

# Some classes of convex functions on time scales

B. O. Fagbemi and A. A. Mogbademu

Department of Mathematics, University of Lagos, Nigeria.

E-mail: opetoyinbo@yahoo.com, amogbademu@unilag.edu.ng

## Abstract

The notion of convexity of a function on an interval of a generic time scales has proved to be of significant interest to many authors in this area of research. Here we introduce and discuss some new classes of convex functions over a time scale. Consequently, various interconnections that exists among them and the relationships of their properties on classical intervals are provided. Interesting possible areas of applications of our results are also given.

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## 1 Introduction

The development of the theory of time scales, was introduced by Stephen Hilger (1988) in his PhD thesis, as a theory capable of containing difference and differential calculus in a consistent way.

The investigations are not only significant in the theoretical research of differential and difference equations, but also crucial in many computational and numerical applications. See [7].

The general idea is to prove a result for a dynamic equation where the domain of the unknown function is called a time scale, which is an arbitrary closed subset of the reals. By choosing the time scale to be the set of real numbers  $\mathbb{R}$ , the general result yields a result containing the ordinary derivative or integral. Further, when we choose the time scale to be the set of integers  $Z$ , the same general result yields a result for difference equations or integral. Hence, all results that are proved on a general time scale include results for both differential and difference equations (see [12]).

For a good introduction to the theory of time scales, see [3].

## 2 Preliminaries

A time scale  $\mathbb{T}$  is an arbitrary nonempty closed subset of  $\mathbb{R}$  (together with the topology of the subspace of  $\mathbb{R}$ ).

Two mappings  $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{R}$  satisfying  $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \forall t \in \mathbb{T}$ , and  $\rho(t) = \sup\{s \in \mathbb{T} : s \leq t\} \forall t \in \mathbb{T}$ , are called *jump operators*.

The jump operators  $\sigma$  and  $\rho$  allow the classification of points in the following way.

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If  $\sigma(t) > t$  for all  $t \in \mathbb{T}$ ,  $t$  is *right-scattered*. If  $\rho(t) < t$  for all  $t \in \mathbb{T}$ ,  $t$  is *left-scattered*. Points that are simultaneously right-scattered and left-scattered are called *isolated*. If  $t < \sup \mathbb{T}$  and  $\sigma(t) = t$  for all  $t \in \mathbb{T}$ ,  $t$  is *right-dense*. If  $t > \inf \mathbb{T}$  and  $\rho(t) = t$  for all  $t \in \mathbb{T}$ ,  $t$  is *left-dense*. Points that are right-dense and left-dense at the same time are called *dense*.

The mappings  $\mu, \nu : \mathbb{T} \rightarrow [0, +\infty]$  defined by  $\mu(t) = \sigma(t) - t \forall t \in \mathbb{T}$  and  $\nu(t) = t - \rho(t) \forall t \in \mathbb{T}$  are called the *forward and backward graininess functions* respectively.

A function  $f$  defined on  $\mathbb{T}$  is called right-dense continuous (or rd-continuous)(we write  $f \in C_{rd}$ ) if it is continuous at the right-dense points and its left-sided limits exist (finite) at all left-dense points;  $f$  is left-dense continuous (or ld-continuous)(we write  $f \in C_{ld}$ ) if it is continuous at the left-dense points and its right-sided limits exists(finite) at all right-dense points. Thus, the set of *continuous functions* on  $\mathbb{T}$  contains both  $C_{rd}$  and  $C_{ld}$ .

Throughout this paper, we will denote a time scale by  $\mathbb{T}$ , and for any  $I$ , interval of  $\mathbb{R}$  (open or closed, finite or infinite),  $I_{\mathbb{T}} = I \cap \mathbb{T}$ , a time scale interval.

A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is *delta differentiable* on  $\mathbb{T}^k$ , provided

$$f^{\Delta}(t) = \lim_{s \rightarrow t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \quad (2.1)$$

exists, where  $s \rightarrow t, s \in \mathbb{T} \setminus \{\sigma(t)\}$  for all  $t \in \mathbb{T}^k$ .

