SOME WEIGHTED INTEGRAL INEQUALITIES FOR OPERATOR SUB/SUPERADDITIVE FUNCTIONS ON HILBERT SPACES

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ABSTRACT. Assume that $f : [0, \infty) \to \mathbb{R}$ is operator subadditive (superadditive) on $[0, \infty)$. Then for all A, $B \geq 0$ and a symmetric Lebesgue integrable and nonnegative function $p : [0, 1] \to [0, \infty)$,

$$\frac{1}{2} f(A + B) \int_0^1 p(t)\,dt \leq (\geq) \int_0^1 p(t) f((1 - t) A + tB)\,dt$$
$$\leq (\geq) \int_0^1 p(t) f(tA)\,dt + \int_0^1 p(t) f(tB)\,dt.$$

In particular, for $p \equiv 1$, we have

$$\frac{1}{2} f(A + B) \leq (\geq) \int_0^1 f((1 - t) A + tB)\,dt$$
$$\leq (\geq) \int_0^1 f(tA)\,dt + \int_0^1 f(tB)\,dt.$$

Some particular operator inequalities for power and logarithmic functions are also given.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1 - \lambda) A + \lambda B) \leq (\geq) (1 - \lambda) f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [7] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

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In [5] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f : I \to \mathbb{R}$

$$f\left(\frac{A+B}{2}\right) \le \int_0^1 f((1-s)A + sB)\,ds \le \frac{f(A)+f(B)}{2}, \tag{1.2}$$

where A, B are selfadjoint operators with spectra included in I.

For recent inequalities for operator convex functions see [1]-[6] and [8]-[19].

Moslehian and Najafi [12] have shown that for $A,\ B \ge 0$, if $AB + BA \ge 0$, then for any operator monotone function $f(t) \ge 0$ on $[0,\infty)$ satisfies the operator subadditive property

$$f(A+B) \le f(A) + f(B). \tag{1.3}$$

If, in general, a continuous function $f : [0,\infty) \to \mathbb{R}$ satisfies the inequality (1.3) for any nonnegative operators A, B, then we call is *operator subadditive*. If the sign of the inequality in (1.3) is reversed, the function f is called operator superadditive.

Motivated by the above results, we establish in this paper some weighted and unweighted Hermite-Hadamard type inequalities for operator subadditive or superadditive functions. Some particular operator inequalities for power and logarithmic functions are also given.

2. Main Results

We have:

Theorem 1. *Assume that $f : [0,\infty) \to \mathbb{R}$ is operator subadditive (superadditive) on $[0,\infty)$. Then for all $A,\ B \ge 0$ and a symmetric Lebesgue integrable and nonnegative function $p : [0,1] \to [0,\infty)$,*

$$\begin{aligned} \frac{1}{2} f(A+B) \int_0^1 p(t)\,dt &\le (\ge) \int_0^1 p(t) f((1-t)A + tB)\,dt \\ &\le (\ge) \int_0^1 p(t) f(tA)\,dt + \int_0^1 p(t) f(tB)\,dt. \end{aligned} \tag{2.1}$$

In particular, for $p \equiv 1$, we have

$$\begin{aligned} \frac{1}{2} f(A+B) &\le (\ge) \int_0^1 f((1-t)A + tB)\,dt \\ &\le (\ge) \int_0^1 f(tA)\,dt + \int_0^1 f(tB)\,dt. \end{aligned} \tag{2.2}$$

Proof. From the subadditivity of f we have for $A,\ B \ge 0$ and $t \in [0,1]$ that

$$\begin{aligned} f(A+B) &= f((1-t)A + tB + tA + (1-t)B) \\ &\le f((1-t)A + tB) + f(tA + (1-t)B) \\ &\le f((1-t)A) + f(tB) + f(tA) + f((1-t)B). \end{aligned}$$

If we multiply this inequality by $p(t) \geq 0$ and integrate over $t \in [0,1]$, then we get

$$f(A+B)\int_0^1 p(t)\,dt \tag{2.3}$$
$$\leq \int_0^1 p(t) f((1-t)A+tB)\,dt + \int_0^1 p(t) f(tA+(1-t)B)\,dt$$
$$\leq \int_0^1 p(t) f((1-t)A)\,dt + \int_0^1 p(t) f(tB)\,dt$$
$$+ \int_0^1 p(t) f(tA)\,dt + \int_0^1 p(t) f((1-t)B)\,dt.$$

