SOME WEIGHTED INTEGRAL INEQUALITIES FOR SUB/SUPERADDITIVE FUNCTIONS ON LINEAR SPACES

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Abstract. Assume that $f : C \to \mathbb{R}$ is subadditive (superadditive) and hemi-Lebesgue integrable on C, a cone in the linear space X with $0 \in C$. Then for all x, $y \in C$ and a symmetric Lebesgue integrable and nonnegative function $p : [0, 1] \to [0, \infty)$,

$$\frac{1}{2} f(x+y) \int_0^1 p(t)\, dt \leq (\geq) \int_0^1 p(t) f((1-t)x + ty)\, dt$$
$$\leq (\geq) \int_0^1 p(t) f(tx)\, dt + \int_0^1 p(t) f(ty)\, dt.$$

In particular, for $p \equiv 1$, we have

$$\frac{1}{2} f(x+y) \leq (\geq) \int_0^1 f((1-t)x + ty)\, dt \leq (\geq) \int_0^1 f(tx)\, dt + \int_0^1 f(ty)\, dt.$$

Some particular inequalities related to Jensen's dicrete inequality for convex functions are also given.

1. Introduction

Let X be a real linear space, $x, y \in X$, $x \neq y$ and let $[x, y] := \{(1-\lambda)x + \lambda y,\ \lambda \in [0, 1]\}$ be the *segment* generated by x and y. We consider the function $f : [x, y] \to \mathbb{R}$ and the attached function $\varphi_{(x,y)} : [0, 1] \to \mathbb{R}$, $\varphi_{(x,y)}(t) := f[(1-t)x + ty]$, $t \in [0, 1]$.

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[x, y] \subset X$:

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\, dt \leq \frac{f(x) + f(y)}{2}, \tag{HH}$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $\varphi(x, y) : [0, 1] \to \mathbb{R}$

$$\varphi_{(x,y)}\left(\frac{1}{2}\right) \leq \int_0^1 \varphi_{(x,y)}(t)\, dt \leq \frac{\varphi_{(x,y)}(0) + \varphi_{(x,y)}(1)}{2}.$$

For other related results see the monograph on line [6]. For some recent results in linear spaces, see [1], [2] and [14]-[17].

By the convexity of f we have for all $t \in [0, 1]$ that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f[(1-t)x + ty] + f[(1-t)y + tx]}{2} \leq \frac{f(x) + f(y)}{2}.$$

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If we multiply this inequality by $p:[0,1]\to[0,\infty)$, a Lebesgue integrable function on $[0,1]$, and integrate on $[0,1]$ over $t\in[0,1]$, then we get

$$\begin{aligned} f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt &\le \frac{\int_0^1 f[(1-t)x+ty]\,p(t)\,dt+\int_0^1 f[(1-t)y+tx]\,p(t)\,dt}{2} \\ &\le \frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt. \end{aligned} \tag{1.1}$$

If p is symmetric on $[0,1]$, namely $p(t)=p(1-t)$ for $t\in[0,1]$, then by changing the variable $s=1-t$, we get

$$\begin{aligned} \int_0^1 f[(1-t)y+tx]\,p(t)\,dt &= \int_0^1 f[sy+(1-s)x]\,p(1-s)\,dt \\ &= \int_0^1 f[(1-t)x+ty]\,p(t)\,dt \end{aligned}$$

and by (1.1) we obtain the *Féjer's inequality*

$$\begin{aligned} f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt &\le \int_0^1 f[(1-t)x+ty]\,p(t)\,dt \\ &\le \frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt. \end{aligned} \tag{1.2}$$

If $(X;\|\cdot\|)$ is a normed linear space, then $f(x)=\|x\|^r$, $r\ge 1$ is convex and by (1.2) we get

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|^r\int_0^1 p(t)\,dt &\le \int_0^1 \|(1-t)x+ty\|^r\,p(t)\,dt \\ &\le \frac{\|x\|^r+\|y\|^r}{2}\int_0^1 p(t)\,dt, \end{aligned} \tag{1.3}$$

for all x, $y\in X$.

For $r=1$ we get

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|\int_0^1 p(t)\,dt &\le \int_0^1 \|(1-t)x+ty\|\,p(t)\,dt \\ &\le \frac{\|x\|+\|y\|}{2}\int_0^1 p(t)\,dt, \end{aligned} \tag{1.4}$$

for all x, $y\in X$.

Let X be a linear space. A subset $C\subseteq X$ is called a *convex cone* in X provided the following conditions hold:

(i) x, $y\in C$ imply $x+y\in C$;
(ii) $x\in C$, $\alpha\ge 0$ imply $\alpha x\in C$.