We state the following remark using inequality (2.1) and its nabla equivalence.

**Remark 2.1** (i) If  $\mathbb{T} = \mathbb{R}$ , then  $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$  becomes the total differential operator (ordinary derivative).

(ii) If  $\mathbb{T} = \mathbb{Z}$ , then

$$f^{\Delta}(t) = f(t+1) - f(t)$$

and

$$f^{\Delta^r}(t) = f^{\Delta^{r-1}}(t+1) - f^{\Delta^r}(t)$$

are the forward and r-th forward difference operators;

$$f^{\Delta}(t) = f\left(t + \frac{1}{2}\right) - f\left(t - \frac{1}{2}\right)$$

is the central difference operator;

$$f^{\nabla}(t) = f(t) - f(t-1)$$

and

$$f^{\nabla^r}(t) = f^{\nabla^{r-1}}(t) - f^{\nabla^r}(t-1)$$

are the backward and r-th backward difference operators respectively (see [4]).

It is known that rd-continuous functions possess an *antiderivative*, i.e., there exists a function  $F : \mathbb{T} \rightarrow \mathbb{R}$  with  $F^{\Delta}(t) = f(t)$  for all  $t \in \mathbb{T}^k$ , and in this case, the delta integral is defined by

$$\int_s^t f(\tau) \Delta \tau = F(t) - F(s), \quad \forall s, t \in \mathbb{T}.$$

The nabla integral is analogously defined in [3] and consequently, the diamond- $\alpha$  integral (see [12]).

The notion of convexity of a function on an interval of a generic time scale was introduced by Mozyrska and Torres [9] thus;

**Definition 2.1.** [9] Let  $I$  be an interval in  $\mathbb{R}$  such that the set  $I_{\mathbb{T}}$  is a nonempty subset of  $\mathbb{T}$ . A function  $f$  defined and continuous on  $I_{\mathbb{T}}$  is called convex on  $I_{\mathbb{T}}$  if for any  $t_1, t, t_2 \in I_{\mathbb{T}}$ ,

$$(t_2 - t)f(t_1) + (t_1 - t_2)f(t) + (t - t_1)f(t_2) \geq 0. \quad (2.2)$$

An equivalent definition to that of Mozyrska and Torres [9] defined above was consequently given in Dinu [5];

**Definition 2.2.** [5] A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is called *convex* on  $I_{\mathbb{T}}$ , if

$$f(\lambda t + (1 - \lambda)s) \leq \lambda f(t) + (1 - \lambda)f(s), \quad (2.3)$$

for all  $t, s \in I_{\mathbb{T}}$  and all  $\lambda \in [0, 1]$  such that  $\lambda t + (1 - \lambda)s \in I_{\mathbb{T}}$ .

Motivated by the works of these authors [5, 9], we introduce and discuss some classes of convex functions on time scales.

### 3 Some classes of convex functions on time scales

In understanding of convex functions, the basis is that of Jensen convex or mid-point convex functions, which deals with the arithmetic mean (see [10]). We shall state its analogue on time scales.

**Definition 3.1.** Let  $I_{\mathbb{T}} \subset \mathbb{T}$  be a time scale interval. A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is called convex in the Jensen sense or *J-convex* or mid-point convex on  $I_{\mathbb{T}}$  or  $f \in J(I_{\mathbb{T}})$  if for all  $t_1, t_2 \in I_{\mathbb{T}}$ , the inequality,

$$f\left(\frac{t_1 + t_2}{2}\right) \leq \frac{f(t_1) + f(t_2)}{2} \quad (3.1)$$

holds.

**Remark 3.1.**(i) If  $\mathbb{T} = \mathbb{R}$ , then our version is the same as the classical Jensen inequality. However, if  $\mathbb{T} = \mathbb{Z}$ , then it reduces to the well-known arithmetic-geometric mean inequality. See [8].

(ii) The extensions of the inequality (3.1) to the convex combination of finitely many points and next to random variables associated to arbitrary probability spaces are known as the discrete Jensen and integral inequalities on time scales respectively (see [8]).

