Effectiveness of Cannon and Composite sets of Polynomials of two complex variables in Faber regions

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Abstract. The paper obtains conditions for effectiveness of Cannon and Composite sets of polynomials of two complex variables in Faber regions. It generalizes to these regions the results of Nassif on composite sets in balls of centre origin whose constituents are also Cannon sets.

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1. **Introduction**

Let C^k; k=1,2, be a Faber curve in the z_k- plane and suppose that the corresponding Faber transformation is

$$z_k = \phi_k(t_k) = t_k + \sum_{n=1}^{\infty} a_n^k t_k^{-n} \tag{1.1}$$

where the function $\phi_k(t_k)$is regular and one-to-one for $|t_k| > T_k$. Thus the curve C^k is the map in the z_k-plane of the crcle $|t_k| > \gamma_k$ by the transformation (1.1) where $\gamma_k > T_k$. For $r > T_k$the map of the circle $|t_k| = r$ is the curve C_r^k so that C^k is actually $C_{\gamma_k}^k$. Following Breadze [2] the product set

$$B_{r_1,r_2} = D(C'_{r_1} x D(C''_{r_2}), r_k > T_k \tag{1.2}$$

and its closure $\bar{B}_{r_1,r_2}$ are called Faber regions in the space C^2 of the two complex variables z_1 and z_2. For studies on Faber transformation and regions (see Newns [4],Ullman [5]).An open polycylinder in C^2 is the open connected set

$$\Gamma_{r_1,r_2} = \{(z_1, z_2) : |z_k| < r_k\}; r_k > 0, \tag{1.3}$$

and its closure is denoted $\bar{\Gamma}_{r_1,r_2}$. By reasoning as in [1], [2] and [4], we take as a Banach space the class of functions regular in Faber regions B_{r_1,r_2} or $\bar{B}_{r_1,r_2}$, and let a norm defined on this space be given by

$$S(F) = sup_{\bar{B}_{\rho_1,\rho_2}} |F(z_1, z_2)| = M'(F; \rho_1, \rho_2) \tag{1.4}$$

for all functions F regular in $B_{r_1,r_2}, \rho_k < r_k$ and M' is the maximum modulus in Faber regions. Since $F(z_1, z_2)$ is regular in $\bar{B}_{R_1,R_2}$ then it admits the Faber series

$$F(z_1, z_2) = \sum_{m;n=0}^{\infty} a_{m,n} f_m^1(z_1) f_n^2(z_2) \tag{1.5}$$

where

$$a_{m,n} = \frac{-1}{4\pi^2} \int_{|t_1|=r_1} \int_{|t_2|=r_2} F\{\phi_1(t_1), \phi_2(t_2)\} t_1^{-m-1} t_2^{-n-1} dt_1 dt_2 \tag{1.6}$$

$m, n \geq 0, T_k < r_k < R_k$,
and

$$\{f_m^1(z_1)\} \text{ and } \{f_n^2(z_2)\} \tag{1.7}$$

are the sets of Faber polynomials corresponding to the respective transformations $z_1 = \phi_1(t_1)$ and $z_2 = \phi_2(t_2)$.
In view of the representation (1.5), the base for this space is the set $\{f_m^1(z_1).f_n^2(z_2)\}$. Thus, if $\{P_j(z_1, z_2)\}$ is a sequence of basic set of polynomials of the two complex variables z_1 and z_2, we can write the unique representation

$$f_m^1(z_1).f_n^2(z_2) = \sum_{j=0} \Pi_j^{m,n} P_j(z_1, z_2), m, n \geq 0. \tag{1.8}$$

If $F(z_1, z_2)$ is a function regular in B_{r_1,r_2} then the basic series associated with F is given by

$$F(z_1, z_2) \sum_{j=0} \Pi_j(F) P_j(z_1, z_2), \tag{1.9}$$

where

$$\Pi_j(F) = \sum_{m,n=0} a_{m,n} \Pi_j^{m,n}. \tag{1.10}$$

In order to formulate conditions for effectiveness of basic sets in Faber regions we shall consider only Cannon sets of polynomials, for which, with analogy to the single variable case, the number $N_{m,n}$ of non-zero coefficients in (1.8) satisfy the condition that

$$\lim_{m+n\infty} N_{m,n}^{\frac{1}{m+n}} = 1.$$

The Cannon sum of the bassic set $\{P_j(z_1, z_2)\}$ for the Faber region $B_{\rho_1,\rho_2,\rho_k<r_k}$ is given by

$$\Omega_{m,n}(\rho_1, \rho_2) = \rho_1^n \rho_2^m \sum_{j=0} |\Pi_j^{m,n}| M'(P_j, \rho_1, \rho_2). \qquad (1.11)$$

