REVERSES AND REFINEMENTS OF FIRST FÉJER'S INEQUALITY FOR TWICE DIFFERENTIABLE CONVEX FUNCTIONS

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Abstract. In this paper we provide upper and lower bounds for the first Féjer's difference

$$\int_a^b p(t) f(t) \, dt - f\left(\frac{a+b}{2}\right) \int_a^b p(t) \, dt$$

in the case of twice differentiable convex functions under various assumptions for the second derivative f'' and $p : [a,b] \to [0,\infty)$ a Lebesgue integrable and symmetric function on $[a,b]$.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(t) dt \le \frac{f(a)+f(b)}{2}, \quad a,\ b \in \mathbb{R},\ a < b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [29]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [23]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [29]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality*. For a monograph devoted to this result see [27]. The recent survey paper [26] provides other related results.

Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ and assume that $f'_+(a)$ and $f'_-(b)$ are finite. We recall the following improvement and reverse inequality for the first Hermite-Hadamard result that has been established in [24]

$$\begin{aligned} 0 &\le \frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a) \\ &\le \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \le \frac{1}{8}(b-a)\left[f'_-(b) - f'_+(a)\right]. \end{aligned} \tag{1.2}$$

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The following inequality that provides a reverse and improvement of the second Hermite-Hadamard result has been obtained in [25]

$$
\begin{aligned}
0 &\leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a) \\
&\leq \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{8}(b-a)\left[f'_{-}(b)-f'_{+}(a)\right].
\end{aligned}
\tag{1.3}
$$

The constant $\frac{1}{8}$ is best possible in both (1.2) and (1.3).

In 1906, Féjer [28], while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. *Consider the integral $\int_a^b f(t)\,p(t)\,dt$, where f is a convex function in the interval (a,b) and p is a positive function in the same interval such that*

$$
p(a+t)=p(b-t),\quad 0\leq t\leq \frac{1}{2}(b-a),
$$

i.e., $y=p(t)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the t-axis. Under those conditions the following inequalities are valid:

$$
f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \leq \int_a^b f(t)\,p(t)\,dt \leq \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt.
\tag{1.4}
$$

If f is concave on (a,b), then the inequalities reverse in (1.4).

We have the following refinement and reverse of Fejer's first inequality:

Theorem 2. *Let f be a convex function on I and $a,\ b\in I$, with $a<b$. If $p:[a,b]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t)=p(t)$ for all $t\in[a,b]$, then*

$$
\begin{aligned}
0 &\leq \frac{1}{2}\int_a^b \left|t-\frac{a+b}{2}\right| p(t)\,dt\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right] \\
&\leq \int_a^b p(t)\,f(t)\,dt-\left(\int_a^b p(t)\,dt\right) f\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{2}\int_a^b \left|t-\frac{a+b}{2}\right| p(t)\,dt\left[f'_{-}(b)-f'_{+}(a)\right].
\end{aligned}
\tag{1.5}
$$

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Theorem 3. *Let $I\subset\mathbb{R}$ be a closed interval, $a\in I$ and let n be a positive integer. If $f:I\longrightarrow\mathbb{C}$ is such that the n-derivative $f^{(n)}$ is absolutely continuous on I, then for each $x\in I$*

$$
f(x)=T_n(f;a,x)+R_n(f;a,x),
\tag{1.6}
$$

where $T_n(f;c,y)$ is Taylor's polynomial, i.e.,

$$
T_n(f;a,x):=\sum_{k=0}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a).
\tag{1.7}
$$

Note that $f^{(0)} := f$ *and* $0! := 1$ *and the remainder is given by*

$$R_n(f; a, x) := \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)\, dt. \tag{1.8}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For related results, see [1]-[5], [10]-[13], [17]-[18] and [21].

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0,1]$ that

$$\int_c^d h(t)\, dt = (d-c) \int_0^1 h((1-s)c + sd)\, ds.$$

Therefore,

$$\begin{aligned}
&\int_a^x f^{(n+1)}(t)(x-t)^n\, dt \\
&= (x-a) \int_0^1 f^{(n+1)}((1-s)a + sx)(x - (1-s)a - sx)^n\, ds \\
&= (x-a)^{n+1} \int_0^1 f^{(n+1)}((1-s)a + sx)(1-s)^n\, ds.
\end{aligned}$$

The identity (1.6) can then be written as

$$\begin{aligned}
f(x) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k \\
&\quad + \frac{1}{n!}(x-a)^{n+1} \int_0^1 f^{(n+1)}((1-s)a + sx)(1-s)^n\, ds
\end{aligned} \tag{1.9}$$

for all x, $a \in I$.