A functional $h:C\to\mathbb{R}$ is called *superadditive (subadditive)* on C if

(iii) $h(x+y)\ge(\le)\,h(x)+h(y)$ for any x, $y\in C$

and *nonnegative (strictly positive)* on C if, obviously, it satisfies

(iv) $h(x)\ge(>)\,0$ for each $x\in C$.

The functional h is *s-positive homogeneous* on C, for a given $s>0$, if

(v) $h(\alpha x) = \alpha^s h(x)$ for any $\alpha \geq 0$ and $x \in C$.

In [9] we obtained further results concerning the quasilinearity of some composite functionals:

Theorem 1. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C. If $h : C \to [0, \infty)$ is a superadditive (subadditive) functional on C and $p,\ q \geq 1$ $(0 < p,\ q < 1)$ then the functional*

$$\Psi_{p,q} : C \to [0, \infty), \ \Psi_{p,q}(x) = h^q(x)\, v^{q\left(1-\frac{1}{p}\right)}(x) \tag{1.5}$$

is superadditive (subadditive) on C.

Theorem 2. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C. If $h : C \to [0, \infty)$ is a superadditive functional on C and $0 < p,\ q < 1$ then the functional*

$$\Phi_{p,q} : C \to [0, \infty), \ \Phi_{p,q}(x) = \frac{v^{q\left(1-\frac{1}{p}\right)}(x)}{h^q(x)} \tag{1.6}$$

is subadditive on C.

The following result holds [11].

Theorem 3. *Let C be a convex cone in the linear space X and $v : C \to (0, \infty)$ an additive functional on C.*

(i) If $p \geq q \geq 0$, $p \geq 1$ and $h : C \to [0, \infty)$ is superadditive on C, then the new mapping

$$\Lambda_{p,q} : C \to [0, \infty), \ \Lambda_{p,q}(x) := v^{\frac{p-q}{p}}(x)\, h^q(x) \tag{1.7}$$

is superadditive on C;

(ii) If $p \leq q$, $p \in (0, 1)$ and $h : C \to [0, \infty)$ is subadditive on C, then the mapping $\Lambda_{p,q}$ is subadditive on C.

Now, if we assume that $p \geq q \geq 0$, $p \geq 1$, then by denoting $r := \frac{q}{p} \in [0, 1]$, we deduce that the functional

$$\Theta_{p,r}(x) := v^{1-r}(x)\, h^{pr}(x)$$

is superadditive, provided v is additive and h is superadditive on C. In particular, the functional

$$\Upsilon_t(x) := v^{\frac{1}{2}}(x)\, h^t(x)$$

is superadditive for $t \geq \frac{1}{2}$.

If $p \leq q$, $p \in (0, 1)$ and if we denote $s := \frac{q}{p} \in [1, \infty)$, then the functional

$$\digamma_{p,s}(x) := \frac{h^{sp}(x)}{v^{s-1}(x)}$$

is subadditive provided v is additive and h is subadditive on C. In particular, the functional

$$\Xi_z(x) := \frac{h^z(x)}{v(x)}$$

is subadditive for $z \in (0, 2)$.

Motivated by the above results, in this paper we establish some weighted integral inequalities for subadditive (superadditive) functions defined on cones from linear spaces that are hemi-Lebesgue integrable.

2. Main Results

A superadditive or a subadditive function f defined on a cone C in the linear space X will be called *hemi-Lebesgue integrable* on C if for any $x \in C$ the function $[0,1] \ni t \mapsto f(tx) \in \mathbb{R}$ is Lebesgue integrable on $[0,1]$.

Theorem 4. *Assume that $f : C \to \mathbb{R}$ is subadditive (superadditive) and hemi-Lebesgue integrable on C, a cone in the linear space X with $0 \in C$. Then for all x, $y \in C$ and a symmetric Lebesgue integrable and nonnegative function $p : [0,1] \to [0,\infty)$,*

$$\frac{1}{2} f(x+y) \int_0^1 p(t)\,dt \le (\ge) \int_0^1 p(t) f((1-t)x + ty)\,dt \le (\ge) \int_0^1 p(t) f(tx)\,dt + \int_0^1 p(t) f(ty)\,dt. \tag{2.1}$$

In particular, for $p \equiv 1$, we have

$$\frac{1}{2} f(x+y) \le (\ge) \int_0^1 f((1-t)x+ty)\,dt \le (\ge) \int_0^1 f(tx)\,dt + \int_0^1 f(ty)\,dt. \tag{2.2}$$

