BOUNDS FOR THE DIFFERENCE OF WEIGHTED AND INTEGRAL MEANS FOR CONVEX FUNCTIONS

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Abstract. Let f be a convex function on I and $a,\ b\in I$ with $a<b$. Also, let $p:[a,b]\to[0,\infty)$ be a Lebesgue integrable and symmetric weight on $[a,b]$, namely $p(b+a-t)=p(t)$ for all $t\in[a,b]$.

In this paper we provide some upper and lower bounds for the difference

$$\frac{1}{\int_a^b p(x)\,dx}\int_a^b f(x)\,p(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx,$$

where f and p are as above. Some examples are given as well.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)dt\le\frac{f(a)+f(b)}{2},\quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [7]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [7]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality*. For a monograph devoted to this result see [5]. The recent survey paper [4] provides other related results.

Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and assume that $f'_+(a)$ and $f'_-(b)$ are finite. We recall the following improvement and reverse inequality for the first Hermite-Hadamard result that has been established in [2]

$$\begin{aligned}0&\le\frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right](b-a)\\&\le\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\le\frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right].\end{aligned} \tag{1.2}$$

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The following inequality that provides a reverse and improvement of the second Hermite-Hadamard result has been obtained in [3]

$$0 \le \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right) - f'_{-}\left(\frac{a+b}{2}\right)\right](b-a) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{8}(b-a)\left[f'_{-}(b) - f'_{+}(a)\right]. \tag{1.3}$$

The constant $\frac{1}{8}$ is best possible in both (1.2) and (1.3).

In 1906, Fejér [6], while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. *Consider the integral $\int_a^b f(t)\,p(t)\,dt$, where f is a convex function in the interval (a,b) and p is a positive function in the same interval such that*

$$p(a+t) = p(b-t), \quad 0 \le t \le \frac{1}{2}(b-a),$$

i.e., $y = p(t)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b), 0\right)$ and is normal to the t-axis. Under those conditions the following inequalities are valid:

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \le \int_a^b f(t)\,p(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt. \tag{1.4}$$

If f is concave on (a,b), then the inequalities reverse in (1.4).

In this paper we provide some upper and lower bounds for the difference

$$\frac{1}{\int_a^b p(x)\,dx}\int_a^b f(x)\,p(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx,$$

where f is convex and p is nonnegative integrable on $[a,b]$. Some examples are given as well.

2. Main Results

We start to the following identity of interest:

Lemma 1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$ such that $\int_a^b g(x)\,dx \neq 0$, then*

$$\begin{aligned} &\frac{1}{\int_a^b g(x)\,dx}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{(b-a)\int_a^b g(x)\,dx} \\ &\times \int_a^b g(x)\left(\int_a^b (x-u)\left(\int_0^1 f'((1-t)x+tu)\,dt\right)du\right)dx \\ &= \frac{1}{(b-a)\int_a^b g(x)\,dx}\int_a^b\int_a^b\int_0^1 g(x)(x-u)\,f'((1-t)x+tu)\,dt\,du\,dx. \end{aligned} \tag{2.1}$$

Proof. For all x, $u \in [a,b]$ we have

$$f(x) - f(u) = \int_u^x f'(y)\,dy = (x-u)\int_0^1 f'((1-t)u + tx)\,dt.$$

If we take the integral mean over u on $[a,b]$ we get

$$f(x) - \frac{1}{b-a}\int_a^b f(u)\,du = \frac{1}{b-a}\int_a^b (x-u)\left(\int_0^1 f'((1-t)u+tx)\,dt\right)du$$

for all $x \in [a,b]$.

If we take the weighted integral mean $\frac{1}{\int_a^b g(x)dx}\int_a^b g(x)(\cdot)\,dx$, we get

$$\frac{1}{\int_a^b g(x)\,dx}\int_a^b g(x) f(x)\,dx - \frac{1}{b-a}\int_a^b f(u)\,du \frac{1}{\int_a^b g(x)\,dx}\int_a^b g(x)\,dx$$
$$= \frac{1}{\int_a^b g(x)\,dx}\int_a^b g(x)\left[\frac{1}{b-a}\int_a^b (x-u)\left(\int_0^1 f'((1-t)u+tx)\,dt\right)du\right]dx$$
$$= \frac{1}{(b-a)\int_a^b g(x)\,dx}\int_a^b g(x)\left(\int_a^b (x-u)\left(\int_0^1 f'((1-t)x+tu)\,dt\right)du\right)dx,$$

which proves the first equality in (2.1).