By the symmetry of p and changing the variable, we have

$$\int_0^1 p(t) f(tA+(1-t)B)\,dt = \int_0^1 p(1-s) f((1-s)A+sB)\,ds = \int_0^1 p(t) f((1-t)A+tB)\,dt,$$

$$\int_0^1 p(t) f((1-t)A)\,dt = \int_0^1 p(1-s) f(sA)\,ds = \int_0^1 p(t) f(tA)\,dt$$

and

$$\int_0^1 p(t) f((1-t)B)\,dt = \int_0^1 p(t) f(tB)\,dt.$$

Then by (2.3) we obtain

$$f(A+B)\int_0^1 p(t)\,dt \leq 2\int_0^1 p(t) f((1-t)A+tB)\,dt$$
$$\leq 2\int_0^1 p(t) f(tA)\,dt + 2\int_0^1 p(t) f(tB)\,dt,$$

which is equivalent to (2.1). □

Remark 1. *We observe, for the simple symmetrical weight* $p(t) = \left|t-\frac{1}{2}\right|$, $t \in [0,1]$, *we get from (2.1) that*

$$\frac{1}{8} f(A+B) \leq (\geq) \int_0^1 \left|t-\frac{1}{2}\right| f((1-t)A+tB)\,dt \tag{2.4}$$
$$\leq (\geq) \int_0^1 \left|t-\frac{1}{2}\right| f(tA)\,dt + \int_0^1 \left|t-\frac{1}{2}\right| f(tB)\,dt,$$

while for $p(t) = t(1-t)$, $t \in [0,1]$, *we obtain*

$$\frac{1}{12} f(A+B) \leq (\geq) \int_0^1 t(1-t) f((1-t)A+tB)\,dt \tag{2.5}$$
$$\leq (\geq) \int_0^1 t(1-t) f(tA)\,dt + \int_0^1 t(1-t) f(tB)\,dt$$

for $A, B \geq 0$, *where* $f : [0,\infty) \to \mathbb{R}$ *is subadditive (superadditive) on* $[0,\infty)$.

Definition 1. *The function* f *defined on* $[0,\infty)$ *is called convex-starshaped if* $f(tA) \leq tf(A)$ *for all* $t \in [0,1]$ *and* $A \geq 0$. *It is called concave-starshaped if* $f(tA) \geq tf(A)$ *for all* $t \in [0,1]$ *and* $A \geq 0$.

Corollary 1. *With the assumptions of Theorem 1 and, in addition, f is convex-starshaped (concave-starshaped), then for all A, $B \geq 0$*

$$f\left(\frac{A+B}{2}\right)\int_0^1 p(t)\,dt \leq (\geq)\,\frac{1}{2}f(A+B)\int_0^1 p(t)\,dt \tag{2.6}$$
$$\leq (\geq)\int_0^1 p(t)\,f((1-t)A+tB)\,dt$$
$$\leq (\geq)\int_0^1 p(t)\,f(tA)\,dt+\int_0^1 p(t)\,f(tB)\,dt$$
$$\leq (\geq)\,[f(A)+f(B)]\int_0^1 tp(t)\,dt.$$

In particular, for $p \equiv 1$, we have

$$f\left(\frac{A+B}{2}\right) \leq (\geq)\,\frac{1}{2}f(A+B) \leq (\geq)\int_0^1 f((1-t)A+tB)\,dt \tag{2.7}$$
$$\leq (\geq)\int_0^1 f(tA)\,dt+\int_0^1 f(tB)\,dt \leq (\geq)\,\frac{f(A)+f(B)}{2}$$

for all A, $B \geq 0$.