Proof. From the subadditivity of f we have for x, $y \in C$ and $t \in [0,1]$ that

$$\begin{aligned} f(x+y) &= f((1-t)x + ty + tx + (1-t)y) \\ &\le f((1-t)x+ty) + f(tx + (1-t)y) \\ &\le f((1-t)x) + f(ty) + f(tx) + f((1-t)y). \end{aligned}$$

If we multiply this inequality by $p(t) \ge 0$ and integrate over $t \in [0,1]$ we get

$$\begin{aligned} & f(x+y) \int_0^1 p(t)\,dt \\ & \le \int_0^1 p(t) f((1-t)x+ty)\,dt + \int_0^1 p(t) f(tx+(1-t)y)\,dt \\ & \le \int_0^1 p(t) f((1-t)x)\,dt + \int_0^1 p(t) f(ty)\,dt \\ & + \int_0^1 p(t) f(tx)\,dt + \int_0^1 p(t) f((1-t)y)\,dt. \end{aligned} \tag{2.3}$$

By the symmetry of p and changing the variable, we have

$$\begin{aligned} \int_0^1 p(t) f(tx+(1-t)y)\,dt &= \int_0^1 p(1-s) f((1-s)x+sy)\,ds \\ &= \int_0^1 p(t) f((1-t)x+ty)\,dt, \end{aligned}$$

$$\int_0^1 p(t) f((1-t)x)\,dt = \int_0^1 p(1-s) f(sx)\,ds = \int_0^1 p(t) f(tx)\,dt$$

and

$$\int_0^1 p(t) f((1-t)y)\,dt = \int_0^1 p(t) f(ty)\,dt.$$

Then by (2.3) we obtain

$$\begin{aligned}f(x+y)\int_0^1 p(t)\,dt &\le 2\int_0^1 p(t)\,f((1-t)x+ty)\,dt\\ &\le 2\int_0^1 p(t)\,f(tx)\,dt+2\int_0^1 p(t)\,f(ty)\,dt,\end{aligned}$$

which is equivalent to (2.1). □

Remark 1. *We observe, for the simple symmetrical weight $p(t)=\left|t-\frac{1}{2}\right|$, $t\in[0,1]$, we get from (2.1) that*

$$\begin{aligned}\frac{1}{8}f(x+y) &\le (\ge)\int_0^1\left|t-\frac{1}{2}\right| f((1-t)x+ty)\,dt\\ &\le (\ge)\int_0^1\left|t-\frac{1}{2}\right| f(tx)\,dt+\int_0^1\left|t-\frac{1}{2}\right| f(ty)\,dt,\end{aligned}\tag{2.4}$$

while for $p(t)=t(1-t)$, $t\in[0,1]$, we get

$$\begin{aligned}\frac{1}{12}f(x+y) &\le (\ge)\int_0^1 t(1-t)\,f((1-t)x+ty)\,dt\\ &\le (\ge)\int_0^1 t(1-t)\,f(tx)\,dt+\int_0^1 t(1-t)\,f(ty)\,dt\end{aligned}\tag{2.5}$$

for x, $y\in C$, where $f:C\to\mathbb{R}$ is subadditive (superadditive) and hemi-Lebesgue integrable on C, a cone C in the linear space X with $0\in C$.

Definition 1. *Let C be a cone in the linear space X with $0\in C$. The function f defined on C is called convex-starshaped if $f(tx)\le tf(x)$ for all $t\in[0,1]$ and $x\in C$. It is called concave-starshaped if $f(tx)\ge tf(x)$ for all $t\in[0,1]$ and $x\in C$.*

Corollary 1. *With the assumptions of Theorem 4 and, in addition, f is convex-starshaped (concave-starshaped), then for all x, $y\in C$*

$$\begin{aligned}f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt &\le (\ge)\,\frac{1}{2}f(x+y)\int_0^1 p(t)\,dt\\ &\le (\ge)\int_0^1 p(t)\,f((1-t)x+ty)\,dt\\ &\le (\ge)\int_0^1 p(t)\,f(tx)\,dt+\int_0^1 p(t)\,f(ty)\,dt\\ &\le (\ge)\,[f(x)+f(y)]\int_0^1 tp(t)\,dt.\end{aligned}\tag{2.6}$$

In particular, for $p\equiv 1$, we have

$$\begin{aligned}f\left(\frac{x+y}{2}\right) &\le (\ge)\,\frac{1}{2}f(x+y)\le (\ge)\int_0^1 f((1-t)x+ty)\,dt\\ &\le (\ge)\int_0^1 f(tx)\,dt+\int_0^1 f(ty)\,dt\le (\ge)\,\frac{f(x)+f(y)}{2}\end{aligned}\tag{2.7}$$

for all x, $y\in C$.