**Definition 3.2.** A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is a *P-function* on  $I_{\mathbb{T}}$  or  $f \in P(I_{\mathbb{T}})$  if  $f$  is nonnegative, and for all  $t_1, t_2 \in I_{\mathbb{T}}$ , and  $\omega \in [0, 1]$ , we have

$$f(\omega t_1 + (1 - \omega)t_2) \leq f(t_1) + f(t_2). \quad (3.2)$$

Obviously,  $P(I_{\mathbb{T}})$  is contained in the class of Godunova Levin functions on time scales. Also,  $P(I_{\mathbb{T}})$  contains all nonnegative monotone, convex and quasi-convex functions on  $I_{\mathbb{T}}$ , i.e, nonnegative functions satisfying

$$f(\omega t_1 + (1 - \omega)t_2) \leq \max\{f(t_1) + f(t_2)\}$$

for all  $t_1, t_2 \in I_{\mathbb{T}}$ , and  $\omega \in [0, 1]$ .

In order to unify the concepts of some classes of convex functions such as Godunova-Lein convex,  $s$ -Godunova-Levin convex,  $s$ -convex function (in the first and second sense) with the classes defined above for functions of time scale variables,  $\mathbb{T}$ , we introduce the concept of  $h$ -convex functions on time scales as follows:

Assume that  $I_{\mathbb{T}}$  and  $J_{\mathbb{T}}$  are intervals in  $\mathbb{T}$ ,  $[0, 2] \subseteq J_{\mathbb{T}}$  and functions  $h$  and  $f$  are real non-negative functions defined on  $I_{\mathbb{T}}$  and  $J_{\mathbb{T}}$  respectively.

**Definition 3.3.** Let  $h : J_{\mathbb{T}} \rightarrow \mathbb{R}$  with  $h$  not identical to zero. We say that  $f : I_{\mathbb{T}} \rightarrow \mathbb{R}$  is an  $h$ -convex function on  $I_{\mathbb{T}}$  or  $f$  belongs to the class  $SX(h, I_{\mathbb{T}})$  if for all  $t_1, t_2 \in I_{\mathbb{T}}$ ,  $f$  is non-negative, we have

$$f(\omega t_1 + (1 - \omega)t_2) \leq h(\omega)f(t_1) + h(1 - \omega)f(t_2), \quad (3.3)$$

for all  $\omega \in (0, 1)$  such that  $\omega t_1 + (1 - \omega)t_2 \in I_{\mathbb{T}}$ .

**Remark 3.2.** (i) If inequality (3.3) is reversed, then  $f$  is said to be  $h$ -concave on  $I_{\mathbb{T}}$  i.e  $f \in SV(h, I_{\mathbb{T}})$  (see [6, 14]).

(ii) Obviously, if  $h(\omega) = \omega$ , then all non-negative convex functions on  $I_{\mathbb{T}}$  belong to  $SX(h, I_{\mathbb{T}})$  and all non-negative concave functions on  $I_{\mathbb{T}}$  belong to  $SV(h, I_{\mathbb{T}})$ ; if  $h(\omega) = \frac{1}{2}$ , then  $SX(h, I_{\mathbb{T}})$  reduces to the class  $J(I_{\mathbb{T}})$ ; and if  $h(\omega) = 1$ , then  $SX(h, I_{\mathbb{T}}) \supseteq P(I_{\mathbb{T}})$ .

**Definition 3.4.** A function  $f : I_{\mathbb{T}} \subset \mathbb{T} \rightarrow \mathbb{R}$  is said to belong to the class  $MT(I_{\mathbb{T}})$  if  $f$  is nonnegative and for all  $t_1, t_2 \in I_{\mathbb{T}}$  and  $\omega \in (0, 1)$  satisfies the inequality:

$$f(\omega t_1 + (1 - \omega)t_2) \leq \frac{\sqrt{\omega}}{2\sqrt{1 - \omega}}f(t_1) + \frac{\sqrt{1 - \omega}}{2\sqrt{\omega}}f(t_2), \quad (3.4)$$

**Remark 3.3.** (i) If we set  $\omega = \frac{1}{2}$ , inequality (3.4) reduces to the inequality (3.1).

(ii) If  $\mathbb{T} = \mathbb{R}$ , and  $I_{\mathbb{T}} = I$ , we obtain some definitions for classical  $MT$ -convex function (see [11, 13]).

