Complex Opial type inequalities

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Abstract

We establish here complex Opial type inequalities for analytic functions from a complex numbers domain into the set of complex numbers.

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1 Introduction

This article is greatly motivated by the article of Z. Opial [4].

Theorem 1 *(Opial, 1960) Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then*

$$\int_0^h |x(t)\, x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt. \tag{1}$$

In the last inequality the constant $\frac{h}{4}$ is the best possible.
Equality holds for the function

$$x(t) = t \quad \text{on} \quad \left[0, \frac{h}{2}\right]$$

and

$$x(t) = h - t \quad \text{on} \quad \left[\frac{h}{2}, h\right].$$

Opial type inequalities have applications in establishing uniqueness of solution to initial value problems in differential equations, see [5], also find upper bounds to such solutions.

We are also inspired by the author's monographs [1], [2], to continue our search for Opial type inequalities in the complex numbers setting.

2 Background

See also [3].

Let γ be a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We notice that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$l(\gamma) = \int_{\gamma_{u,w}} |dz| := \int_a^b |z'(t)|\,dt.$$

We mention also the triangle inequality for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\, l(\gamma), \tag{2}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

S. Dragomir in [3] proved the following useful complex Taylor's formula with remainder over a non-necessarily convex domain D.

Theorem 2 *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and $y, x \in D$. Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = x$ and $z(b) = y$ then*

$$f(y) = \sum_{k=0}^{n} \frac{(y-x)^k}{k!} f^{(k)}(x) + \frac{1}{n!}\int_{\gamma_{x,y}} (y-z)^n f^{(n+1)}(z)\,dz, \tag{3}$$

for $n \in \mathbb{Z}_+$.

3 Main Results

A complex Opial type inequality follows

Theorem 3 *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and let $x, y, w \in D$. Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = x$, $z(c) = y$, and $z(b) = w$, where $c \in [a,b]$ is floating. Assume that $f^{(k)}(x) = 0$, $k = 0,1,...,n$, $n \in \mathbb{Z}_+$, and $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

1)

$$\left| \int_a^b f(z(t)) f^{(n+1)}(z(t)) z'(t) \, dt \right| \leq \int_a^b |f(z(t))| \left| f^{(n+1)}(z(t)) \right| |z'(t)| \, dt \leq$$

$$\frac{1}{2^{\frac{1}{q}} n!} \left[\int_a^b \left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)| \, dt \right) |z'(c)| \, dc \right]^{\frac{1}{p}} \cdot \tag{4}$$

$$\left(\int_a^b \left| f^{(n+1)}(z(t)) \right|^q |z'(t)| \, dt \right)^{\frac{2}{q}},$$

equivalently it holds

2)

$$\left| \int_{\gamma_{x,w}} f(z) f^{(n+1)}(z) \, dz \right| \leq \int_{\gamma_{x,w}} |f(z)| \left| f^{(n+1)}(z) \right| |dz| \leq$$

$$\frac{1}{2^{\frac{1}{q}} n!} \left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c) - z|^{pn} |dz| \right) |z'(c)| \, dc \right]^{\frac{1}{p}} \left(\int_{\gamma_{x,w}} \left| f^{(n+1)}(z) \right|^q |dz| \right)^{\frac{2}{q}}. \tag{5}$$

Proof. By (3) we obtain

$$f(y) = \frac{1}{n!} \int_{\gamma_{x,y}} (y - z)^n f^{(n+1)}(z) \, dz, \quad n \in \mathbb{Z}_+.$$

Then by triangle's and Hölder's inequalities we have

$$|f(y)| \leq \frac{1}{n!} \int_{\gamma_{x,y}} |y - z|^n \left| f^{(n+1)}(z) \right| |dz| =$$

$$\frac{1}{n!} \int_a^c |y - z(t)|^n \left| f^{(n+1)}(z(t)) \right| |z'(t)| \, dt \leq \tag{6}$$

$$\frac{1}{n!} \left(\int_a^c |y - z(t)|^{pn} |z'(t)| \, dt \right)^{\frac{1}{p}} \left(\int_a^c \left| f^{(n+1)}(z(t)) \right|^q |z'(t)| \, dt \right)^{\frac{1}{q}}.$$

We set

$$\rho(c) := \int_a^c \left| f^{(n+1)}(z(t)) \right|^q |z'(t)| \, dt, \quad a \leq c \leq b, \tag{7}$$

then $\rho(a) = 0$, and

$$\rho'(c) = \left| f^{(n+1)}(z(c)) \right|^q |z'(c)| \geq 0.$$

That is

$$\left| f^{(n+1)}(z(c)) \right| |z'(c)|^{\frac{1}{q}} = (\rho'(c))^{\frac{1}{q}}. \tag{8}$$

Hence it holds

$$|f(z(c))| \left| f^{(n+1)}(z(c)) \right| |z'(c)|^{\frac{1}{q}} |z'(c)|^{\frac{1}{p}} \leq \tag{9}$$

$$\frac{1}{n!} \left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)| \, dt \right)^{\frac{1}{p}} (\rho(c) \rho'(c))^{\frac{1}{q}} |z'(c)|^{\frac{1}{p}}.$$