If $(X;\|\cdot\|)$ is a normed space, then the function f is subadditive and convex-starshaped, and for all x, $y\in X$ we have

$$\left\|\frac{x+y}{2}\right\|\int_0^1 p(t)\,dt\le\int_0^1 p(t)\,\|(1-t)\,x+ty\|\,dt \le[\|x\|+\|y\|]\int_0^1 tp(t)\,dt. \tag{2.8}$$

Since p is symmetric, then

$$\int_0^1 tp(t)\,dt=\int_0^1(1-t)\,p(1-t)\,dt=\int_0^1(1-t)\,p(t)\,dt =\int_0^1 p(t)\,dt-\int_0^1 tp(t)\,dt,$$

which shows that

$$\int_0^1 tp(t)\,dt=\frac{1}{2}\int_0^1 p(t)\,dt.$$

Therefore the inequality (2.8) is the same with the inequality (1.4).

Remark 2. *If f is a function of real variable defined on $[0,\infty)$ that is subadditive (superadditive) and continuous, and since for $0<a<b$*

$$\int_0^1 p(t)\,f((1-t)\,a+ty)\,dt=\frac{1}{b-a}\int_a^b p\left(\frac{u-a}{b-a}\right)f(u)\,du,$$

$$\int_0^1 p(t)\,f(ta)\,dt=\frac{1}{a}\int_0^a p\left(\frac{v}{a}\right)f(v)\,dv$$

and

$$\int_0^1 p(t)\,f(tb)\,dt=\frac{1}{b}\int_0^b p\left(\frac{v}{b}\right)f(v)\,dv,$$

then by (2.1) we get

$$\frac{1}{2}f(a+b)\int_0^1 p(t)\,dt\le(\ge)\,\frac{1}{b-a}\int_a^b p\left(\frac{u-a}{b-a}\right)f(u)\,du \le(\ge)\,\frac{1}{a}\int_0^a p\left(\frac{v}{a}\right)f(v)\,dv+\frac{1}{b}\int_0^b p\left(\frac{v}{b}\right)f(v)\,dv \tag{2.9}$$

for any symmetric Lebesgue integrable and nonnegative function $p:[0,1]\to[0,\infty)$.

In particular, if $p\equiv 1$, then we have the inequality

$$\frac{1}{2}f(a+b)\le(\ge)\,\frac{1}{b-a}\int_a^b f(u)\,du\le(\ge)\,\frac{1}{a}\int_0^a f(v)\,dv+\frac{1}{b}\int_0^b f(v)\,dv, \tag{2.10}$$

which was obtained in [18].

Corollary 2. *Let C be a convex cone in the linear space X with $0\in C$ and $v:C\to(0,\infty)$ an additive functional on C. Assume that $h:C\to[0,\infty)$ is a* *superadditive (subadditive) functional on C and $p,\ q \geq 1$ $(0 < p,\ q < 1)$. If h and v are hemi-Lebesgue integrable on C, then*

$$\begin{aligned}(2.11)\qquad & \frac{1}{2}h^{q}(x+y)\,v^{q\left(1-\frac{1}{p}\right)}(x+y)\int_0^1 w(t)\,dt \\ & \leq (\geq) \int_0^1 w(t)\,h^{q}((1-t)\,x+ty)\,v^{q\left(1-\frac{1}{p}\right)}((1-t)\,x+ty)\,dt \\ & \leq (\geq) \int_0^1 w(t)\,h^{q}(tx)\,v^{q\left(1-\frac{1}{p}\right)}(tx)\,dt + \int_0^1 w(t)\,h^{q}(ty)\,v^{q\left(1-\frac{1}{p}\right)}(ty)\,dt,\end{aligned}$$

where $x,\ y \in C$ and a symmetric Lebesgue integrable and nonnegative function $w : [0,1] \to [0,\infty)$.

In particular, we have

$$\begin{aligned}(2.12)\qquad & \frac{1}{2}h^{q}(x+y)\,v^{q\left(1-\frac{1}{p}\right)}(x+y) \\ & \leq (\geq) \int_0^1 h^{q}((1-t)\,x+ty)\,v^{q\left(1-\frac{1}{p}\right)}((1-t)\,x+ty)\,dt \\ & \leq (\geq) \int_0^1 h^{q}(tx)\,v^{q\left(1-\frac{1}{p}\right)}(tx)\,dt + \int_0^1 h^{q}(ty)\,v^{q\left(1-\frac{1}{p}\right)}(ty)\,dt,\end{aligned}$$

where $x,\ y \in C$.