We also have the double integral inequalities:

Theorem 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator subadditive (superadditive) on $[0,\infty)$. Then for all A, $B \geq 0$ and symmetric Lebesgue integrable and nonnegative functions p, $q:[0,1]\to[0,\infty)$, we have*

$$\frac{1}{2}f(A+B)\int_0^1 p(t)\,dt\int_0^1 q(t)\,dt \tag{2.8}$$
$$\leq (\geq)\int_0^1\int_0^1 p(t)\,q(s)\,f((1-t-s+2ts)A+(s+t-2st)B)\,dtds$$
$$\leq (\geq)\int_0^1\int_0^1 p(t)\,q(s)\,f(t(1-s)A+tsB)\,dtds$$
$$+\int_0^1\int_0^1 p(t)\,q(s)\,f(tsA+t(1-s)B)\,dtds$$
$$\leq (\geq)\,2\int_0^1\int_0^1 p(t)\,q(s)\,f(tsA)\,dtds+2\int_0^1\int_0^1 p(t)\,q(s)\,f(tsB)\,dtds.$$

In particular, for p, $q \equiv 1$, we have

$$\frac{1}{2}f(A+B) \tag{2.9}$$
$$\leq (\geq)\int_0^1\int_0^1 f((1-t-s+2ts)A+(s+t-2st)B)\,dtds$$
$$\leq (\geq)\int_0^1\int_0^1 f(t(1-s)A+tsB)\,dtds+\int_0^1\int_0^1 f(tsA+t(1-s)B)\,dtds$$
$$\leq (\geq)\,2\int_0^1\int_0^1 f(tsA)\,dtds+2\int_0^1\int_0^1 f(tsB)\,dtds.$$

Proof. If we replace A with $(1-s)A+sB$ and B with $sA+(1-s)B$, $s\in[0,1]$ in the inequality (2.1), then we get

$$\begin{aligned}&\frac{1}{2}f(A+B)\int_0^1 p(t)\,dt\\&\leq(\geq)\int_0^1 p(t)\,f((1-t)((1-s)A+sB)+t(sA+(1-s)B))\,dt\\&\leq(\geq)\int_0^1 p(t)\,f(t((1-s)A+sB))\,dt+\int_0^1 p(t)\,f(t(sA+(1-s)B))\,dt.\end{aligned}$$

If we multiply this inequality by $q(t)\geq 0$, $s\in[0,1]$, integrate and use Fubini's theorem, then we get

$$\begin{aligned}&\frac{1}{2}f(A+B)\int_0^1 p(t)\,dt\int_0^1 q(t)\,dt\\&\leq(\geq)\int_0^1\int_0^1 p(t)\,q(s)\\&\times f((1-t)((1-s)A+sB)+t(sA+(1-s)B))\,dtds\\&\leq(\geq)\int_0^1\int_0^1 p(t)\,q(s)\,f(t((1-s)A+sB))\,dtds\\&+\int_0^1\int_0^1 p(t)\,q(s)\,f(t(sA+(1-s)B))\,dtds.\end{aligned}\tag{2.10}$$

Observe that

$$\begin{aligned}&\int_0^1\int_0^1 p(t)\,q(s)\,f((1-t)((1-s)A+sB)+t(sA+(1-s)B))\,dtds\\&=\int_0^1\int_0^1 p(t)\,q(s)\,f((1-t)(1-s)A+(1-t)sB+tsA+t(1-s)B)\,dtds\\&=\int_0^1\int_0^1 p(t)\,q(s)\,f((1-t-s+2ts)A+(s+t-2st)B)\,dtds,\end{aligned}$$

$$\begin{aligned}&\int_0^1\int_0^1 p(t)\,q(s)\,f(t((1-s)A+sB))\,dtds\\&=\int_0^1\int_0^1 p(t)\,q(s)\,f(t(1-s)A+tsB)\,dtds\\&\leq\int_0^1\int_0^1 p(t)\,q(s)\,f(t(1-s)A)\,dtds+\int_0^1\int_0^1 p(t)\,q(s)\,f(tsB)\\&=\int_0^1\int_0^1 p(t)\,q(1-u)\,f(tuA)\,dtdu+\int_0^1\int_0^1 p(t)\,q(s)\,f(tsB)\\&=\int_0^1\int_0^1 p(t)\,q(u)\,f(tuA)\,dtdu+\int_0^1\int_0^1 p(t)\,q(s)\,f(tsB)\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \int_0^1 p(t) q(s) f(t(sA + (1-s)B)) dtds \\
&= \int_0^1 \int_0^1 p(t) q(s) f(tsA + t(1-s)B) dtds \\
&\leq \int_0^1 \int_0^1 p(t) q(s) f(tsA) dtds + \int_0^1 \int_0^1 p(t) q(s) f(t(1-s)B) dtds \\
&= \int_0^1 \int_0^1 p(t) q(s) f(tsA) dtds + \int_0^1 \int_0^1 p(t) q(1-u) f(tuB) dtds \\
&= \int_0^1 \int_0^1 p(t) q(s) f(tsA) dtds + \int_0^1 \int_0^1 p(t) q(u) f(tuB) dtdu.
\end{aligned}$$