In this paper we provide upper and lower bounds for the first Féjer's difference

$$\int_a^b p(t) f(t)\, dt - f\left(\frac{a+b}{2}\right) \int_a^b p(t)\, dt$$

in the case of twice differentiable convex functions under various assumptions for the second derivative f'' and $p : [a,b] \to [0,\infty)$ a Lebesgue integrable and symmetric function on $[a,b]$.

2. Main Results

We have:

Theorem 4. *Let* f *be a twice differentiable convex function on* I *and* a, $b \in I$, *with* $a < b$. *If* $p : [a,b] \to [0,\infty)$ *is Lebesgue integrable and symmetric, namely*

$p(b+a-t)=p(t)$ for all $t\in[a,b]$, then

$$0\leq\inf_{t\in[a,b]}\left(\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds\right)\times\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt\leq\int_a^b p(t)f(t)\,dt-f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt\leq\sup_{t\in[a,b]}\left(\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds\right)\times\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt. \tag{2.1}$$

In particular, if $p\equiv 1$, then

$$0\leq\frac{1}{12}(b-a)^3\inf_{t\in[a,b]}\left(\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds\right)\leq\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)(b-a)\leq\frac{1}{12}(b-a)^3\sup_{t\in[a,b]}\left(\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds\right). \tag{2.2}$$

Proof. We have from (1.9) for $n=2$ that

$$f(x)=f(c)+f'(c)(x-c)+(x-c)^2\int_0^1 f''((1-s)c+sx)(1-s)\,ds$$

for all $x,\ c\in[a,b]$, where f is such that f' is absolutely continuos on $[a,b]$.

If we replace c with $\frac{a+b}{2}$ and x with t, then we get

$$f(t)=f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)\left(t-\frac{a+b}{2}\right)+\left(t-\frac{a+b}{2}\right)^2\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds \tag{2.3}$$

for all $t\in[a,b]$.

If we multiply (2.3) with $p(t)\geq 0$ and integrate, then we get

$$\int_a^b p(t)f(t)\,dt=f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt+f'\left(\frac{a+b}{2}\right)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)dt+\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2\left(\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds\right)dt. \tag{2.4}$$

Since the function $p(t)\left(t-\frac{a+b}{2}\right)$ is asymmetric on $[a,b]$, hence

$$\int_a^b p(t)\left(t-\frac{a+b}{2}\right)dt=0$$

and by (2.4) we get

$$\begin{aligned}(2.5)\qquad &\int_a^b p(t) f(t)\,dt - f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \\ &= \int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2\left(\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds\right)dt.\end{aligned}$$

Observe that for all $t\in[a,b]$ we have

$$\begin{aligned}0 &\le \inf_{t\in[a,b]}\left(\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds\right)\\ &\le \int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds\\ &\le \sup_{t\in[a,b]}\left(\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds\right)\end{aligned}$$

and by the equality (2.5) we get (2.1).

Since

$$\int_a^b \left(t-\frac{a+b}{2}\right)^2 dt = \frac{1}{12}(b-a)^3,$$

hence by (2.1) we get (2.2). □

Corollary 1. *With the assumptions of Theorem 4 and if there exists the constants* $\Gamma>\gamma>0$ *such that* $\Gamma\ge f''(x)\ge\gamma$ *for almost every* $x\in(a,b)$, *then*

$$\begin{aligned}(2.6)\quad 0 &\le \frac{1}{2}\gamma\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt \le \int_a^b p(t) f(t)\,dt - f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt\\ &\le \frac{1}{2}\Gamma\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt\end{aligned}$$

and

$$(2.7)\qquad 0\le \frac{1}{24}(b-a)^3\gamma \le \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \le \frac{1}{24}\Gamma(b-a)^3.$$