The second equality follows by Fubini's theorem. □

We recall the following Čebyšev's weighted inequality for synchronous (asynchronous) integrable functions h, $k : [a,b] \to \mathbb{R}$ and integrable $q : [a,b] \to [0,\infty)$

$$\frac{1}{\int_a^b q(x)\,dx}\int_a^b h(x) k(x) q(x)\,dx \tag{2.2}$$
$$\geq (\leq) \frac{1}{\int_a^b q(x)\,dx}\int_a^b h(x) q(x)\,dx \frac{1}{\int_a^b q(x)\,dx}\int_a^b k(x) q(x)\,dx.$$

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $p : [a,b] \to [0,\infty)$ a Lebesgue integrable symmetric function on $[a,b]$ such that $\int_a^b p(x)\,dx > 0$, then*

$$\frac{1}{\int_a^b p(x)\,dx}\int_a^b f(x) p(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \tag{2.3}$$
$$\leq \frac{1}{(b-a)\int_a^b p(x)\,dx}$$
$$\times \int_a^b\int_a^b\int_0^1 p(x)\left(x - \frac{a+b}{2}\right) f'((1-t)x+tu)\,dtdudx$$
$$\leq \frac{1}{2(b-a)\int_a^b p(x)\,dx}\int_a^b p(x)\left|x-\frac{a+b}{2}\right|dx$$
$$\times \int_a^b\int_0^1 \left[f'((1-t)b+tu) - f'((1-t)a+tu)\right]dtdu$$
$$\leq \frac{1}{2}\frac{f'_-(b) - f'_+(a)}{b-a}\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left|x-\frac{a+b}{2}\right|dx.$$

Proof. Let $t\in(0,1)$ and $x\in[a,b]$. Since f is a convex function on $[a,b]$, then the function $[a,b]\ni u\mapsto f'((1-t)x+tu)$ is monotonic nondecreasing on $[a,b]$. By Čebyšev's unweighted inequality we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b (x-u)\, f'((1-t)x+tu)\,du\\ &\le \frac{1}{b-a}\int_a^b (x-u)\,du\frac{1}{b-a}\int_a^b f'((1-t)x+tu)\,du\\ &=\left(x-\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b f'((1-t)x+tu)\,du.\end{aligned}$$

If we integrate over t and use Fubini's theorem we get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b (x-u)\left(\int_0^1 f'((1-t)x+tu)\,dt\right)du\\ &\le\left(x-\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b\int_0^1 f'((1-t)x+tu)\,dtdu\end{aligned}$$

for all $x\in[a,b]$

If we take the weighted integral mean, we get

$$\begin{aligned}&\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\int_a^b (x-u)\left(\int_0^1 f'((1-t)x+tu)\,dt\right)du\\ &\le\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left(x-\frac{a+b}{2}\right)\left(\frac{1}{b-a}\int_a^b\int_0^1 f'((1-t)x+tu)\,dtdu\right)dx\\ &=\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b\int_a^b\int_0^1 p(x)\left(x-\frac{a+b}{2}\right)f'((1-t)x+tu)\,dtdudx\end{aligned}$$

and by the representation (2.1) we obtain the desired result (2.3).

Since the functions $[a,b]\ni x\mapsto f'((1-t)x+tu)$ and $x-\frac{a+b}{2}$ have the same monotonicity, then by the weighted Čebyšev's inequality we have

$$\begin{aligned}&\int_a^b p(x)\left(x-\frac{a+b}{2}\right)f'((1-t)x+tu)\,dx\\ &\ge\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left(x-\frac{a+b}{2}\right)dx\int_a^b p(x)\,f'((1-t)x+tu)\,dx=0\end{aligned}$$

since the function $p(x)\left(x-\frac{a+b}{2}\right)$ is asymmetric on $[a,b]$ and therefore