Proof. Observe, by Theorem 1, that the functional

$$\Psi_{p,q} : C \to [0,\infty)\,,\ \Psi_{p,q}(x) = h^{q}(x)\,v^{q\left(1-\frac{1}{p}\right)}(x)$$

is superadditive (subadditive) on C.

If we write Theorem 4 for the function $f = \Psi_{p,q}$ and $p = w$, we get (2.11). □

Corollary 3. *Let C be a convex cone in the linear space X with $0 \in C$ and $v : C \to (0,\infty)$ an additive functional on C. Assume that $h : C \to [0,\infty)$ is a superadditive functional on C and $0 < p,\ q < 1$. If h and v are hemi-Lebesgue integrable on C, then*

$$\begin{aligned}(2.13)\qquad & \frac{1}{2}\frac{v^{q\left(1-\frac{1}{p}\right)}(x+y)}{h^{q}(x+y)}\int_0^1 w(t)\,dt \\ & \leq \int_0^1 w(t)\,\frac{v^{q\left(1-\frac{1}{p}\right)}((1-t)\,x+ty)}{h^{q}((1-t)\,x+ty)}dt \\ & \leq \int_0^1 w(t)\,\frac{v^{q\left(1-\frac{1}{p}\right)}(tx)}{h^{q}(tx)}dt + \int_0^1 w(t)\,\frac{v^{q\left(1-\frac{1}{p}\right)}(ty)}{h^{q}(ty)}dt,\end{aligned}$$

where $x,\ y \in C$, for a symmetric Lebesgue integrable and nonnegative function $w : [0,1] \to [0,\infty)$.

In particular, for $w \equiv 1$, we have

$$\begin{aligned}(2.14)\qquad \frac{1}{2}\frac{v^{q\left(1-\frac{1}{p}\right)}(x+y)}{h^{q}(x+y)} & \leq \int_0^1 \frac{v^{q\left(1-\frac{1}{p}\right)}((1-t)\,x+ty)}{h^{q}((1-t)\,x+ty)}dt \\ & \leq \int_0^1 \frac{v^{q\left(1-\frac{1}{p}\right)}(tx)}{h^{q}(tx)}dt + \int_0^1 \frac{v^{q\left(1-\frac{1}{p}\right)}(ty)}{h^{q}(ty)}dt,\end{aligned}$$

where $x,\ y \in C$.

Similar results may be obtained by the use of Theorem 3 and its consequences, however we do not provide them here.

We also have the double integral inequalities:

Theorem 5. *Assume that $f : C \to \mathbb{R}$ is subadditive (superadditive) and hemi-Lebesgue integrable on C, a cone in the linear space X with $0 \in C$. Then for all x, $y \in C$ and symmetric Lebesgue integrable and nonnegative functions p, $q : [0,1] \to [0,\infty)$, we have*

$$
\begin{aligned}
&\frac{1}{2} f(x+y) \int_0^1 p(t)\,dt \int_0^1 q(t)\,dt \\
&\le \int_0^1 \int_0^1 p(t)\,q(s)\,f((1-t-s+2ts)\,x+(s+t-2st)\,y)\,dtds \\
&\le \int_0^1 \int_0^1 p(t)\,q(s)\,f(t(1-s)\,x+tsy)\,dtds \\
&+ \int_0^1 \int_0^1 p(t)\,q(s)\,f(tsx+t(1-s)\,y)\,dtds \\
&\le 2\int_0^1 \int_0^1 p(t)\,q(s)\,f(tsx)\,dtds + 2\int_0^1 \int_0^1 p(t)\,q(s)\,f(tsy)\,dtds.
\end{aligned} \tag{2.15}
$$

In particular, for p, $q \equiv 1$, we have

$$
\begin{aligned}
&\frac{1}{2} f(x+y) \\
&\le \int_0^1 \int_0^1 f((1-t-s+2ts)\,x+(s+t-2st)\,y)\,dtds \\
&\le \int_0^1 \int_0^1 f(t(1-s)\,x+tsy)\,dtds + \int_0^1 \int_0^1 f(tsx+t(1-s)\,y)\,dtds \\
&\le 2\int_0^1 \int_0^1 f(tsx)\,dtds + 2\int_0^1 \int_0^1 f(tsy)\,dtds.
\end{aligned} \tag{2.16}
$$