Proof. From (2.1) we get

$$\begin{aligned}0 &\le \gamma\left(\int_0^1 (1-s)\,ds\right)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt\\ &\le \int_a^b p(t) f(t)\,dt - \left(\int_a^b p(t)\,dt\right) f\left(\frac{a+b}{2}\right)\\ &\le \Gamma\left(\int_0^1 (1-s)\,ds\right)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt,\end{aligned}$$

which is equivalent to (2.6). □

Corollary 2. *With the assumptions of Theorem 4 and if f'' is monotonic nondecreasing on (a,b), then*

$$\begin{aligned}
0 &\leq \frac{2}{b-a}\left[\frac{2}{b-a}\left(f(a)-f\left(\frac{a+b}{2}\right)\right)+f'\left(\frac{a+b}{2}\right)\right] \\
&\quad\times\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt \\
&\leq \int_a^b p(t)\,f(t)\,dt - f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \\
&\leq \frac{2}{b-a}\left[\frac{2}{b-a}\left(f(b)-f\left(\frac{a+b}{2}\right)\right)-f'\left(\frac{a+b}{2}\right)\right] \\
&\quad\times\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
0 &\leq \frac{1}{6}\left[\frac{2}{b-a}\left(f(a)-f\left(\frac{a+b}{2}\right)\right)+f'\left(\frac{a+b}{2}\right)\right](b-a)^2 \\
&\leq \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \\
&\leq \frac{1}{6}\left[\frac{2}{b-a}\left(f(b)-f\left(\frac{a+b}{2}\right)\right)-f'\left(\frac{a+b}{2}\right)\right](b-a)^2 .
\end{aligned} \tag{2.9}$$

Proof. Observe that, by the monotonicity of f'', we have for all $t\in[a,b]$

$$\begin{aligned}
&\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds \\
&\geq \int_0^1 f''\left((1-s)\frac{a+b}{2}+sa\right)(1-s)\,ds \\
&= \int_0^1 f''\left(\frac{a+b}{2}-s\frac{b-a}{2}\right)(1-s)\,ds \\
&= -\frac{2}{b-a}\int_0^1 (1-s)\,d\left(f'\left(\frac{a+b}{2}-s\frac{b-a}{2}\right)\right) \\
&= -\frac{2}{b-a}\left[(1-s)\,f'\left(\frac{a+b}{2}-s\frac{b-a}{2}\right)\bigg|_0^1+\int_0^1 f'\left(\frac{a+b}{2}-s\frac{b-a}{2}\right)ds\right] \\
&= -\frac{2}{b-a}\left[\int_0^1 f'\left(\frac{a+b}{2}-s\frac{b-a}{2}\right)ds - f'\left(\frac{a+b}{2}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{2}{b-a}\left[-\frac{2}{b-a}\int_0^1 df\left(\frac{a+b}{2}-s\frac{b-a}{2}\right)-f'\left(\frac{a+b}{2}\right)\right] \\
&= -\frac{2}{b-a}\left[-\frac{2}{b-a}\left[f\left(\frac{a+b}{2}-s\frac{b-a}{2}\right)\bigg|_0^1\right]-f'\left(\frac{a+b}{2}\right)\right] \\
&= \frac{2}{b-a}\left[f'\left(\frac{a+b}{2}\right)+\frac{2}{b-a}\left(f(a)-f\left(\frac{a+b}{2}\right)\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds \\
&\leq \int_0^1 f''\left((1-s)\frac{a+b}{2}+sb\right)(1-s)\,ds \\
&= \int_0^1 f''\left(\frac{a+b}{2}+s\frac{b-a}{2}\right)(1-s)\,ds \\
&= \frac{2}{b-a}\int_0^1 (1-s)\,df'\left(\frac{a+b}{2}+s\frac{b-a}{2}\right) \\
&= \frac{2}{b-a}\left[(1-s)\,f'\left(\frac{a+b}{2}+s\frac{b-a}{2}\right)\Big|_0^1+\int_0^1 f'\left(\frac{a+b}{2}+s\frac{b-a}{2}\right)ds\right] \\
&= \frac{2}{b-a}\left[\int_0^1 f'\left(\frac{a+b}{2}+s\frac{b-a}{2}\right)ds-f'\left(\frac{a+b}{2}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{b-a}\left[\frac{2}{b-a}\int_0^1 df\left(\frac{a+b}{2}+s\frac{b-a}{2}\right)-f'\left(\frac{a+b}{2}\right)\right] \\
&= \frac{2}{b-a}\left[\frac{2}{b-a}\left[f\left(\frac{a+b}{2}+s\frac{b-a}{2}\right)\Big|_0^1\right]-f'\left(\frac{a+b}{2}\right)\right] \\
&= \frac{2}{b-a}\left[\frac{2}{b-a}\left(f(b)-f\left(\frac{a+b}{2}\right)\right)-f'\left(\frac{a+b}{2}\right)\right].
\end{aligned}$$