Some simple examples of functions which are  $MT$ -convex on  $I_{\mathbb{T}}$  but are not convex on  $I_{\mathbb{T}}$  are:

i) The functions  $f, g : (1, \infty) \cap [0, 1] \subset \mathbb{T} \rightarrow \mathbb{R}$ ,  $f(t) = t^p$ ,  $g(t) = (1+t)^p$ ,  $p \in (0, \frac{1}{1000})$ .

ii) A function  $h : [1, \frac{3}{2}] \subset \mathbb{T} \rightarrow \mathbb{R}$ ,  $h(t) = (1 + t_2)^m$ ,  $m \in (0, \frac{1}{100})$ .

Now, we define a new general class of convex functions which we will call  $\phi_{h-s, \mathbb{T}}$ -convex functions on time scales.

**Definition 3.5.** Let  $\mathbb{T}$  be a time scale and  $h : \mathbb{J}_{\mathbb{T}} \rightarrow \mathbb{R}$  a real valued function, where  $\mathbb{J}_{\mathbb{T}} \subset \mathbb{T}$ . For all  $s \in [0, 1]$ ,  $\omega \in (0, 1)$  and  $\phi$ , a given real-valued function, with  $\phi(t) \in \text{id}_{I_{\mathbb{T}}}$  - an identity function in  $I_{\mathbb{T}}$ , then  $f : I_{\mathbb{T}} \rightarrow \mathbb{R}$  is  $\phi_{h-s, \mathbb{T}}$ -convex on time scales if for all  $t_1, t_2 \in I_{\mathbb{T}}$ ,

$$f(\omega\phi(t_1) + (1 - \omega)\phi(t_2)) \leq \left(\frac{h(\omega)}{\omega}\right)^{-s} f(\phi(t_1)) + \left(\frac{h(1 - \omega)}{1 - \omega}\right)^{-s} f(\phi(t_2)). \quad (3.5)$$

**Remark 3.4.** We observe that:

- (i) If  $s = 1$ ,  $h(\omega) = 1$  and  $\phi(t_1) = t_1$ , then  $f \in SX(I_{\mathbb{T}})$ , i.e,  $f$  is convex on time scales (see [5, 9]).
- (ii) If  $s = 1$ ,  $h(\omega) = 1$ , where  $\omega = \frac{1}{2}$  and  $\phi(t_1) = t_1$ , then  $f \in J(I_{\mathbb{T}})$  is mid-point convex on time scales.
- (iii) If  $s = 0$  and  $\phi(t_1) = t_1$ , then  $f \in P(I_{\mathbb{T}})$  is  $P$ -convex on time scales.
- (iv) If  $h(\omega) = \omega^{\frac{s}{s+1}}$  and  $\phi(t_1) = t_1$ , then  $f \in SX(h, I_{\mathbb{T}})$  is  $h$ -convex on time scales.
- (v) If  $s = 1$ ,  $h(\omega) = 2\sqrt{\omega(1-\omega)}$  and  $\phi(t_1) = t_1$ , then  $f \in MT(I_{\mathbb{T}})$  is  $MT$ -convex on time scales.

Moreover, suppose we denote by  $SX(\phi_{h-s, \mathbb{T}})$  the class of  $\phi_{h-s, \mathbb{T}}$  convex functions on time scales, then it is easy to see that:  $P(I_{\mathbb{T}}) \subseteq SX(\phi_{h-1}, I_{\mathbb{T}}) = SX(\phi_h, I_{\mathbb{T}}) = MT(\phi_{h-1}, I_{\mathbb{T}})$  for  $0 \leq s \leq 1$ , whenever  $\phi$  is the identity function.

If inequality (3.5) is reversed, then  $f$  is  $\phi_{h-s, \mathbb{T}}$  concave, that is,  $f \in SV(\phi_{h-s, \mathbb{T}})$ .