Proof. If we replace x with $(1-s)\,x+sy$ and y with $sx+(1-s)\,y$, $s \in [0,1]$ in the inequality (2.1), then we get

$$
\begin{aligned}
&\frac{1}{2} f(x+y) \int_0^1 p(t)\,dt \\
&\le (\ge) \int_0^1 p(t)\,f((1-t)((1-s)\,x+sy)+t(sx+(1-s)\,y))\,dt \\
&\le (\ge) \int_0^1 p(t)\,f(t((1-s)\,x+sy))\,dt + \int_0^1 p(t)\,f(t(sx+(1-s)\,y))\,dt.
\end{aligned}
$$

If we multiply this inequality by $q(t) \geq 0$, $s \in [0,1]$, integrate and use Fubini's theorem, then we get

$$
\begin{aligned}
(2.17)\quad & \frac{1}{2} f(x+y) \int_0^1 p(t)\,dt \int_0^1 q(t)\,dt \\
& \leq (\geq) \int_0^1 \int_0^1 p(t)\,q(s)\,f((1-t)((1-s)x+sy)+t(sx+(1-s)y))\,dtds \\
& \leq (\geq) \int_0^1 \int_0^1 p(t)\,q(s)\,f(t((1-s)x+sy))\,dtds \\
& + \int_0^1 \int_0^1 p(t)\,q(s)\,f(t(sx+(1-s)y))\,dtds.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& \int_0^1 \int_0^1 p(t)\,q(s)\,f((1-t)((1-s)x+sy)+t(sx+(1-s)y))\,dtds \\
& = \int_0^1 \int_0^1 p(t)\,q(s)\,f((1-t)(1-s)x+(1-t)sy+tsx+t(1-s)y)\,dtds \\
& = \int_0^1 \int_0^1 p(t)\,q(s)\,f((1-t-s+2ts)x+(s+t-2st)y)\,dtds,
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^1 \int_0^1 p(t)\,q(s)\,f(t((1-s)x+sy))\,dtds \\
& = \int_0^1 \int_0^1 p(t)\,q(s)\,f(t(1-s)x+tsy)\,dtds \\
& \leq \int_0^1 \int_0^1 p(t)\,q(s)\,f(t(1-s)x)\,dtds + \int_0^1 \int_0^1 p(t)\,q(s)\,f(tsy) \\
& = \int_0^1 \int_0^1 p(t)\,q(1-u)\,f(tux)\,dtdu + \int_0^1 \int_0^1 p(t)\,q(s)\,f(tsy) \\
& = \int_0^1 \int_0^1 p(t)\,q(u)\,f(tux)\,dtdu + \int_0^1 \int_0^1 p(t)\,q(s)\,f(tsy)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^1 \int_0^1 p(t)\,q(s)\,f(t(sx+(1-s)y))\,dtds \\
& = \int_0^1 \int_0^1 p(t)\,q(s)\,f(tsx+t(1-s)y)\,dtds \\
& \leq \int_0^1 \int_0^1 p(t)\,q(s)\,f(tsx)\,dtds + \int_0^1 \int_0^1 p(t)\,q(s)\,f(t(1-s)y)\,dtds \\
& = \int_0^1 \int_0^1 p(t)\,q(s)\,f(tsx)\,dtds + \int_0^1 \int_0^1 p(t)\,q(1-u)\,f(tuy)\,dtds \\
& = \int_0^1 \int_0^1 p(t)\,q(s)\,f(tsx)\,dtds + \int_0^1 \int_0^1 p(t)\,q(u)\,f(tuy)\,dtdu.
\end{aligned}
$$

By utilising (2.17) we get the desired result (2.15). $\square$

3. Some Results Related to Jensen's Inequality

Let C be a convex subset of the real linear space X and let $\varphi : C \rightarrow \mathbb{R}$ be a convex mapping. Here we consider the following well-known form of *Jensen's discrete inequality*:

$$\varphi\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \leq \frac{1}{P_I}\sum_{i\in I} p_i \varphi\left(x_i\right), \tag{3.1}$$

where I denotes a finite subset of the set $\mathbb{N}$ of natural numbers, $x_i \in C$, $p_i \geq 0$ for $i \in I$ and $P_I := \sum_{i\in I} p_i > 0$.

Let us fix $I \in \mathcal{P}_f(\mathbb{N})$ (the class of finite parts of $\mathbb{N}$) and $x_i \in C$ $(i \in I)$. Now consider the functional $f_I : S_+(I) \rightarrow \mathbb{R}$ given by

$$f_I(\mathbf{p}) := \sum_{i\in I} p_i \varphi\left(x_i\right) - P_I \varphi\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \geq 0, \tag{3.2}$$

where $S_+(I) := \left\{\mathbf{p} = (p_i)_{i\in I} \middle| p_i \geq 0,\ i \in I \text{ and } P_I > 0\right\}$ and f is convex on C.