By utilising (2.10) we get the desired result (2.8). □

For nonnegative operators A, B and a symmetric Lebesgue integrable and non-negative function $p : [0,1] \to [0,\infty)$ we define the function of two variables

$$f_p(A,B) := \int_0^1 p(t) f((1-t)A + tB) dt, \tag{2.11}$$

where $f : [0,\infty) \to \mathbb{R}$ is continuous.

Lemma 1. *Assume that $f : [0,\infty) \to \mathbb{R}$ is operator subadditive (superadditive) on $[0,\infty)$ and a symmetric Lebesgue integrable and nonnegative function $p : [0,1] \to [0,\infty)$, then $f_p(\cdot,\cdot)$ is operator subadditive (superadditive) as a function of two variables.*

Proof. Assume that A, B and C, D are nonnegative, then

$$\begin{aligned}
f_p((A,B) + (C,D)) &= f_p(A+C, B+D) \\
&= \int_0^1 p(t) f((1-t)(A+C) + t(B+D)) dt \\
&= \int_0^1 p(t) f((1-t)A + (1-t)C + tB + tD) dt \\
&= \int_0^1 p(t) f((1-t)A + tB + (1-t)C + tD) dt \\
&\leq \int_0^1 p(t) [f((1-t)A + tB) + f((1-t)C + tD)] dt \\
&= \int_0^1 p(t) f((1-t)A + tB) dt + \int_0^1 p(t) f((1-t)C + tD) dt \\
&= f_p(A,B) + f_p(C,D),
\end{aligned}$$

which proves that $f_p(\cdot,\cdot)$ is operator subadditive. □

Theorem 3. *Assume that $f : [0,\infty) \to \mathbb{R}$ is operator subadditive (superadditive) on $[0,\infty)$. Then for all A, B, C, $D \geq 0$ and symmetric Lebesgue integrable and* *nonnegative functions* p, $q : [0,1] \to [0,\infty)$, *we have*

$$\begin{aligned}
(2.12)\quad & \frac{1}{4}\int_0^1 q(s)\,ds\int_0^1 p(t)\,dt f(A+C+B+D) \\
&\le \frac{1}{2}\int_0^1 q(s)\,ds\int_0^1 p(t)\,f((1-t)A+tB+(1-t)C+tD)\,dt \\
&\le \int_0^1\int_0^1 q(s)\,p(t)\,f((1-t)((1-s)A+sC)+t((1-s)B+sD))\,dtds \\
&\le \int_0^1\int_0^1 q(s)\,p(t)\,f((1-t)sA+tsB)\,dtds \\
&+\int_0^1\int_0^1 q(s)\,p(t)\,f((1-t)sC+tsD)\,dtds \\
&\le \int_0^1\int_0^1 q(s)\,p(t)\,f(tsA)\,dtds+\int_0^1\int_0^1 q(s)\,p(t)\,f(tsB)\,dtds \\
&+\int_0^1\int_0^1 q(s)\,p(t)\,f(tsC)\,dtds+\int_0^1\int_0^1 q(s)\,p(t)\,f(tsD)\,dtds.
\end{aligned}$$

In particular, for p, $q \equiv 1$, *we have*

$$\begin{aligned}
(2.13)\quad & \frac{1}{4}f(A+C+B+D) \\
&\le \frac{1}{2}\int_0^1 p(t)\,f((1-t)A+tB+(1-t)C+tD)\,dt \\
&\le \int_0^1\int_0^1 f((1-t)((1-s)A+sC)+t((1-s)B+sD))\,dtds \\
&\le \int_0^1\int_0^1 f((1-t)sA+tsB)\,dtds \\
&+\int_0^1\int_0^1 f((1-t)sC+tsD)\,dtds \\
&\le \int_0^1\int_0^1 f(tsA)\,dtds+\int_0^1\int_0^1 f(tsB)\,dtds \\
&+\int_0^1\int_0^1 f(tsC)\,dtds+\int_0^1\int_0^1 f(tsD)\,dtds.
\end{aligned}$$