Next, we give an example of our newly introduced generalized class of  $\phi_{h-s, \mathbb{T}}$  convex functions on time scales.

**Example 3.1.** Consider the function  $f$  to be a non-negative convex function on  $I_{\mathbb{T}}$  and  $h$ , a non negative function on  $I_{\mathbb{T}}$  satisfying

$$h(\omega) \leq \omega^{1-\frac{1}{s}}, \quad s \in (0, 1], \quad \omega \in (0, 1).$$

Then, we may have that

$$\begin{aligned} f(\lambda t + (1-\lambda)s) &\leq \lambda f(t) + (1-\lambda)f(s) \\ &\leq \left(\frac{h(\omega)}{\omega}\right)^{-s} f(\phi(t_1)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f(\phi(t_2)), \end{aligned}$$

showing that  $f \in SX(\phi_{h-s, \mathbb{T}})$ .

**Remark 3.5.** Example 3.1 implies all convex functions are examples of our newly defined class of  $\phi_{h-s, \mathbb{T}}$ -convex function on  $I_{\mathbb{T}}$  provided that the condition  $h(\omega) \leq \omega^{1-\frac{1}{s}}$  is satisfied.

Any non-negative concave function  $f$  belongs to the class  $SV(\phi_{h-s, \mathbb{T}})$  i.e. is  $\phi_{h-s, \mathbb{T}}$ -concave provided  $h$  satisfies  $h(\omega) \geq \omega^{1-\frac{1}{s}}$  for any  $\omega \in (0, 1)$  and  $s \in (0, 1]$ .

The following definition is useful in defining another form of inequality (3.5) on time scales.

**Definition 3.6.** A function  $h : J_{\mathbb{T}} \rightarrow \mathbb{R}$  is said to be a *supermultiplicative* function on  $J_{\mathbb{T}} \subset \mathbb{T}$  if for all  $m, n \in J_{\mathbb{T}}$ ,

$$h(mn) \geq h(m)h(n). \quad (3.6)$$

$h$  is said to be a *submultiplicative* function on time scales if the inequality (3.6) is reversed and respectively a *multiplicative* function on time scales if the equality holds in (3.6).

If  $h$  is a supermultiplicative or submultiplicative function on time scales, then some very interesting results arise for  $\phi_{h-s, \mathbb{T}}$ -convex function. In

that case, we assume results for composition hold in the context of time scales.

**Proposition 3.1.** Let  $h : J_{\mathbb{T}} \rightarrow \mathbb{R}$  be a non negative function on  $J_{\mathbb{T}} \subset \mathbb{T}$  and let  $f : \mathbb{T} \rightarrow \mathbb{R}$  be a function such that  $f \in SX(\phi_{h-s, \mathbb{T}})$ , where  $\phi(t) = t$ . Then for all  $s \in [0, 1]$ ,  $t_1, t_2, t_3 \in \mathbb{T}$  such that  $t_1 < t_2 < t_3$  and  $t_3 - t_1, t_3 - t_2, t_3 - t_2, t_3 - t_2, \in J_{\mathbb{T}}$ , then the following inequality holds:

$$\begin{aligned} & [(t_3 - t_1), (t_2 - t_1), (t_3 - t_2)]^{-s} f(t_1) - [(t_3 - t_2), (t_2 - t_1), (t_3 - t_2)]^{-s} f(t_2) \\ & + [(t_3 - t_1), (t_3 - t_2), (t_2 - t_1)]^s f(t_3) \geq 0. \end{aligned} \quad (3.7)$$

If the function  $h$  is submultiplicative and  $f \in SX(\phi_{h-s, \mathbb{T}})$ , then the inequality (3.7) is reversed.