Integrating (9) and by Hölder's inequality we obtain

$$\int_a^b |f(z(c))| \left| f^{(n+1)}(z(c)) \right| |z'(c)| \, dc \leq$$

$$\frac{1}{n!} \int_a^b \left[\left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)| \, dt \right) |z'(c)| \right]^{\frac{1}{p}} (\rho(c) \rho'(c))^{\frac{1}{q}} \, dc \leq$$

$$\frac{1}{n!} \left\{ \int_a^b \left[\left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)| \, dt \right) |z'(c)| \right] dc \right\}^{\frac{1}{p}} \left(\int_a^b \rho(c) \rho'(c) \, dc \right)^{\frac{1}{q}} =$$

$$\frac{1}{n!} \left\{ \int_a^b \left[\left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)| \, dt \right) |z'(c)| \right] dc \right\}^{\frac{1}{p}} \frac{\rho(b)^{\frac{2}{q}}}{2^{\frac{1}{q}}}, \tag{10}$$

proving the claim. ■

We continue with an extreme case

Proposition 4 *All here are as in Theorem 3 but with $p = 1$, $q = \infty$. Then*

1)

$$\left| \int_a^b f(z(t)) f^{(n+1)}(z(t)) z'(t) \, dt \right| \leq \int_a^b |f(z(t))| \left| f^{(n+1)}(z(t)) \right| |z'(t)| \, dt \leq$$

$$\left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c) - z|^n |dz| \right) |z'(c)| \, dc \right] \left\| f^{(n+1)} \right\|_{\gamma_{x,y},\infty}^2, \tag{11}$$

equivalently it holds

2)

$$\left| \int_{\gamma_{x,w}} f(z) f^{(n+1)}(z) \, dz \right| \leq \int_{\gamma_{x,w}} |f(z)| \left| f^{(n+1)}(z) \right| |dz| \leq$$

$$\left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c) - z|^n |dz| \right) |z'(c)| \, dc \right] \left\| f^{(n+1)} \right\|_{\gamma_{x,y},\infty}^2. \tag{12}$$

Proof. By (3) we obtain again

$$f(y)=\frac{1}{n!}\int_{\gamma_{x,y}}(y-z)^n f^{(n+1)}(z)\,dz,\ \ n\in\mathbb{N}. \tag{13}$$

Hence it holds

$$|f(y)|\leq\frac{1}{n!}\int_{\gamma_{x,y}}|y-z|^n\left|f^{(n+1)}(z)\right||dz|\leq$$

$$\left(\int_{\gamma_{x,y}}|y-z|^n|dz|\right)\left\|f^{(n+1)}\right\|_{\gamma_{x,y},\infty}. \tag{14}$$

Therefore we have

$$|f(y)|\left|f^{(n+1)}(y)\right|\leq\left(\int_{\gamma_{x,y}}|y-z|^n|dz|\right)\left\|f^{(n+1)}\right\|^2_{\gamma_{x,y},\infty}. \tag{15}$$

That is

$$|f(z(c))|\left|f^{(n+1)}(z(c))\right||z'(c)|\leq\left(\int_{\gamma_{x,y}}|z(c)-z|^n|dz|\right)|z'(c)|\left\|f^{(n+1)}\right\|^2_{\gamma_{x,y},\infty}. \tag{16}$$

Consequently by integration of (16) we derive

$$\int_a^b|f(z(c))|\left|f^{(n+1)}(z(c))\right||z'(c)|\,dc\leq$$

$$\left\{\int_a^b\left[\left(\int_{\gamma_{x,y}}|z(c)-z|^n|dz|\right)|z'(c)|\right]dc\right\}\left\|f^{(n+1)}\right\|^2_{\gamma_{x,y},\infty}, \tag{17}$$

proving the claim. ■

A typical case follows:

Corollary 5 *(to Theorem 3 when $p=q=2$) We have*

1)

$$\left|\int_a^b f(z(t))\,f^{(n+1)}(z(t))\,z'(t)\,dt\right|\leq\int_a^b|f(z(t))|\left|f^{(n+1)}(z(t))\right||z'(t)|\,dt\leq \tag{18}$$

$$\frac{1}{\sqrt{2}n!}\left[\int_a^b\left(\int_a^c|z(c)-z(t)|^{2n}|z'(t)|\,dt\right)|z'(c)|\,dc\right]^{\frac{1}{2}}\cdot$$

$$\left(\int_a^b\left|f^{(n+1)}(z(t))\right|^2|z'(t)|\,dt\right),$$

equivalently it holds

2)

$$\left|\int_{\gamma_{x,w}} f(z)\, f^{(n+1)}(z)\, dz\right| \leq \int_{\gamma_{x,w}} |f(z)| \left|f^{(n+1)}(z)\right| |dz| \leq$$

$$\frac{1}{\sqrt{2}n!}\left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c)-z|^{2n} |dz|\right) |z'(c)|\, dc\right]^{\frac{1}{2}} \left(\int_{\gamma_{x,w}} \left|f^{(n+1)}(z)\right|^2 |dz|\right). \tag{19}$$

We finish with

Corollary 6 *(to Theorem 3, $n = 0$ case) Here we assume that $f(x) = 0$. Then*

$$\left|\int_{\gamma_{x,w}} f(z)\, f'(z)\, dz\right| \leq \int_{\gamma_{x,w}} |f(z)|\, |f'(z)|\, |dz| \leq$$

$$2^{-\frac{1}{q}} \left(\int_{\gamma_{x,w}} l(\gamma_{x,z})\, |dz|\right)^{\frac{1}{p}} \left(\int_{\gamma_{x,w}} |f'(z)|^q\, |dz|\right)^{\frac{2}{q}}. \tag{20}$$

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