We observe that $S_+(I)$ is a convex cone and the functional J_I is nonnegative and positive homogeneous on $S_+(I)$.

Lemma 1 ([13])**.** *The functional $f_I(\cdot)$ is a superadditive functional on $S_+(I)$.*

We have for $\mathbf{p}$, $\mathbf{q} \in S_+(I)$ that

$$f_I(\mathbf{p}+\mathbf{q}) := \sum_{i\in I}(p_i+q_i)\varphi(x_i) - (P_I+Q_I)\varphi\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)x_i\right),$$

and for a symmetric nonnegative Lebesgue integrable function $w : [0,1] \rightarrow [0,\infty)$ we have

$$\begin{aligned}
&\int_0^1 w(t) f_I((1-t)\mathbf{p}+t\mathbf{q})\,dt \\
&= \sum_{i\in I}\int_0^1 w(t)((1-t)p_i+tq_i)\,dt\varphi(x_i) \\
&- \int_0^1 w(t)((1-t)P_I+tQ_I)\varphi\left(\frac{1}{(1-t)P_I+tQ_I}\sum_{i\in I}((1-t)p_i+tq_i)x_i\right)dt \\
&= \sum_{i\in I} p_i\varphi(x_i)\int_0^1 w(t)(1-t)\,dt + \sum_{i\in I} q_i\varphi(x_i)\int_0^1 w(t)\,tdt \\
&- \int_0^1 w(t)((1-t)P_I+tQ_I)\varphi\left(\frac{1}{(1-t)P_I+tQ_I}\sum_{i\in I}((1-t)p_i+tq_i)x_i\right)dt \\
&= \left[\sum_{i\in I} p_i\varphi(x_i) + \sum_{i\in I} q_i\varphi(x_i)\right]\int_0^1 w(t)\,tdt \\
&- \int_0^1 w(t)((1-t)P_I+tQ_I)\varphi\left(\frac{1}{(1-t)P_I+tQ_I}\sum_{i\in I}((1-t)p_i+tq_i)x_i\right)dt,
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 w(t) f_I(t\mathbf{p})\,dt \\
&= \sum_{i\in I} p_i\varphi(x_i)\int_0^1 tw(t)\,dt - P_I\varphi\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\int_0^1 tw(t)\,dt \\
&= \int_0^1 tw(t)\,dt\left[\sum_{i\in I}p_i\varphi(x_i) - P_I\varphi\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 w(t) f_I(t\mathbf{q})\,dt \\
&= \sum_{i\in I} q_i\varphi(x_i)\int_0^1 tw(t)\,dt - Q_I\varphi\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\int_0^1 tw(t)\,dt \\
&= \left[\sum_{i\in I}q_i\varphi(x_i) - Q_I\varphi\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]\int_0^1 tw(t)\,dt.
\end{aligned}$$

From the inequality (2.1) we have

$$\begin{aligned}
\frac{1}{2}f_I(\mathbf{p}+\mathbf{q})\int_0^1 w(t)\,dt &\geq \int_0^1 w(t) f_I((1-t)\mathbf{p}+t\mathbf{q})\,dt \\
&\geq \int_0^1 w(t) f_I(t\mathbf{p})\,dt + \int_0^1 w(t) f_I(t\mathbf{q})\,dt,
\end{aligned} \tag{3.3}$$

for all $\mathbf{p}, \mathbf{q} \in S_+(I)$.

Therefore

$$\begin{aligned}
&\frac{1}{2}\left[\sum_{i\in I}(p_i+q_i)\varphi(x_i) - (P_I+Q_I)\varphi\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)x_i\right)\right] \\
&\geq \left[\sum_{i\in I}p_i\varphi(x_i) + \sum_{i\in I}q_i\varphi(x_i)\right]\frac{\int_0^1 w(t)\,tdt}{\int_0^1 w(t)\,dt} \\
&- \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)((1-t)P_I + tQ_I) \\
&\times \varphi\left(\frac{1}{(1-t)P_I+tQ_I}\sum_{i\in I}((1-t)p_i+tq_i)x_i\right)dt \\
&\geq \left[\sum_{i\in I}p_i\varphi(x_i) - P_I\varphi\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]\frac{\int_0^1 w(t)\,tdt}{\int_0^1 w(t)\,dt} \\
&+ \left[\sum_{i\in I}q_i\varphi(x_i) - Q_I\varphi\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]\frac{\int_0^1 w(t)\,tdt}{\int_0^1 w(t)\,dt}
\end{aligned} \tag{3.4}$$

for all $\mathbf{p}, \mathbf{q} \in S_+(I)$.