We can easily deduce from (1.8), (1.1) and the formula of Newns [4, p.749] that

$$\begin{aligned}\Omega_{m,n}(\rho_1, \rho_2) &\geq \rho_1^n \rho_2^m M'(f_m^1; C_{\rho_1}^1) M'(f_m^2; C_{\rho_2}^2) \\ &> (\rho_1\rho_2)^{m+n}\{1 - \tfrac{l_1}{2\Pi\Delta_1}(\tfrac{T_1'}{\rho_1})^m\}\{1 - \tfrac{l_2}{2\Pi\Delta_2}(\tfrac{T_2'}{\rho_2})^n\}\end{aligned} \qquad (1.12)$$

where l_k is the length of the curve $C_{T_k'}^k; T_k' > T_k$ and $\Delta_k(> 0)$ is the distance between the curves $C_{r_k}^k$ and $C_{T_k'}^k$. If the Cannon function of the same set for the same Faber region is defined as

$$\Omega(\rho_1, \rho_2) = \lim_{m+n\to\infty} sup\{\Omega_{m,n}(\rho_1, \rho_2)\}^{\frac{1}{m+n}} \qquad (1.13)$$

then (1.12) yields

$$\Omega(\rho_1, \rho_2) \geq \rho_1\rho_2. \qquad (1.14)$$

We observe that if the series

$$\sum_{m,n=0}^{\infty} |a_{m,n}| \{\sum_{j=0} |\Pi_j^{m,n}| M'(P_j, \rho_1, \rho_2)\}$$

converges, then the arrangements of the terms of the series

$$\sum_{m,n=0}^{\infty} a_{m,n} \{\sum_{j=0} \Pi_j^{m,n} P_j(z_1, z_2)\}$$

associated with the function (1.5) which leads to the basic series (1.8) is justifiable and hence the basic series associated with function (1.5) will represnt it in $\bar{B}_{\rho_1,\rho_2}$.

Analogous notation is used for polycylinder (cf. [1]).

2. The required condition for effectiveness of the basic set $\{P_j(z_1, z_2)\}$ in Faber regions is the following.

Theorem 2.1. Let $\{P_j(z_1, z_2)\}$ be a basic set for which the Cannon function is

$$\Omega(R_1, R_2) = \sigma^2 \geq R_1 R_2, R_k > T_k. \tag{2.1}$$

Then $\{P_j(z_1, z_2)\}$ will be effective in B_{R_1,R_2} for the class of functions regular in $\bar{B}_{\rho_1,\rho_2}$, where

$$r_1 = \frac{\sigma^2}{R_2} \geq R_1, r_2 = \frac{\sigma^2}{R_1} \geq R_2. \tag{2.2}$$

We observe from (2.2) that

$$\frac{r_1}{r_2} = \frac{R_1}{R_2}$$

and hence Theorem 2.1 is the analogue for Faber regions of the sufficiency assertion of [1, Theorem 3.2], for polycylinders and may be derived in a manner, replacing the monomials

$$\{z_1^m z_2^n\} \text{by} \{f_m^1(z_1) f_m^2(z_2)\}$$

and, in view of (1.4) replacing the polycylinders Γ_{r_1,r_2} by the Faber regions B_{r_1,r_2}.

We, however, give another proof:

Proof of Theorem 2.1. Let $F(z_1, z_2)$ be any function regular in B_{r_1,r_2} where it admits the expansion

$$F(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} f_m^1(z_1) f_n^2(z_2).$$

Then there are numbers $\rho_k > r_k$ such that the function $F(z_1, z_2)$ is regular in B_{ρ_1,rho_2}. Hence, we can deduce that

$$\lim_{m+n\to\infty} sup\{|a_{m,n}|(\rho_1^{-n}, \rho_2^{-m})\}^{\frac{1}{m+n}} \leq \frac{1}{\rho_1 \rho_2}. \tag{2.3}$$

Now supposing that $\frac{\rho_1}{R_1} \leq \frac{\rho_2}{R_2}$ we obtain from (1.10) that

$$|a_{m,n}|\{\sum_{j=0} |\Pi_j^{m,n}| M'(P_j, \rho_1, \rho_2)\} \leq |a_{m,n}| \rho_1^{-n} \rho_2^{-m} (\frac{\rho_2}{R_2})^{\frac{1}{m+n}} \Omega_{m,n}(R_1, R_2).$$