Proof. If we use Theorem 1 for $f_p(\cdot,\cdot)$ we have

$$\begin{aligned}
(2.14)\quad & \frac{1}{2}f_p((A,B)+(C,D))\int_0^1 q(s)\,ds \\
&\le (\ge)\int_0^1 q(s)\,f_p((1-s)(A,B)+s(C,D))\,dt \\
&\le (\ge)\int_0^1 q(s)\,f_p(s(A,B))\,ds+\int_0^1 q(s)\,f(s(C,D))\,ds
\end{aligned}$$

for all A, B, C, $D \ge 0$.

Observe that

$$\begin{aligned}
&\int_0^1 q(s) f_p((1-s)(A,B)+s(C,D))\,ds \\
&=\int_0^1 q(s) f_p((1-s)A+sC,(1-s)B+sD)\,ds \\
&=\int_0^1 q(s)\left(\int_0^1 p(t) f((1-t)((1-s)A+sC)+t((1-s)B+sD))\,dt\right) ds \\
&=\int_0^1\int_0^1 q(s) p(t) f((1-t)((1-s)A+sC)+t((1-s)B+sD))\,dtds,
\end{aligned}$$

$$\begin{aligned}
\int_0^1 q(s) f_p(s(A,B))\,ds &= \int_0^1 q(s) f_p(sA,sB)\,ds \\
&=\int_0^1 q(s)\left(\int_0^1 p(t) f((1-t)sA+tsB)\,dt\right) ds \\
&=\int_0^1\int_0^1 q(s) p(t) f((1-t)sA+tsB)\,dtds
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 q(s) f(s(C,D))\,ds &= \int_0^1 q(s) f(sC,sD)\,ds \\
&=\int_0^1 q(s)\left(\int_0^1 p(t) f((1-t)sC+tsD)\,dt\right) ds \\
&=\int_0^1\int_0^1 q(s) p(t) f((1-t)sC+tsD)\,dtds.
\end{aligned}$$

If f is operator subadditive (superadditive) on $[0,\infty)$, then we have

$$\begin{aligned}
&\int_0^1\int_0^1 q(s) p(t) f((1-t)sA+tsB)\,dtds \\
&\leq (\geq) \int_0^1\int_0^1 q(s) p(t)[f((1-t)sA)+f(tsB)]\,dtds \\
&=\int_0^1\int_0^1 q(s) p(t) f((1-t)sA)\,dtds+\int_0^1\int_0^1 q(s) p(t) f(tsB)\,dtds \\
&=\int_0^1\int_0^1 q(s) p(1-u) f(usA)\,duds+\int_0^1\int_0^1 q(s) p(t) f(tsB)\,dtds \\
&=\int_0^1\int_0^1 q(s) p(u) f(usA)\,duds+\int_0^1\int_0^1 q(s) p(t) f(tsB)\,dtds \\
&=\int_0^1\int_0^1 q(s) p(t) f(tsA)\,dtds+\int_0^1\int_0^1 q(s) p(t) f(tsB)\,dtds
\end{aligned}$$

and

$$\begin{aligned}&\int_0^1\int_0^1 q(s)\,p(t)\,f((1-t)\,sC+tsD)\,dtds\\&\le(\ge)\int_0^1\int_0^1 q(s)\,p(t)\,[f((1-t)\,sC)+f(tsD)]\,dtds\\&=\int_0^1\int_0^1 q(s)\,p(t)\,f(tsC)\,dtds+\int_0^1\int_0^1 q(s)\,p(t)\,f(tsD)\,dtds.\end{aligned}$$

This proves the second, third and fourth inequalities.