**Proof.** Since  $f \in SX(\phi_{h-s, \mathbb{T}})$ , and  $t_1, t_2, t_3 \in \mathbb{T}$  are points which satisfy assumptions of the proposition. Then

$$\frac{t_3 - t_2}{t_3 - t_1}, \frac{t_2 - t_1}{t_3 - t_1} \in J_{\mathbb{T}}$$

and

$$\frac{t_3 - t_2}{t_3 - t_1} + \frac{t_2 - t_1}{t_3 - t_1} = 1.$$

Also,

$$\begin{aligned} h(t_3 - t_1)^{-s} &= \left( h \left( \frac{t_3 - t_2}{t_3 - t_1} (t_3 - t_1) \right) \right)^{-s} \\ &\geq \left( h \left( \frac{t_3 - t_2}{t_3 - t_1} \right) h(t_3 - t_1) \right)^{-s} \end{aligned}$$

and

$$\begin{aligned} h(t_2 - t_1)^{-s} &= \left( h \left( \frac{t_2 - t_1}{t_3 - t_1} (t_3 - t_1) \right) \right)^{-s} \\ &\geq \left( h \left( \frac{t_2 - t_1}{t_3 - t_1} \right) h(t_3 - t_1) \right)^{-s}. \end{aligned}$$

Let  $h(t_3 - t_1) > 0$ . If in inequality (3.5), we set  $\omega = \frac{t_3 - t_2}{t_3 - t_1}$ ,  $1 - \omega = \frac{t_2 - t_1}{t_3 - t_1}$ ,  $a = t_1, b = t_3$ , then we have  $t_2 = \omega a + (1 - \omega)b$  and

$$f(t_2) \leq \left( \frac{h(t_3 - t_2)}{\frac{h(t_3 - t_1)}{t_3 - t_2}} \right)^{-s} f(t_1) + \left( \frac{h(t_3 - t_2)}{\frac{h(t_3 - t_1)}{t_3 - t_1}} \right)^{-s} f(t_3). \quad (3.8)$$

Multiplying inequality (3.8) by  $\left( \frac{t_3 - t_2}{t_3 - t_1} \right)^{-s} (h(t_3 - t_1))^{-s}$  and further multiplication by  $(t_3 - t_1)^{-s} (t_2 - t_1)^{-s}$  with rearrangement yields (3.7).

**Remark 3.6.** (i) Inequality (3.7) can alternatively be used in the definition 3.5.

(ii) If we consider inequality (3.7) with  $h(t) = h_{-1}(t) = t^2$ ,  $s = 1$ , we obtain an alternate definition of Godunova-Levin function on time scales.

(iii) Inequality (3.7) is equivalent to definition 14 of [9] with  $h(t) = 1$ ,  $s = 1$  considering points  $t_1, t_2 \in I_{\mathbb{T}}$  with  $t_1 < t_2$  and  $t \in I_{\mathbb{T}}$  such that  $t_1 < t < t_2$  and  $t = \omega t_1 + (1 - \omega)t_2$ .

**Theorem 3.1.** Let  $f : I_{\mathbb{T}} \rightarrow \mathbb{R}$  be defined and  $\Delta_{\phi_{h-s, \mathbb{T}}}$  differentiable function on  $I_{\mathbb{T}}^k$ . If  $f^{\Delta_{\phi_{h-s, \mathbb{T}}}}$  is nondecreasing (nonincreasing) on  $I_{\mathbb{T}}^k$ , then  $f$  is  $\phi_{h-s, \mathbb{T}}$  convex (concave) on  $I_{\mathbb{T}}$ .

**Proof.** Using the affinity of  $f$  on  $I_{\mathbb{T}}$ , i.e,  $I_{\mathbb{T}}$  consisting of less than three points, we want to prove the relation

$$\frac{f(\phi(t)) - f(\phi(t_1))}{t - t_1} \leq \frac{f(\phi(t_2)) - f(\phi(t))}{t_2 - t}. \quad (3.9)$$

Let  $t_1 \leq \gamma_1 < \xi_2$ . From the mean value Theorem, we have existence of points  $\xi_1, \gamma_1 \in [t_1, t]_{\mathbb{T}}$  and  $\xi_2, \gamma_2 \in [t, t_2]_{\mathbb{T}}$  such that

$$f^{\Delta\phi_{h-s, \mathbb{T}}}(\xi_1) \leq \frac{f(\phi(t)) - f(\phi(t_1))}{t - t_1} \leq f^{\Delta\phi_{h-s, \mathbb{T}}}(\gamma_1)$$

and

$$f^{\Delta\phi_{h-s, \mathbb{T}}}(\xi_2) \leq \frac{f(\phi(t_2)) - f(\phi(t))}{t_2 - t} \leq f^{\Delta\phi_{h-s, \mathbb{T}}}(\gamma_2). \quad (3.10)$$