$$\int_a^b p(x)\left(x-\frac{a+b}{2}\right)dx=0.$$

Observe since f is convex, hence

$$f'((1-t)a+tu)\le f'((1-t)x+tu)\le f'((1-t)b+tu)$$

and we have

$$\begin{aligned}
0 &\leq \int_a^b p(x)\left(x-\frac{a+b}{2}\right) f'((1-t)x+tu)\,dx \\
&= \int_a^b p(x)\left(x-\frac{a+b}{2}\right) \\
&\times \left[f'((1-t)x+tu)-\frac{f'((1-t)b+tu)+f'((1-t)a+tu)}{2}\right] dx \\
&\leq \int_a^b p(x)\left|x-\frac{a+b}{2}\right| \\
&\times \left|f'((1-t)x+tu)-\frac{f'((1-t)b+tu)+f'((1-t)a+tu)}{2}\right| dx \\
&\leq \frac{f'((1-t)b+tu)-f'((1-t)a+tu)}{2}\int_a^b p(x)\left(x-\frac{a+b}{2}\right) dx.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b\int_a^b\int_0^1 p(x)\left|x-\frac{a+b}{2}\right| f'((1-t)x+tu)\,dtdudx \\
&\leq \frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b p(x)\left|x-\frac{a+b}{2}\right| dx \\
&\times \int_a^b\int_0^1 \frac{f'((1-t)b+tu)-f'((1-t)a+tu)}{2} dtdu,
\end{aligned}$$

which proves the second inequality in (2.3).

The last part follows by the fact that

$$0 \leq f'((1-t)b+tu)-f'((1-t)a+tu) \leq f'_-(b)-f'_+(a)$$

for all $t\in(0,1)$ and $u\in[a,b]$. □

Corollary 1. *With the assumptions of Theorem 2 and if f is differentiable and the derivative f' is Lipschitzian with constant $L>0$, namely*

$$|f'(u)-f'(v)| \leq L|u-v| \text{ for all } u,v\in(a,b)$$

then

$$
\begin{aligned}
(2.4)\qquad & \frac{1}{\int_a^b g(x)\,dx}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx \\
& \le \frac{1}{(b-a)\int_a^b p(x)\,dx} \\
& \times \int_a^b\int_a^b\int_0^1 p(x)\left(x-\frac{a+b}{2}\right)f'((1-t)\,x+tu)\,dtdudx \\
& \le \frac{1}{2(b-a)\int_a^b p(x)\,dx}\int_a^b p(x)\left|x-\frac{a+b}{2}\right|dx \\
& \times \int_a^b\int_0^1 \left[f'((1-t)\,b+tu)-f'((1-t)\,a+tu)\right]dtdu \\
& \le \frac{1}{4}L(b-a)\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left|x-\frac{a+b}{2}\right|dt.
\end{aligned}
$$

Proof. Observe that

$$
\begin{aligned}
& \int_a^b\int_0^1 \left[f'((1-t)\,b+tu)-f'((1-t)\,a+tu)\right]dtdu \\
& \le L\int_a^b\int_0^1 \left[(1-t)\,b+tu-(1-t)\,a-tu\right]dtdu \\
& = L(b-a)\int_a^b\int_0^1 (1-t)\,dr=\frac{1}{2}L(b-a)^2
\end{aligned}
$$

and by (2.3) we obtain the desired result (2.4). □

Theorem 3. *Let* $f:[a,b]\to\mathbb{R}$ *be a convex function and* $p:[a,b]\to[0,\infty)$ *a Lebesgue integrable function on* $[a,b]$ *such that* $\int_a^b p(x)\,dx>0,$ *then*

$$
\begin{aligned}
(2.5)\qquad & \frac{1}{b-a}\int_a^b\left(\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,(x-u)\,dx\right) \\
& \times\left(\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left(\int_0^1 f'((1-t)\,x+tu)\,dt\right)dx\right)du \\
& \le \frac{1}{\int_a^b g(x)\,dx}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx.
\end{aligned}
$$

Proof. Using Čebyšev's weighted inequality for synchronous functions in the variable x, $x-u$ and $f'((1-t)\,x+tu)$ we have

$$
\begin{aligned}
& \frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,(x-u)\,f'((1-t)\,x+tu)\,dx \\
& \ge \frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,(x-u)\,dx \\
& \times\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,f'((1-t)\,x+tu)\,dx.
\end{aligned}
$$

If we take the integral mean on $[a,b]$ this inequality, then we get

$$\begin{aligned}
&\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b\int_a^b p(x)(x-u)f'((1-t)x+tu)\,dxdu\\
&\geq \frac{1}{b-a}\int_a^b\left(\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)(x-u)\,dx\right)\\
&\times\left(\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)f'((1-t)x+tu)\,dx\right)du.
\end{aligned}$$