We also have

$$\begin{aligned}&\int_0^1 q(s)\,ds\int_0^1 p(t)\,f((1-t)\,A+tB+(1-t)\,C+tD)\,dt\\&=\int_0^1 q(s)\,ds\int_0^1 p(t)\,f((1-t)\,(A+C)+t\,(B+D))\,dt\\&\ge(\le)\,\frac{1}{2}\int_0^1 q(s)\,ds\int_0^1 p(t)\,f(A+C+B+D)\,,\end{aligned}$$

which proves the first inequality in (2.12). □

3. Some Related Results

In [15] the authors obtained the following result that generalizes the result of Moslehian and Najafi [12]:

Lemma 2. *Let $A_i \ge 0$, $(i=1,2,\cdots,n)$. Then the following are equivalent*

(i) *We have the inequality*

$$\sum_{1\le i\ne j\le n} A_iA_j\ge 0; \tag{3.1}$$

(ii) *For every operator convex function $f(t)$ on $[0,\infty)$ with $f(0)\le 0$ we have*

$$f\left(\sum_{i=1}^n A_i\right)\ge\sum_{i=1}^n g(A_i)\,; \tag{3.2}$$

(iii) *For every non-negative operator monotone function $f(t)$ on $[0,\infty)$*

$$f\left(\sum_{i=1}^n A_i\right)\le\sum_{i=1}^n f(A_i)\,. \tag{3.3}$$

We have:

Theorem 4. *Assume that A, $B\ge 0$ with $AB+BA\ge 0$. Let $f:[0,\infty)\to\mathbb{R}$ be an operator convex function with $f(0)\le 0$, then for any symmetric Lebesgue integrable and nonnegative function $p:[0,1]\to[0,\infty)$,*

$$\begin{aligned}\frac{1}{2}f(A+B)\int_0^1 p(t)\,dt&\ge\int_0^1 p(t)\,f((1-t)\,A+tB)\,dt\\&\ge\int_0^1 p(t)\,f(tA)\,dt+\int_0^1 p(t)\,f(tB)\,dt.\end{aligned} \tag{3.4}$$

In particular,

$$\frac{1}{2}f(A+B)\geq\int_0^1 f((1-t)A+tB)\,dt\geq\int_0^1 f(tA)\,dt+\int_0^1 f(tB)\,dt. \tag{3.5}$$

Proof. Consider the set

$$\mathcal{D}:=\{(A,B)\,|A,\ B\geq 0\text{ and }AB+BA\geq 0\}.$$

We observe that for $(A,B)\in\mathcal{D}$ and $t\in[0,1]$, $(tA,(1-t)B)\in\mathcal{D}$.
Also

$$((1-t)A+tB,tA+(1-t)B)\in\mathcal{D},$$

since

$$\begin{aligned}&((1-t)A+tB)(tA+(1-t)B)+(tA+(1-t)B)((1-t)A+tB)\\&=(1-t)tA^2+t^2BA+(1-t)^2AB+t(1-t)B^2\\&+(1-t)tA^2+(1-t)^2BA+t^2AB+t(1-t)B^2\\&=2(1-t)tA^2+\left[t^2+(1-t)^2\right](AB+BA)+2t(1-t)B^2\\&\geq 0.\end{aligned}$$

By Lemma 2 we have

$$\begin{aligned}&f((1-t)A+tB+tA+(1-t)B)\\&\geq f((1-t)A+tB)+f(tA+(1-t)B),\end{aligned}$$

namely

$$f(A+B)\geq f((1-t)A+tB)+f(tA+(1-t)B)$$

for all $t\in[0,1]$.

Also by the same lemma,

$$f((1-t)A+tB)\geq f((1-t)A)+f(tB)$$

for all $t\in[0,1]$.

Now, by multiplying with $p(t)\geq 0$ and using a similar argument to the one from the proof of Theorem 1, we get the desired result (3.4). □

The case of operator monotone functions is as follows:

Theorem 5. *Assume that A, $B\geq 0$ with $AB+BA\geq 0$. Let $f:[0,\infty)\to[0,\infty)$ be an operator monotone function, then for any symmetric Lebesgue integrable and nonnegative function $p:[0,1]\to[0,\infty)$,*

$$\begin{aligned}\frac{1}{2}f(A+B)\int_0^1 p(t)\,dt&\leq\int_0^1 p(t)f((1-t)A+tB)\,dt\\&\leq\int_0^1 p(t)f(tA)\,dt+\int_0^1 p(t)f(tB)\,dt.\end{aligned} \tag{3.6}$$

