ &\leq A^{-1}+\int_T A_t^{-1}AA_t^{-1}d\mu(t)-\int_T A_t^{-1}d\mu(t) \end{aligned} \tag{4.1}$$

for all $A>0$.

The first inequality in (4.1) is equivalent to

$$A^{-1}\leq \frac{1}{2}\left[\int_T A_t^{-1}d\mu(t)+A^{-1}\left(\int_T A_t d\mu(t)\right)A^{-1}\right] \tag{4.2}$$

while the second inequality is equivalent to

$$\int_T A_t^{-1}d\mu(t)\leq \frac{1}{2}\left[A^{-1}+\int_T A_t^{-1}AA_t^{-1}d\mu(t)\right]. \tag{4.3}$$

From (2.2) we have the reverse Jensen's inequality

$$\begin{aligned} 0&\leq \int_T A_t^{-1}d\mu(t)-\left(\int_T A_s d\mu(s)\right)^{-1} \\ &\leq \int_T A_t^{-1}\left(\int_T A_s d\mu(s)\right)A_t^{-1}d\mu(t)-\int_T A_t^{-1}d\mu(t). \end{aligned} \tag{4.4}$$

The second inequality in (4.4) is equivalent to

$$\int_T A_t^{-1}d\mu(t)\leq \frac{1}{2}\left[\int_T A_t^{-1}\left(\int_T A_s d\mu(s)\right)A_t^{-1}d\mu(t)+\left(\int_T A_s d\mu(s)\right)^{-1}\right]. \tag{4.5}$$

If S is a positive operator satisfying the equality

$$\int_T A_t^{-1}SA_t^{-1}d\mu(t)=\int_T A_t^{-1}d\mu(t), \tag{Sl}$$

then we have the Slater type inequality

$$0\leq S^{-1}-\int_T A_t^{-1}d\mu(t)\leq S^{-1}\left(\int_T A_t d\mu(t)\right)S^{-1}-S^{-1}. \tag{4.6}$$

The second inequality in (4.6) is equivalent to

$$S^{-1}\leq \frac{1}{2}\left[S^{-1}\left(\int_T A_t d\mu(t)\right)S^{-1}+\int_T A_t^{-1}d\mu(t)\right]. \tag{4.7}$$

From (3.8) we also have the inequalities

$$\begin{aligned} 0&\leq \int_0^1 ((1-t)A+tB)^{-1}dt-\left(\frac{A+B}{2}\right)^{-1} \\ &\leq \frac{B^{-1}+A^{-1}}{2}-\int_0^1 ((1-t)A+tB)^{-1}dt \\ &\leq \frac{1}{8}\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right] \end{aligned} \tag{4.8}$$

for all $A,\, B>0$.

We note that the function $f(x)=-\ln x$ is operator convex on $(0,\infty)$. The ln function is operator Fréchet differentiable with the following explicit formula for the derivative (cf. Pedersen [16, p. 155]):

$$D\ln(T)(S)=\int_0^\infty (s1_H+T)^{-1}S(s1_H+T)^{-1}ds \tag{4.9}$$

for $T,\, S>0$.

If $(A_t)_{t\in T}$ is a bounded continuous field $(A_t)_{t\in T}$ of positive operators in $\mathcal{B}(H)$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ and such that $\int_T \mathbf{1}d\mu(t)=\mathbf{1}$, then by (2.1) we have the double inequality

$$\begin{aligned}
&\int_0^\infty (s1_H+A)^{-1}A(s1_H+A)^{-1}ds-\ln A\\
&-\int_0^\infty \left(s1_H+\int_T A_t d\mu(t)\right)^{-1}A\left(s1_H+\int_T A_t d\mu(t)\right)^{-1}ds\\
&\le -\int_T \ln(A_t)\,d\mu(t)\\
&\le \int_0^\infty (s1_H+A_t)^{-1}A(s1_H+A_t)^{-1}ds-\ln A\\
&-\int_0^\infty (s1_H+A_t)^{-1}A_t(s1_H+A_t)^{-1}ds
\end{aligned} \tag{4.10}$$

for all A positive operators.

From the first inequality in (4.10) we have

$$\begin{aligned}
&\int_0^\infty (s1_H+A)^{-1}A(s1_H+A)^{-1}ds+\int_T \ln(A_t)\,d\mu(t)\\
&\le \ln A+\int_0^\infty \left(s1_H+\int_T A_t d\mu(t)\right)^{-1}A\left(s1_H+\int_T A_t d\mu(t)\right)^{-1}ds
\end{aligned} \tag{4.11}$$

while from the second inequality

$$\begin{aligned}
&\ln A+\int_0^\infty (s1_H+A_t)^{-1}A_t(s1_H+A_t)^{-1}ds\\
&\le \int_T \ln(A_t)\,d\mu(t)+\int_0^\infty (s1_H+A_t)^{-1}A(s1_H+A_t)^{-1}ds
\end{aligned} \tag{4.12}$$

for all A positive operators.

We have the reverse Jensen's inequality

$$\begin{aligned}
0&\le \ln\left(\int_T A_s d\mu(s)\right)-\int_T \ln(A_t)\,d\mu(t)\\
&\le \int_T\int_0^\infty (s1_H+A_t)^{-1}\left(\int_T A_s d\mu(s)\right)(s1_H+A_t)^{-1}ds d\mu(t)\\
&-\int_T\int_0^\infty (s1_H+A_t)^{-1}A_t(s1_H+A_t)^{-1}ds d\mu(t).
\end{aligned} \tag{4.13}$$

If $S>0$ is an operator satisfying the equality

$$\int_T\int_0^{\infty}(s1_H+A_t)^{-1}S(s1_H+A_t)^{-1}dsd\mu(t) \tag{4.14}$$
$$=\int_T\int_0^{\infty}(s1_H+A_t)^{-1}A_t(s1_H+A_t)^{-1}dsd\mu(t),$$

then we have the Slater type inequality

$$0\le\int_T\ln(A_t)\,d\mu(t)-\ln(S) \tag{4.15}$$
$$\le\int_0^{\infty}(s1_H+S)^{-1}\left(\int_T A_t d\mu(t)\right)(s1_H+S)^{-1}ds$$
$$-\int_0^{\infty}(s1_H+S)^{-1}S(s1_H+S)^{-1}ds.$$

By (3.8) we then get the following reverse of Hermite-Hadamard inequalities

$$0\le\ln\left(\frac{A+B}{2}\right)-\int_0^1\ln((1-t)A+tB)\,dt \tag{4.16}$$
$$\le\int_0^1\ln((1-t)A+tB)\,dt-\frac{\ln B+\ln A}{2}$$
$$\le\frac{1}{8}\left[\int_0^{\infty}(s1_H+A)^{-1}(B-A)(s1_H+A)^{-1}ds\right.$$
$$\left.-\int_0^{\infty}(s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}ds\right]$$

for all A, $B>0$.

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