Therefore, by (2.1) we get (2.8). □

Corollary 3. *With the assumptions of Theorem 4 and if f'' is convex on (a,b), then*

$$\begin{aligned}
0 &\leq \frac{1}{2}\inf_{t\in[a,b]} f''\left(\frac{a+b+t}{3}\right)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt \\
&\leq \int_a^b p(t)\,f(t)\,dt - f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \\
&\leq \frac{1}{3}\left(f''\left(\frac{a+b}{2}\right)+\frac{1}{2}\sup_{t\in[a,b]} f''(t)\right)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt.
\end{aligned} \tag{2.10}$$

In particular, if $p\equiv 1$, then

$$\begin{aligned}
0 &\leq \frac{1}{24}(b-a)^3\inf_{t\in[a,b]} f\left(\frac{a+b+t}{3}\right) \\
&\leq \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \\
&\leq \frac{1}{36}(b-a)^3\left(f''\left(\frac{a+b}{2}\right)+\frac{1}{2}\sup_{t\in[a,b]} f''(t)\right).
\end{aligned} \tag{2.11}$$

Proof. If f'' is convex on (a,b), then by Jensen's integral inequality we have for $t\in[a,b]$ that

$$\begin{aligned}
&\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds \\
&\geq \int_0^1 (1-s)\,ds f''\left(\frac{\int_0^1\left[(1-s)\frac{a+b}{2}+st\right](1-s)\,ds}{\int_0^1(1-s)\,ds}\right) \\
&= \frac{1}{2}f''\left(\frac{\frac{a+b}{2}\int_0^1(1-s)^2\,ds+t\int_0^1 s(1-s)\,ds}{\frac{1}{2}}\right) \\
&= \frac{1}{2}f''\left(\frac{\frac{a+b}{6}+\frac{t}{6}}{\frac{1}{2}}\right)=\frac{1}{2}f''\left(\frac{a+b+t}{3}\right).
\end{aligned}$$

Also, by the convexity of f'' we have

$$\begin{aligned}
&\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds \\
&\leq \int_0^1\left[(1-s)f''\left(\frac{a+b}{2}\right)+sf''(t)\right](1-s)\,ds \\
&= \frac{1}{3}f''\left(\frac{a+b}{2}\right)+\frac{1}{6}f''(t)
\end{aligned}$$

for $t\in[a,b]$.

Therefore, by (2.1) we get (2.10). $\square$

We also have:

Theorem 5. *Let f be a twice differentiable convex function on I and $a,b\in I$, with $a<b$ while $p:[a,b]\to[0,\infty)$ is Lebesgue integrable and symmetric. If f'' is convex on $[a,b]$, then*

$$\begin{aligned}
0&\leq\frac{1}{2}f''\left(\frac{a+b}{2}\right)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2dt \\
&\leq\int_a^b p(t)f(t)\,dt-f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \\
&\leq\frac{1}{3}\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{4}\right]\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2dt.
\end{aligned}\tag{2.12}$$

In particular, we have

$$\begin{aligned}
0&\leq\frac{1}{24}f''\left(\frac{a+b}{2}\right)(b-a)^3 \\
&\leq\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)(b-a) \\
&\leq\frac{1}{36}\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{4}\right](b-a)^3.
\end{aligned}\tag{2.13}$$

Proof. From (2.5) and Fubini theorem we have

$$\begin{aligned}(2.14)\qquad & \int_a^b p(t) f(t)\,dt - f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \\ &= \int_0^1 \left(\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 f''\left((1-s)\frac{a+b}{2}+st\right)dt\right)(1-s)\,ds \\ &= K.\end{aligned}$$