Since  $t_1 < \gamma_1 < \xi_2$  and from the assumption that  $f^{\Delta\phi_{h-s, \mathbb{T}}}(\gamma_1) \leq f^{\Delta\phi_{h-s, \mathbb{T}}}(\xi_2)$ , inequality (3.9) holds:

$$\frac{f(\phi(t)) - f(\phi(t_1))}{t - t_1} \leq f^{\Delta\phi_{h-s, \mathbb{T}}}(\gamma_1) \leq f^{\Delta\phi_{h-s, \mathbb{T}}}(\xi_2) \leq \frac{f(\phi(t_2)) - f(\phi(t))}{t_2 - t} \quad (3.11)$$

for nondecreasing  $f^{\Delta\phi_{h-s, I_{\mathbb{T}}}}$  and

$$\frac{f(\phi(t)) - f(\phi(t_1))}{t - t_1} \geq f^{\Delta\phi_{h-s, \mathbb{T}}}(\gamma_1) \geq f^{\Delta\phi_{h-s, \mathbb{T}}}(\xi_2) \geq \frac{f(\phi(t_2)) - f(\phi(t))}{t_2 - t} \quad (3.12)$$

for nonincreasing  $f^{\Delta\phi_{h-s, \mathbb{T}}}$ .

The inequality (3.11) is equivalent with the inequality (3.5) and with the  $\phi_{h-s, \mathbb{T}}$ -convexity of  $f$ , while the inequality (3.12) is equivalent with the  $\phi_{h-s, \mathbb{T}}$ -concavity of  $f$ .

It is obvious that the nabla version of the Theorem 3.1 holds for nondecreasing (nonincreasing)  $f^{\Delta\phi_{h-s, \mathbb{T}}}$ .

A natural question of interest can be asked: *Can the generalized class of convex function (3.5) be continuous on time Scales?* The answer to this is affirmative.

We first discuss the geometrical interpretation of a  $\phi_{h-s, I_{\mathbb{T}}}$  convexity on time scales in order to justify this claim.

$\phi_{h-s, I_{\mathbb{T}}}$  convexity of a function  $f : I_{\mathbb{T}} \rightarrow \mathbb{R}$  on time scales geometrically means that the points of the graph of  $f(\phi(t))$  are under the chord (or on the chord) joining the endpoints  $(\phi(t_1), f(\phi(t_1)))$  and  $(\phi(t_2), f(\phi(t_2)))$  for every  $t_1, t_2 \in I_{\mathbb{T}}$ . Then,

$$f(\phi(t)) \leq f(\phi(t_1)) + \frac{f(\phi(t_2)) - f(\phi(t_1))}{\phi(t_2) - \phi(t_1)}(\phi(t) - \phi(t_1)) \quad (3.13)$$

for all  $\phi(t) \in [\phi(t_1), \phi(t_2)]$  and all  $\phi(t_1), \phi(t_2) \in I_{\mathbb{T}}$ .

This shows that convex functions are majorized locally (i.e. on any compact subinterval) by affine functions.

Theorem 3.1 shows that the newly introduced, generalized class of  $\phi_{h-s, \mathbb{T}}$ -convex function in Definition 3.5, being differentiable, is convex.

Next, we state and prove a result that shows that our newly introduced class of  $\phi_{h-s, \mathbb{T}}$ -convex function, is convex if and only if is midpoint convex on time scales.

**Theorem 3.2.** Let  $f : I_{\mathbb{T}} \rightarrow \mathbb{R}$  be a continuous function on  $I_{\mathbb{T}}$ . Then  $f$  is  $\phi_{h-s, \mathbb{T}}$  convex on  $I_{\mathbb{T}}$  if and only if  $f$  is midpoint convex on  $I_{\mathbb{T}}$ .