Therefore, applying (2.1), (2.2) and (2.3), we easily derive the inequality

$$\lim_{m+n\to\infty} sup[|a_{m,n}|\{\sum_{j=0} |\Pi_j^{m,n}| M'(P_j, R_1, R_2)\}]^{\frac{1}{m+n}} \leq \frac{\rho_2}{R_2}\sigma^2 < 1.$$

Similarly if $\frac{\rho_2}{R_2} \leq \frac{\rho_1}{R_1}$ we can obtain the inequality

$$\lim_{m+n\to\infty} sup[|a_{m,n}|\{\sum_{j=0} |\Pi_j^{m,n}|M'(P_j, R_1, R_2)\}]^{\frac{1}{m+n}} \leq \frac{R_2}{\rho_2} < 1.$$

Hence the series

$$\sum_{m,n=0}^{\infty} |a_{m,n}|\{\sum_{j=0} |\Pi_j^{m,n}|M'(P_j, R_1, R_2)\}$$

converges and thus the basic series associated with $F(z_1, z_2)$ represents it in $\bar{B}_{R_1,R_2}$, and the theorem is proved.

The analogue for Faber regions of the results of [1, Theorem 3.4] for polycylinders and proved along the same lines is the following.

Theorem 2.2. A necessary condition for the effectiveness of the Cannon set $\{P_j(z_1, z_2)\}$ of polynomials in B_{R_1,R_2} for the class of functions regular in $\bar{B}_{r_1,r_2}$ $R_k \leq r_k$

$$\Omega(R_1, R_2) \leq \max(R_1r_2, R_2r_1). \tag{2.4}$$

Letting r_k tend to R_k in Theorems 2.1 and 2.2 and appealing to the relation (2.1) we obtain the familiar Cannon's theorem for Faber regions in the form.

Theorem 2.3. The Cannon set $\{P_j(z_1, z_2)\}$ of polynomials wil be effective in $\bar{B}_{R_1,R_2}$ if and only if

$$\Omega(R_1, R_2) = R_1R_2.$$

3. We now consider effectiveness of composite sets of polynomials of two complex variables z_1 and z_2 in Faber regions.

Let $\{P_\mu^1(z_1)\}$ and $\{P_v^2(z_2)\}$ be basic sets of polynomials. If for any mode of arrangement we write

$$P_\mu^1(z_1).P_v^2(z_2) = P_j(z_1, z_2), \tag{3.1}$$

then $\{P_j(z_1, z_2)\}$ is called the composite set of polynomials whose constituents are the sets $\{P_\mu^1(z_1)\}$ and $\{P_v^2(z_2)\}$, (c.f. [3]).
Following [3], suppose that the Faber polynomials $\{f_m^1(z_1)\}$ and $\{f_n^2(z_2)\}$ admit the unique representations

$$f_m^1(z_1) = \sum_\mu \Pi_{m,\mu}^1 P_\mu^1(z_1); f_m^1(z_1) = \sum_\mu \Pi_{m,\mu}^1 P_\mu^1(z_1) \tag{3.2}$$

then the product $f^1_m(z_1)f^2_n(z_2)$ will admit the unique representation

$$f^1_m(z_1)f^2_n(z_2) = \sum_j \Pi^{m,n}_j P_j(z_1, z_2) \tag{3.3}$$

where, besides (3.1) we have

$$\Pi^{m,n}_j = \Pi^1_{m,\mu}\Pi^2_{n,v} \tag{3.4}$$

so that $\{P_j(z_1, z_2)\}$ is infact a basic set of polynomials of the two complex variables z_1 and z_2.
We shall suppose each of the basic sets $\{P^1_\mu(z_1)\}$ and $\{P^2_v(z_2)\}$ is Cannon set in the sense that if N^1_m and N^2_n are the numbers of nonzero coefficients in the representations (3.2), then

$$\lim_{m\to\infty} {N^1_m}^{\frac{1}{m}} = 1; \lim_{m\to\infty} {N^2_n}^{\frac{1}{n}} = 1.$$

Therefore if $N_{m,n}$ is the number of nonzero coefficients in the representation (3.3), we should have $N_{m,n} \leq N^1_m N^2_m$ and hence $\lim_{m+n\to\infty} N_{m,n}^{\frac{1}{m+n}} = 1$. Hence, it follows that the composite set $\{P_j(z_1, z_2)\}$ is also a Cannon set.
It has been shown in [3, Theorem 5] that the composite set $\{P_j(z_1, z_2)\}$ whose constituents are Cannon sets $\{P^1_\mu(z_1), P^2_v(z_2)\}$ will be effective in the ball $\bar{S}_R = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 \leq R^2\}, R > 0$ if and only if, each of the constituent sets is effective in the disk $|z_k| \leq r$ for $0 < r \leq R$. To generalize this result to Faber regions, we write for the Cannon sum of the Cannon set $\{P^1_\mu(z_1)\}$ in $\bar{D}(C'_{r_1})$,

$$\omega^1_n(C'_{r_1}) = \sum_{\mu=0} |\Pi^1_{n,\mu}| M(P^1_\mu; C'_{r_1}), \tag{3.5}$$

and in view of (3.2) the Cannon function is given by

$$\lambda^1(C'_{r_1}) = \lim_{n\to\infty} sup\{\omega_n(C'_{r_1})\}^{\frac{1}{n}}. \tag{3.6}$$