Also, by taking the integral on $[0,1]$ over t and use Fubini's theorem, then we get

$$\begin{aligned}
&\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b\int_a^b\int_0^1 p(x)(x-u)f'((1-t)x+tu)\,dtdxdu\\
&\geq \frac{1}{b-a}\int_a^b\left(\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)(x-u)\,dx\right)\\
&\times\left(\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left(\int_0^1 f'((1-t)x+tu)\,dt\right)dx\right)du\\
&= \frac{1}{b-a}\int_a^b\left(\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)(x-u)\,dx\right)\\
&\times\left(\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left(\int_0^1 f'((1-t)x+tu)\,dt\right)dx\right)du.
\end{aligned}$$

By making use of the representation (2.1) we deduce the inequality (2.5). □

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $p:[a,b]\to[0,\infty)$ a Lebesgue integrable function on $[a,b]$ such that $\int_a^b p(x)\,dx>0$, then*

$$\begin{aligned}
&\frac{1}{2}\frac{1}{(b-a)\int_a^b p(x)\,dx}\\
&\times\int_a^b p(x)\{(x-a)[f(x)-f(a)]-(b-x)[f(b)-f(x)]\}\,dxdt\\
&\leq \frac{1}{\int_a^b g(x)\,dx}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\\
&\leq \frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b p(x)f'(x)\left(x-\frac{a+b}{2}\right)dx.
\end{aligned} \tag{2.6}$$

Proof. Observe that

$$\begin{aligned} (2.7) \qquad & \int_a^b (x-u) f'((1-t)x+tu)\,du \\ &= \int_a^x (x-u) f'((1-t)x+tu)\,du + \int_x^b (x-u) f'((1-t)x+tu)\,du \\ &= \int_a^x (x-u) f'((1-t)x+tu)\,du - \int_x^b (u-x) f'((1-t)x+tu)\,du. \end{aligned}$$

Since f' is monotonic nondecreasing, hence

$$\begin{aligned} \int_a^x (x-u) f'((1-t)x+ta)\,du &\le \int_a^x (x-u) f'((1-t)x+tu)\,du \\ &\le \int_a^x (x-u) f'((1-t)x+tx)\,du, \end{aligned}$$

which is equivalent to

$$\begin{aligned} f'((1-t)x+ta) \int_a^x (x-u)\,du &\le \int_a^x (x-u) f'((1-t)x+tu)\,du \\ &\le f'(x) \int_a^x (x-u)\,du, \end{aligned}$$

namely

$$\begin{aligned} (2.8) \qquad \frac{1}{2} f'((1-t)x+ta)(x-a)^2 &\le \int_a^x (x-u) f'((1-t)x+tu)\,du \\ &\le \frac{1}{2} f'(x)(x-a)^2. \end{aligned}$$

Also

$$\begin{aligned} -\int_x^b (u-x) f'((1-t)x+tb)\,du &\le -\int_x^b (u-x) f'((1-t)x+tu)\,du \\ &\le -\int_x^b (u-x) f'((1-t)x+tx)\,du, \end{aligned}$$

which is equivalent to

$$\begin{aligned} -f'((1-t)x+tb) \int_x^b (u-x)\,du &\le -\int_x^b (u-x) f'((1-t)x+tu)\,du \\ &\le -f'(x) \int_x^b (u-x)\,du, \end{aligned}$$

namely

$$\begin{aligned} (2.9) \qquad -\frac{1}{2} f'((1-t)x+tb)(b-x)^2 &\le -\int_x^b (u-x) f'((1-t)x+tu)\,du \\ &\le -\frac{1}{2} f'(x)(b-x)^2. \end{aligned}$$

By using (2.7)-(2.9) we get

$$\begin{aligned}
&\frac{1}{2}f'\left((1-t)x+ta\right)(x-a)^2-\frac{1}{2}f'\left((1-t)x+tb\right)(b-x)^2\\
&\le\int_a^b(x-u)f'\left((1-t)x+tu\right)du\\
&\le\frac{1}{2}f'(x)(x-a)^2-\frac{1}{2}f'(x)(b-x)^2,
\end{aligned}$$

namely

$$\begin{aligned}
&\frac{1}{2}f'\left((1-t)x+ta\right)(x-a)^2-\frac{1}{2}f'\left((1-t)x+tb\right)(b-x)^2\\
&\le\int_a^b(x-u)f'\left((1-t)x+tu\right)du\\
&\le f'(x)(b-a)\left(x-\frac{a+b}{2}\right)
\end{aligned}$$

for all $x\in[a,b]$ and $t\in[0,1]$.