Since for all $s \in [0,1]$ the function $[a,b] \ni t \mapsto f''\left((1-s)\frac{a+b}{2}+st\right)$ is convex and the function $[a,b] \ni t \mapsto p(t)\left(t-\frac{a+b}{2}\right)^2$ is symmetric on $[a,b]$, then by Féjer inequality we have

$$\begin{aligned}(2.15)\quad & f''\left(\frac{a+b}{2}\right)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt \\ &= f''\left((1-s)\frac{a+b}{2}+s\frac{a+b}{2}\right)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt \\ &\le \int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 f''\left((1-s)\frac{a+b}{2}+st\right)dt \\ &\le \frac{f''\left((1-s)\frac{a+b}{2}+sa\right)+f''\left((1-s)\frac{a+b}{2}+sb\right)}{2}\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt \\ &\le \left[(1-s)f''\left(\frac{a+b}{2}\right)+s\frac{f''(a)+f''(b)}{2}\right]\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt.\end{aligned}$$

If we multiply (2.15) by $(1-s)$ and integrate, then we get

$$\begin{aligned}& f''\left(\frac{a+b}{2}\right)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt\int_0^1(1-s)\,ds \\ &\le K \\ &\le \left[f''\left(\frac{a+b}{2}\right)\int_0^1(1-s)^2\,ds+\frac{f''(a)+f''(b)}{2}\int_0^1(1-s)\,s\,ds\right] \\ &\times\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt \\ &= \left[\frac{1}{3}f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{12}\right]\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt,\end{aligned}$$

which is equivalent to (2.12). □

3. An Example for Symmetric Functions

Consider the symmetric function $p(t) = \left|t-\frac{a+b}{2}\right|$, $t \in [a,b]$. Observe that

$$\int_a^b p(t)\,dt = \int_a^b \left|t-\frac{a+b}{2}\right| dt = \frac{1}{4}(b-a)^2$$

and

$$\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt = \int_a^b \left|t-\frac{a+b}{2}\right|\left(t-\frac{a+b}{2}\right)^2 dt$$
$$= \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)^3 dt = \frac{1}{32}(b-a)^4.$$

Let f be a twice differentiable convex function on I and $a,\, b \in I$, with $a<b$, then by (2.1) we get

$$\begin{aligned}(3.1)\qquad 0 &\le \frac{1}{32}(b-a)^4 \inf_{t\in[a,b]}\left(\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds\right)\\ &\le \int_a^b \left|t-\frac{a+b}{2}\right| f(t)\,dt - \frac{1}{4}f\left(\frac{a+b}{2}\right)(b-a)^2\\ &\le \frac{1}{32}(b-a)^4 \sup_{t\in[a,b]}\left(\int_0^1 f''\left((1-s)\frac{a+b}{2}+st\right)(1-s)\,ds\right).\end{aligned}$$

If there exists the constants $\Gamma > \gamma > 0$ such that $\Gamma \ge f''(x) \ge \gamma$ for almost every $x \in (a,b)$, then by (2.6)

$$\begin{aligned}(3.2)\qquad 0 &\le \frac{1}{64}\gamma(b-a)^4 \le \int_a^b \left|t-\frac{a+b}{2}\right| f(t)\,dt - \frac{1}{4}f\left(\frac{a+b}{2}\right)(b-a)^2\\ &\le \frac{1}{64}\Gamma(b-a)^4.\end{aligned}$$

If f'' is monotonic nondecreasing on (a,b), then by (2.8)

$$\begin{aligned}(3.3)\qquad 0 &\le \frac{1}{32}\left[\frac{2}{b-a}\left(f(a)-f\left(\frac{a+b}{2}\right)\right)+f'\left(\frac{a+b}{2}\right)\right](b-a)^3\\ &\le \int_a^b \left|t-\frac{a+b}{2}\right| f(t)\,dt - \frac{1}{4}f\left(\frac{a+b}{2}\right)(b-a)^2\\ &\le \frac{1}{32}\left[\frac{2}{b-a}\left(f(b)-f\left(\frac{a+b}{2}\right)\right)-f'\left(\frac{a+b}{2}\right)\right](b-a)^3.\end{aligned}$$

If f'' is convex on (a,b), then by (2.10)

$$\begin{aligned}(3.4)\qquad 0 &\le \frac{1}{64}\inf_{t\in[a,b]} f''\left(\frac{a+b+t}{3}\right)(b-a)^4\\ &\le \int_a^b \left|t-\frac{a+b}{2}\right| f(t)\,dt - \frac{1}{4}f\left(\frac{a+b}{2}\right)(b-a)^2\\ &\le \frac{1}{96}\left(f''\left(\frac{a+b}{2}\right)+\frac{1}{2}\sup_{t\in[a,b]} f''(t)\right)(b-a)^4.\end{aligned}$$