Similar notation is adopted for the Cannon set $\{P^2_\mu(z_2)\}$. With this notation our result concerning composite set is the following.
Theorem 3.1. Let $\{P^1_\mu(z_1)\}$ and $\{P^2_v(z_2)\}$ be Cannon sets of polynomials and suppose that $\{P_j(z_1, z_2)\}$ is their composite set. Then the Cannon function of $\{P_j(z_1, z_2)\}$ for their Faber region $\bar{B}_{R_1,R_2}; R_k > T_k$ is given by

$$\Omega(R_1, R_2) = \max\{R_1\lambda^2(C''_{R_2}), R_2\lambda^1(C''_{R_1})). \tag{3.7}$$

Proof. Write

$$S_{m,n}(R_1, R_2) = \sum_{j=0} |\Pi^{m,n}_j| M'(P_j, R_1, R_2) \tag{3.8}$$

then (3.4) yields

$$|\Pi_j^{m,n}| = |\Pi_{m,\mu}^1||\Pi_{n,v}^2|) \tag{3.9}$$

while (3.1) with our notation yields

$$M(P_j, R_1, R_2) = M(P_\mu^1; C'_{R_1}).(M'(P_v^1; C''_{R_2}). \tag{3.10}$$

Introducing (3.9) into (3.8) it follows from (3.5) that

$$S_{m,n}(R_1, R_2) = \sum_{\mu,v} |\Pi_{m,\mu}^1|M'(P_\mu^1; C'_{R_1})|\Pi_{n,v}^2|M'(P_v^2; C''_{R_2}) = \omega_m^1(C'_{R_1})\omega_n^2(C''_{R_2})$$

Hence in view of (1.11) we obtain

$$\Omega_{m,n}(R_1, R_2) = R_1^n R_2^m \omega_m^1(C'_{R_1})\omega_n^2(C''_{R_2}). \tag{3.11}$$

Suppose that

$$\lambda^{(k)}(C_{R_k}^k) = \rho_k \geq R_k$$

and that

$$R_1\lambda^{(2)}(C''_{R_2}) = R_1\rho_2 \leq R_2\rho_1 = R_2\lambda^1(C'_{R_1}),$$

then

$$\frac{R_1}{\rho_1} \leq \frac{R_2}{\rho_2}. \tag{3.12}$$

Let σ_1 be any finite number greater than ρ_1 and choose the number $\sigma_2 > \rho_2$ such that

$$\frac{\sigma_1}{\rho_1} = \frac{\sigma_2}{\rho_2}. \tag{3.13}$$

Then (3.12) and (3.13) together yield

$$\frac{R_1}{\sigma_1} \leq \frac{R_2}{\sigma_2}. \tag{3.14}$$

From the definition (3.16) of $\lambda^(k)(C_{R_k}^k)$ it follows that

$$\omega_n^k(C_{R_k}^k) < K\sigma_k^m; (m \geq 0), K$$

a constant, and hence (3.11) and (3.14) imply that

$$\Omega_{m,n}(R_1, R_2) < K(R_2\sigma_1)^{m+n}; m, n \geq 0.$$

Hence, in the limit as $m + n \to \infty$ we have

$$\Omega(R_1, R_2) = \lim_{m+n\to\infty} sup\{\Omega_{m,n}(R_1, R_2)\}^{\frac{1}{m+n}} \leq R_2\sigma_1$$

and since σ_1 can be taken arbitrarily close to $\lambda^1(C_{R_1}^k)$ we deduce that

$$\Omega(R_1, R_2) = R_2\lambda^1(C'_{R_1}).$$

Similarly, if

$$R_2\lambda^1(C'_{R_1}) \le R_1\lambda^2(C''_{R_2})$$

we can arrive at the inequality

$$\Omega(R_1, R_2) \le R_1\lambda^{(2)}(C''_{R_2}$$

and we conclude that

$$\Omega(R_1, R_2) \le \max\{R_1\lambda^2(C''_{R_2}), R_2\lambda^1(C'_{R_1})\}. \tag{3.15}$$