This implies that

$$\begin{aligned}
&\frac{1}{2}\frac{1}{\int_a^b p(x)\,dx}\int_a^b\int_0^1 p(x)\\
&\times\left[f'\left((1-t)x+ta\right)(x-a)^2-f'\left((1-t)x+tb\right)(b-x)^2\right]dxdt\\
&\le\frac{1}{\int_a^b p(x)\,dx}\int_a^b\int_0^1 p(x)\left(\int_a^b(x-u)f'\left((1-t)x+tu\right)du\right)dxdt\\
&\le\frac{1}{\int_a^b p(x)\,dx}\int_a^b\int_0^1 p(x)f'(x)(b-a)\left(x-\frac{a+b}{2}\right)dxdt,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{1}{2}\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left[(x-a)\int_a^x f'(y)\,dy-(b-x)\int_x^b f'(y)\,dy\right]dxdt\\
&\le\frac{1}{\int_a^b p(x)\,dx}\int_a^b\int_0^1 p(x)\left(\int_a^b(x-u)f'\left((1-t)x+tu\right)du\right)dxdt\\
&\le\frac{1}{\int_a^b p(x)\,dx}\int_a^b\int_0^1 p(x)f'(x)(b-a)\left(x-\frac{a+b}{2}\right)dxdt,
\end{aligned}$$

namely

$$\begin{aligned}
&\frac{1}{2}\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b p(x)\left[(x-a)\left[f(x)-f(a)\right]-(b-x)\left[f(b)-f(a)\right]\right]dxdt\\
&\le\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b\int_0^1 p(x)\left(\int_a^b(x-u)f'\left((1-t)x+tu\right)du\right)dxdt\\
&\le\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b\int_0^1 p(x)f'(x)(b-a)\left(x-\frac{a+b}{2}\right)dxdt,
\end{aligned}$$

By using the identity (2.1) we get the desired result (2.6). □

We have:

Corollary 2. *Let* $f:[a,b]\to\mathbb{R}$ *be a convex function and* $p:[a,b]\to[0,\infty)$ *a Lebesgue integrable symmetric function on* $[a,b]$ *such that* $\int_a^b p(x)\,dx>0$, *then*

$$\begin{aligned}
&\frac{1}{\int_a^b g(x)\,dx}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx \\
&\leq \frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b p(x)\,f'(x)\left(x-\frac{a+b}{2}\right)dx \\
&\leq \frac{1}{2}\left[\frac{f'_-(b)-f'_+(a)}{b-a}\right]\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left|x-\frac{a+b}{2}\right|dx.
\end{aligned} \tag{2.10}$$

Proof. Since the functions $f'(\cdot)$ and $\cdot-\frac{a+b}{2}$ are nondecreasing, then by the weighted integral inequality we have

$$\begin{aligned}
&\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,f'(x)\left(x-\frac{a+b}{2}\right)dx \\
&\geq \frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,f'(x)\,dx\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left(x-\frac{a+b}{2}\right)dx.
\end{aligned}$$

Since $p(\cdot)$ is symmetric, hence $p(x)\left(x-\frac{a+b}{2}\right)$ is asymmetric and we have

$$\int_a^b p(x)\left(x-\frac{a+b}{2}\right)dx=0.$$

Therefore

$$\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,f'(x)\left(x-\frac{a+b}{2}\right)dx\geq 0.$$

We then derive that

$$\begin{aligned}
0 &\leq \frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,f'(x)\left(x-\frac{a+b}{2}\right)dx \\
&= \frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left(f'(x)-\frac{f'_+(a)+f'_-(b)}{2}\right)\left(x-\frac{a+b}{2}\right)dx \\
&= \frac{1}{\int_a^b p(x)\,dx}\left|\int_a^b p(x)\left(f'(x)-\frac{f'_+(a)+f'_-(b)}{2}\right)\left(x-\frac{a+b}{2}\right)dx\right| \\
&\leq \frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left|f'(x)-\frac{f'_+(a)+f'_-(b)}{2}\right|\left|x-\frac{a+b}{2}\right|dx \\
&\leq \frac{1}{2}\left[f'_-(b)-f'_+(a)\right]\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left|x-\frac{a+b}{2}\right|dx,
\end{aligned}$$

which together with (2.6) gives (2.10). □

We also have:

Corollary 3. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $p:[a,b]\to[0,\infty)$ a Lebesgue integrable symmetric function on $[a,b]$ such that $\int_a^b p(x)\,dx>0$. If there exists a constant $L_{\frac{a+b}{2}}>0$, such that*

$$\left|f'(x)-f'\left(\frac{a+b}{2}\right)\right|\le L_{\frac{a+b}{2}}\left|x-\frac{a+b}{2}\right| \tag{2.11}$$

for almost every $x\in[a,b]$, then

$$\begin{aligned}&\frac{1}{\int_a^b g(x)\,dx}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\\&\le\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b p(x)\,f'(x)\left(x-\frac{a+b}{2}\right)dx\\&\le L_{\frac{a+b}{2}}\frac{1}{(b-a)\int_a^b p(x)\,dx}\int_a^b p(x)\left(x-\frac{a+b}{2}\right)^2dx.\end{aligned} \tag{2.12}$$

Proof. We have

$$\begin{aligned}0&\le\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,f'(x)\left(x-\frac{a+b}{2}\right)dx\\&=\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left(f'(x)-f'\left(\frac{a+b}{2}\right)\right)\left(x-\frac{a+b}{2}\right)dx\\&=\frac{1}{\int_a^b p(x)\,dx}\left|\int_a^b p(x)\left(f'(x)-f'\left(\frac{a+b}{2}\right)\right)\left(x-\frac{a+b}{2}\right)dx\right|\\&\le\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left|f'(x)-f'\left(\frac{a+b}{2}\right)\right|\left|x-\frac{a+b}{2}\right|dx\\&\le L_{\frac{a+b}{2}}\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\left(x-\frac{a+b}{2}\right)^2dx\end{aligned}$$

which together with (2.6) gives (2.12). □

3. An Example

We consider the symmetric function $p(x)=\left|x-\frac{a+b}{2}\right|$, $x\in[a,b]$. If f is convex on $[a,b]$, then by (2.3) we get

$$\begin{aligned}&\frac{4}{(b-a)^2}\int_a^b f(x)\left|x-\frac{a+b}{2}\right|dx-\frac{1}{b-a}\int_a^b f(x)\,dx\\&\le\frac{4}{(b-a)^3}\\&\times\int_a^b\int_a^b\int_0^1\left|x-\frac{a+b}{2}\right|\left(x-\frac{a+b}{2}\right)f'((1-t)x+tu)\,dtdudx\\&\le\frac{1}{6}\int_a^b\int_0^1\left[f'((1-t)b+tu)-f'((1-t)a+tu)\right]dtdu\\&\le\frac{1}{6}\left[f'_-(b)-f'_+(a)\right].\end{aligned} \tag{3.1}$$

From the inequality (2.10) we have

$$\frac{4}{(b-a)^2}\int_a^b f(x)\left|x-\frac{a+b}{2}\right|dx-\frac{1}{b-a}\int_a^b f(x)\,dx \tag{3.2}$$
$$\leq \frac{4}{(b-a)^2}\int_a^b f'(x)\left|x-\frac{a+b}{2}\right|\left(x-\frac{a+b}{2}\right)dx$$
$$\leq \frac{1}{6}\left[f'_-(b)-f'_+(a)\right].$$

If there exists a constant $L_{\frac{a+b}{2}}>0$, such that the condition (2.11) holds for f', then by (2.12) we get

$$\frac{4}{(b-a)^2}\int_a^b f(x)\left|x-\frac{a+b}{2}\right|dx-\frac{1}{b-a}\int_a^b f(x)\,dx \tag{3.3}$$
$$\leq \frac{4}{(b-a)^3}\int_a^b f'(x)\left|x-\frac{a+b}{2}\right|\left(x-\frac{a+b}{2}\right)dx$$
$$\leq \frac{1}{8}L_{\frac{a+b}{2}}(b-a).$$

The interested reader can also consider the case of symmetric weight $p(x)=(x-a)(b-x)$, $x\in[a,b]$. We omit the details.

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