In particular,

$$\frac{1}{2}f(A+B)\leq\int_0^1 f((1-t)A+tB)\,dt\leq\int_0^1 f(tA)\,dt+\int_0^1 f(tB)\,dt. \tag{3.7}$$

Remark 2. *Assume that A, $B \geq 0$ with $AB+BA \geq 0$ and p, $q : [0,1] \to [0,\infty)$ be symmetric Lebesgue integrable and nonnegative functions. If $f : [0,\infty) \to \mathbb{R}$ is an operator convex function with $f(0) \leq 0$, then the double integral inequalities in (2.1) and (2.3) hold with the sign " $\geq$ ". If $f : [0,\infty) \to [0,\infty)$ is an operator monotone function then the double integral inequalities in (2.1) and (2.3) hold with the sign " $\leq$ ".*

4. Some Examples

The function $f : [0,\infty) \to [0,\infty)$, $f(x) = x^r$, $r \in (0,1)$ is operator monotone. By Theorem 5 we can state the following:

Proposition 1. *Assume that A, $B \geq 0$ with $AB + BA \geq 0$ and $r \in (0,1)$. Then for any symmetric Lebesgue integrable and nonnegative function $p : [0,1] \to [0,\infty)$,*

$$\frac{1}{2}\left(\int_0^1 p(t)\,dt\right)(A+B)^r \leq \int_0^1 p(t)\left((1-t)A+tB\right)^r dt \leq \left(\int_0^1 p(t)\,t^r dt\right)(A^r + B^r). \tag{4.1}$$

In particular,

$$\frac{1}{2}(A+B)^r \leq \int_0^1 \left((1-t)A+tB\right)^r dt \leq \frac{1}{r+1}(A^r+B^r). \tag{4.2}$$

The function $f : [0,\infty) \to [0,\infty)$, $f(x) = x^s$, $s \in [1,2]$ is operator convex with $f(0) = 0$. By Theorem 5 we can state the following:

Proposition 2. *Assume that A, $B \geq 0$ with $AB + BA \geq 0$ and $s \in [1,2]$. Then for any symmetric Lebesgue integrable and nonnegative function $p : [0,1] \to [0,\infty)$,*

$$\frac{1}{2}\left(\int_0^1 p(t)\,dt\right)(A+B)^s \geq \int_0^1 p(t)\left((1-t)A+tB\right)^s dt \geq \left(\int_0^1 p(t)\,t^r dt\right)(A^s + B^s). \tag{4.3}$$

In particular,

$$\frac{1}{2}(A+B)^s \geq \int_0^1 \left((1-t)A+tB\right)^s dt \geq \frac{1}{s+1}(A^s+B^s). \tag{4.4}$$

In the case of logarithmic function, we observe that $f : (0,\infty) \to \mathbb{R}$, $f(x) = x\ln x$ and $f(0) := 0$ (by continuity) is operator convex on $[0,\infty)$ and by Theorem 4 we have:

Proposition 3. *Assume that A, $B \geq 0$ with $AB+BA \geq 0$. Then for any symmetric Lebesgue integrable and nonnegative function $p : [0,1] \to [0,\infty)$,*

$$\begin{aligned} &\frac{1}{2}\left(\int_0^1 p(t)\,dt\right)(A+B)\ln(A+B) \\ &\geq \int_0^1 p(t)\left((1-t)A+tB\right)\ln\left((1-t)A+tB\right)dt \\ &\geq \int_0^1 p(t)\,tA\ln(tA)\,dt + \int_0^1 p(t)\,tB\ln(tB)\,dt. \end{aligned} \tag{4.5}$$

In particular,

$$\begin{aligned} &\frac{1}{2}(A+B)\ln(A+B) \\ &\geq \int_0^1 p(t)((1-t)A+tB)\ln((1-t)A+tB)\,dt \\ &\geq \int_0^1 tA\ln(tA)\,dt + \int_0^1 tB\ln(tB)\,dt. \end{aligned} \tag{4.6}$$