On the other hand, we have from the inequalities,

$$\begin{aligned} \Omega(R_1, R_2) &= \lim_{m+n\to\infty} sup[R_1^n R_1^m \omega^1_m(C'_{R_1})\omega^1_m(C'_{R_1})]^{\frac{1}{m+n}} \\ &\ge \lim_{m\to\infty} sup[R_2^m \omega^1_m(C'_{R_1})]^{\frac{1}{m}} = R_1\lambda^2(C''_{R_2}) \end{aligned}$$

and

$$\Omega(R_1, R_2) \ge \lim_{n\to\infty} sup\{R_1^n\omega^2_n(C'''_{R_2})\}^{\frac{1}{n}} = R_1\lambda^2(C''_{R_2})$$

that

$$\Omega(R_1, R_2) \ge \max\{R_1\lambda^2(C''_{R_2}), R_2\lambda^1(C'_{R_1})\}. \qquad (3.16).$$

A combination of (3.15) and (3.16) gives the desired equality (3.7) and the theorem is established.

We observe that if the set $\{P^1_\mu(z_1)\}$ is effective in $\bar{D}(C''_{R_1})$ and the set $\{P^2_v(z_2)\}$ is effective in $\bar{D}(C''_{R_2})$, then $\lambda^k(C^k)_{R_k} = R_k$ and hence (3.7) implies that

$$\Omega(R_1, R_2) = R_1R_2.$$

Moreover, if the composite set $\{P_j(z_1, z_2)\}$ is effective in $\bar{B}_{R_1,R_2}$ then the same equation (3.7) yields

$$R_1\lambda^2(C''_{R_2}) \le \Omega(R_1, R_2) = R_1R_2,$$

so that

$$\lambda^2(C''_{R_2}) = R_2.$$

In a similar manner we can deduce that $\lambda^1(C'_{R_2}) = R_1$.

We have, therefore proved the following result which is a generalization of the result of Nassif [3, Theorem 5] for balls:

Theorem 3.2. The composite set $\{P_j(z_1, z_2)\}$ of the Cannon sets $\{P^1_\mu(z_1)\}$ and $\{P^2_\mu(z_2)\}$ is effective in the Faber region $\bar{B}_{R_1,R_2}, R_k > T_k$ if and only if the sets $\{P^1_\mu(z_1)\}$ and $\{P^2_\mu(z_2)\}$ are effective in $\bar{D}(C'_{R_1})$ and $\bar{D}(C''_{R_2})$ respectively.

We note that if the sets $\{P_m^k(z_k)\}$ are not effective in $\bar{D}(C_{R_k}^k)$ so that $\lambda^k(C_{R_k}^k) > R_k$, then we would have

$$\Omega(R_1, R_2) < \lambda^1(C'_{R_1})\lambda^2(C''^k_{R_2}).$$

We claim that the method used in establishing Theorem 3.1 for Faber regions is also valid for polycylinders Γ_{r_1,r_2} with obvious modifications (c.f. [1]). Indeed, the Cannon function $\lambda^k(R_k)$ of the constituent Cannon set $\{P_m^k(z_k)\}$ for the disk $|z_k| \leq R_k$ is given by

$$\lambda^k(R_k) = \lim_{n\to\infty} sup\{\omega_n(R_k)\}^{\frac{1}{n}}$$

where

$$\omega_n(R_k) = \sum_{j=0} |\Pi_{n,j}| M(P_j^k; R_k), R_k > 0$$

and M denotes maximum modulus in polycylinders.

The Cannon function of the composite set $\{P_j(z_1, z_2)\}$ for the closed polycylinder $\bar{\Gamma}_{R_1,R_2}$ is given by

$$\omega(R_1, R_2) = \lim_{n+m\to\infty} sup\{\omega_{m,n}(R_1, R_2)\}^{\frac{1}{n+m}}$$

where

$$\omega_n(R_1, R_2) = R_1^n, R_2^m \sum_{j=0} |\Pi_j^{m,n}| M(P_j; R_1, R_2).$$

With this notation the analogue of Theorem 3.1 for polycylinders is

Theorem 3.3. Let $\{P_\mu^1(z_1)\}$ and $\{P_v^2(z_2)\}$ be Cannon sets of polynomials and suppose that $\{P_j(z_1, z_2)\}$ is their composite set. Then the Cannon function of the set $\{P_j(z_1, z_2)\}$ for the polycylinders $\bar{\Gamma}_{R_1,R_2}; R_k > 0$ is given by

$$\omega(R_1, R_2) = \max\{R_1\lambda^2(R_2), R_2\lambda^1(R_1)\}. \tag{3.17}$$

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