Finally, if f'' is convex on (a,b), then by (2.12) we get

$$
\begin{aligned}
0 &\leq \frac{1}{64} f''\left(\frac{a+b}{2}\right)(b-a)^4 \\
&\leq \int_a^b \left|t-\frac{a+b}{2}\right| f(t)\,dt - \frac{1}{4} f\left(\frac{a+b}{2}\right)(b-a)^2 \\
&\leq \frac{1}{96}\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{4}\right](b-a)^4 .
\end{aligned} \tag{3.5}
$$

4. Examples for Exponential and Logarithm

We consider the exponential function $f(x)=\exp(\alpha x)$, $x\in\mathbb{R}$. We have $f''(x)=\alpha^2\exp(\alpha x)$, which shows that f'' is also convex. Also

$$
\begin{aligned}
E_1(\alpha;a,b) := \alpha^2 &\begin{cases} \exp(\alpha a), & \alpha<0 \\ \\ \exp(\alpha b) & \alpha>0 \end{cases} \\
&\leq f''(x) \\
&\leq \alpha^2 \begin{cases} \exp(\alpha a), & \alpha<0 \\ \\ \exp(\alpha b), & \alpha>0 \end{cases} := E_2(\alpha;a,b)
\end{aligned}
$$

for $x\in[a,b]$.

From the inequality (2.6) we get

$$
\begin{aligned}
0 &\leq \frac{1}{2}E_1(\alpha;a,b)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt \\
&\leq \int_a^b p(t)\exp(\alpha t)\,dt - \exp\left[\alpha\left(\frac{a+b}{2}\right)\right]\int_a^b p(t)\,dt \\
&\leq \frac{1}{2}E_2(\alpha;a,b)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt,
\end{aligned} \tag{4.1}
$$

where $p:[a,b]\to[0,\infty)$ is Lebesgue integrable and symmetric on $[a,b]$.

From the inequality (2.12) we get

$$
\begin{aligned}
0 &\leq \frac{1}{2}\alpha^2\exp\left(\alpha\left(\frac{a+b}{2}\right)\right)\int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt \\
&\leq \int_a^b p(t)\exp(\alpha t)\,dt - \exp\left[\alpha\left(\frac{a+b}{2}\right)\right]\int_a^b p(t)\,dt \\
&\leq \frac{1}{3}\alpha^2\left[\exp\left(\alpha\left(\frac{a+b}{2}\right)\right)+\frac{\exp(\alpha a)+\exp(\alpha b)}{4}\right] \\
&\times \int_a^b p(t)\left(t-\frac{a+b}{2}\right)^2 dt,
\end{aligned} \tag{4.2}
$$

where $p:[a,b]\to[0,\infty)$ is Lebesgue integrable and symmetric on $[a,b]$.

Now, consider the function $f(t):=-\ln t$, $t\in[a,b]\subset(0,\infty)$. This is convex and $f''(t)=\frac{1}{t^2}$, which is also convex on $[a,b]$.

By the inequality (2.6) we have

$$0 \le \frac{1}{2b^2} \int_a^b p(t) \left(t - \frac{a+b}{2} \right)^2 dt \tag{4.3}$$
$$\le \ln \left(\frac{a+b}{2} \right) \int_a^b p(t)\, dt - \int_a^b p(t) \ln t dt$$
$$\le \frac{1}{2a^2} \int_a^b p(t) \left(t - \frac{a+b}{2} \right)^2 dt,$$

where $p : [a, b] \to [0, \infty)$ is Lebesgue integrable and symmetric on $[a, b]$.

From the inequality (2.12) we have

$$0 \le \frac{2}{(a+b)^2} \int_a^b p(t) \left(t - \frac{a+b}{2} \right)^2 dt \tag{4.4}$$
$$\le \ln \left(\frac{a+b}{2} \right) \int_a^b p(t)\, dt - \int_a^b p(t) \ln t dt$$
$$\le \frac{1}{3} \left[\frac{4}{(a+b)^2} + \frac{a^2+b^2}{4a^2b^2} \right] \int_a^b p(t) \left(t - \frac{a+b}{2} \right)^2 dt,$$

where $p : [a, b] \to [0, \infty)$ is Lebesgue integrable and symmetric on $[a, b]$.

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