We observe that $f:(0,\infty)\to\mathbb{R}$, $f(x)=\ln x$ is operator monotone, then by Theorem 5 we have for A, $B>0$ with $AB+BA>0$ that

$$\begin{aligned} \frac{1}{2}\left(\int_0^1 p(t)\,dt\right)\ln(A+B) &\leq \int_0^1 p(t)\ln((1-t)A+tB)\,dt \\ &\leq \int_0^1 p(t)\ln(tA)\,dt + \int_0^1 p(t)\ln(tB)\,dt, \end{aligned} \tag{4.7}$$

for any symmetric Lebesgue integrable and nonnegative function $p:[0,1]\to[0,\infty)$ such that the involved integrals exist.

References

[1] R. P. Agarwal and S. S. Dragomir, A survey of Jensen type inequalities for functions of selfadjoint operators in Hilbert spaces. *Comput. Math. Appl.* 5**9** (2010), no. 12, 3785–3812.

[2] V. Bacak, T. Vildan and R. Türkmen, Refinements of Hermite-Hadamard type inequalities for operator convex functions. *J. Inequal. Appl.* **2013**, 2013:262, 10 pp.

[3] V. Darvish, S. S. Dragomir, H. M. Nazari and A. Taghavi, Some inequalities associated with the Hermite-Hadamard inequalities for operator h-convex functions. *Acta Comment. Univ. Tartu. Math.* **21** (2017), no. 2, 287–297.

[4] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type.* Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[5] S. S. Dragomir, Hermite-Hadamard's type inequalities for operator convex functions. *Appl. Math. Comput.* **218** (2011), no. 3, 766–772.

[6] S. S. Dragomir, Some Hermite-Hadamard type inequalities for operator convex functions and positive maps. *Spec. Matrices* **7** (2019), 38–51. Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 80. [Online http://rgmia.org/papers/v19/v19a80.pdf].

[7] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[8] A. G. Ghazanfari, Hermite-Hadamard type inequalities for functions whose derivatives are operator convex. *Complex Anal. Oper. Theory* **10** (2016), no. 8, 1695–1703.

[9] A. G. Ghazanfari, The Hermite-Hadamard type inequalities for operator s-convex functions. *J. Adv. Res. Pure Math.* **6** (2014), no. 3, 52–61.

[10] J. Han and J. Shi, Refinements of Hermite-Hadamard inequality for operator convex function. *J. Nonlinear Sci. Appl.* **10** (2017), no. 11, 6035–6041.

[11] B. Li, Refinements of Hermite-Hadamard's type inequalities for operator convex functions. *Int. J. Contemp. Math. Sci.* **8** (2013), no. 9-12, 463–467.

[12] M. S. Moslehian and H. Najafi, Around operator monotone functions, *Integral Equations Operator Theory* **71** (2011) 575–582.

[13] G. K. Pedersen, Operator differentiable functions. *Publ. Res. Inst. Math. Sci.* **36** (1) (2000), 139-157.

[14] A. Taghavi, V. Darvish, H. M. Nazari and S. S. Dragomir, Hermite-Hadamard type inequalities for operator geometrically convex functions. *Monatsh. Math.* **181** (2016), no. 1, 187–203.

[15] M. Uchiyama, A. Uchiyama and M. Giga, Superadditivity and derivative of operator functions, *Linear Alg. Appl.* **465** (2015)401–411

[16] M. Vivas Cortez and E. J. Hernández Hernández, Refinements for Hermite-Hadamard type inequalities for operator h-convex function. *Appl. Math. Inf. Sci.* **11** (2017), no. 5, 1299–1307.

[17] M. Vivas Cortez and E. J. Hernández Hernández, On some new generalized Hermite-Hadamard-Fejér inequalities for product of two operator h-convex functions. A*ppl. Math. Inf. Sci.* **11** (2017), no. 4, 983–992.
[18] S.-H. Wang, Hermite-Hadamard type inequalities for operator convex functions on the co-ordinates. *J. Nonlinear Sci. Appl.* **10** (2017), no. 3, 1116–1125
[19] S.-H. Wang, New integral inequalities of Hermite-Hadamard type for operator m-convex and (α, m)-convex functions. *J. Comput. Anal. Appl.* **22** (2017), no. 4, 744–753.

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