If $w \equiv 1$ in (3.4), then we get, after some calculations, that

$$\frac{P_I+Q_I}{2}\varphi\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)x_i\right) \tag{3.5}$$
$$\leq \int_0^1((1-t)P_I+tQ_I)\varphi\left(\frac{1}{(1-t)P_I+tQ_I}\sum_{i\in I}((1-t)p_i+tq_i)x_i\right)dt$$
$$\leq \frac{1}{2}\left[P_I\varphi\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+Q_I\varphi\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]$$

for all $\mathbf{p}$, $\mathbf{q} \in S_+(I)$.

If $(X, \|\cdot\|)$ is a normed space and $\varphi(x) = \|x\|^r$, $r \geq 1$, then φ is convex and by (3.5) we get

$$\frac{(P_I+Q_I)^{r-1}}{2}\left\|\sum_{i\in I}(p_i+q_i)x_i\right\|^r \tag{3.6}$$
$$\leq \int_0^1((1-t)P_I+tQ_I)^{r-1}\|((1-t)p_i+tq_i)x_i\|^r dt$$
$$\leq \frac{1}{2}\left[P_I^{r-1}\left\|\sum_{i\in I}p_ix_i\right\|^r+Q_I^{r-1}\left\|\sum_{i\in I}q_ix_i\right\|^r\right]$$

for all $\mathbf{p}$, $\mathbf{q} \in S_+(I)$ and $x_i \in C$ $(i \in I)$.

If $x_i \in \mathbb{R}$, $(i \in I)$ and $\mathbf{p}$, $\mathbf{q} \in S_+(I)$, then by taking $\varphi(x) = \exp x$, we get

$$\frac{P_I+Q_I}{2}\exp\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)x_i\right) \tag{3.7}$$
$$\leq \int_0^1((1-t)P_I+tQ_I)\exp\left(\frac{1}{(1-t)P_I+tQ_I}\sum_{i\in I}((1-t)p_i+tq_i)x_i\right)dt$$
$$\leq \frac{1}{2}\left[P_I\exp\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+Q_I\exp\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right].$$

If $x_i > 0$, $(i \in I)$ and $\mathbf{p}$, $\mathbf{q} \in S_+(I)$, then by taking $\varphi(x) = -\ln x$ in (3.5) we get

$$\frac{P_I+Q_I}{2}\ln\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)x_i\right) \tag{3.8}$$
$$\geq \int_0^1((1-t)P_I+tQ_I)\ln\left(\frac{1}{(1-t)P_I+tQ_I}\sum_{i\in I}((1-t)p_i+tq_i)x_i\right)dt$$
$$\geq \frac{1}{2}\left[P_I\ln\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+Q_I\ln\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]$$

or, equivalently,

$$\begin{aligned}
(3.9)\qquad & \left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\,x_i\right)^{\frac{P_I+Q_I}{2}} \\
& \geq \exp\left[\int_0^1 ((1-t)\,P_I+tQ_I) \right. \\
& \left. \times \ln\left(\frac{1}{(1-t)\,P_I+tQ_I}\sum_{i\in I}((1-t)\,p_i+tq_i)\,x_i\right)dt\right] \\
& \geq \sqrt{\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)^{P_I}\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)^{Q_I}}
\end{aligned}$$

for $\mathbf{p}$, $\mathbf{q} \in S_+(I)$ and $x_i > 0$, $(i \in I)$.

Define the following functional

$$L_{p,q,I}(\mathbf{p}) := P_I^{\frac{p-q}{p}}\left[\sum_{i\in I}p_if(x_i) - P_If\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^q \qquad (3.10)$$

for $p \geq 1$ and $p \geq q \geq 0$.

The following proposition can be stated via Theorem 3:

Proposition 1. *The functional $L_{p,q,I}(\cdot)$ is superadditive on $S_+(I)$ for any $p \geq 1$ and $p \geq q \geq 0$.*

Remark 3. *We observe that, in particular, the following functionals*

$$L_{p,\alpha,I}(\mathbf{p}) := P_I^{1-\alpha}\left[\sum_{i\in I}p_if(x_i) - P_If\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^{\alpha p}$$

and

$$\tilde{L}_{p,I}(\mathbf{p}) := P_I^{\frac{1}{2}}\left[\sum_{i\in I}p_if(x_i) - P_If\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^{\frac{p}{2}}$$

are superadditive on $S_+(I)$ for any $p \geq 1$ and $\alpha \in (0,1)$.

One can state similar results by utilising the functionals $L_{p,q,I}$, $L_{p,\alpha,I}$ and $\tilde{L}_{p,I}$, however we do not provide the details here.

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