**Proof.** Sufficiently assume for contradiction that  $f$  is not  $\phi_{h-s, I_{\mathbb{T}}}$  convex on  $I_{\mathbb{T}}$ . Thus, there would exist subinterval  $(\phi(a), \phi(b)]$  such that  $f(\phi(t))|[\phi(a), \phi(b)]$  is not under the chord (or on the chord) joining  $(\phi(a), f(\phi(a)))$  and  $(\phi(b), f(\phi(b)))$ , that is, the function

$$f(\psi(t)) = f(\phi(t)) - \frac{f(\phi(b)) - f(\phi(a))}{\phi(b) - \phi(a)}(\phi(t) - \phi(a)) - f(\phi(a)), \quad (3.14)$$

$\phi(t) \in [\phi(a), \phi(b)]$ .

This confirms that  $\xi = \sup\{f(\psi(\phi(t)))|\phi(t) \in [\phi(a), \phi(b)]\} > 0$ .

$\psi(\phi(a)) = \psi(\phi(b)) = 0$  since  $\psi$  is continuous.

Also, let  $K = \inf\{\phi(t) \in [\phi(a), \phi(b)]|\psi(\phi(t)) = \xi\}$ , then we necessarily prove that  $\phi(k) = \xi$  and  $k \in (\phi(a), \phi(b))$ .

For every  $c > 0$  for which  $k + c \in (\phi(a), \phi(b))$ , we have by the definition of  $k$ ,  $\psi(k - c) < \psi(k)$  and  $\psi(k + c) \leq \psi(k)$ .

So that

$$\psi(k) > \frac{\psi(k - c) + \psi(k + c)}{2}.$$

This contradicts the fact that  $\psi$  is midpoint convex on  $I_{\mathbb{T}}$ .

**Remark 3.7.** (i) Theorem 3.1 remains true if the condition of midpoint convexity on time scales is replaced by

$$f((1 - \omega)\phi(t_1) + \omega\phi(t_2)) \leq \left(\frac{h(1 - \omega)}{1 - \omega}\right)^{-s} f(\phi(t_1)) + \left(\frac{h(\omega)}{\omega}\right)^{-s} f(\phi(t_2)),$$

for some  $0 \leq s \leq 1$ ,  $h(\omega) = 1$ ,  $\phi(t_1) = t_1$  and  $\omega = \frac{1}{2}$ .

(ii) If we replace the condition of continuity in Theorem 3.2 by boundedness from above on every compact subinterval of time scales, Theorem 3.2 still holds.

## 4 Applications for convex optimization

Optimization and Economics are ideal disciplines for application of time scales (see [1, 2]). Our notion of  $\phi_{h-s, \mathbb{T}}$  convexity describes a much more general mathematical structure over convexity on a general time scale. We therefore propose a new problem of optimization on time scales for system modelling with both continuous and discrete variables. An optimization problem is  $\phi_{h-s, \mathbb{T}}$  convex provided  $f$  is defined on  $f : I_{\mathbb{T}}^n \rightarrow \mathbb{R}$ , where  $f_0$  is  $\phi_{h-s, \mathbb{T}}$  convex and  $f_1, \dots, f_m$  are convex. For such, we present the form

$$\begin{aligned} & \text{minimize } f_0(\phi(x)) \\ & \text{subject to } f_i(x) \leq 0, i = 1, \dots, m, \end{aligned}$$

satisfying

$$f_i(\omega\phi(t_1) + (1 - \omega)\phi(t_2)) \leq \left(\frac{h(\omega)}{\omega}\right)^{-s} f_i(\phi(t_1)) + \left(\frac{h(1 - \omega)}{1 - \omega}\right)^{-s} f_i(\phi(t_2)), \text{ for}$$

all  $s \in [0, 1]$ ,  $\omega \in (0, 1)$  and  $\phi(t) \in \text{id}_{I_{\mathbb{T}}}$ .

Moreover, since any linear program is a convex optimization problem and hence a  $\phi_{h-s, \mathbb{T}}$  convex optimization problem by Example 3.1, we can consider  $\phi_{h-s, \mathbb{T}}$  convex optimization to be a generalization of linear programming (see [1]).

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