Complex left Caputo fractional inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we present several complex left Caputo type fractional inequalities of well known kinds, such as of Ostrowski, Poincare, Sobolev, Opial and Hilbert-Pachpatte.

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1 Introduction

We are motivated by the following result for functions of complex variable: Complex Ostrowski type inequality

Theorem 1 *(see [3]) Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a, b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\left| f(v)(w-u) - \int_{\gamma} f(z)\, dz \right| \leq$$

$$\|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u|\, |dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w|\, |dz| \leq$$

$$\left[\int_{\gamma_{u,v}} |z-u|\, |dz| + \int_{\gamma_{v,w}} |z-w|\, |dz| \right] \|f'\|_{\gamma_{u,w};\infty},$$

and

$$\left| f(v)(w-u) - \int_{\gamma} f(z)\, dz \right| \leq$$

$$\max_{z \in \gamma_{u,v}} |z-u|\, \|f'\|_{\gamma_{u,v};1} + \max_{z \in \gamma_{v,w}} |z-w|\, \|f'\|_{\gamma_{v,w};1} \leq$$

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$$\max\left\{\max_{z\in\gamma_{u,v}}|z-u|,\max_{z\in\gamma_{v,w}}|z-w|\right\}\|f'\|_{\gamma_{u,w};1}.$$

If $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|f(v)(w-u)-\int_{\gamma}f(z)\,dz\right|\leq$$

$$\left(\int_{\gamma_{u,v}}|z-u|^{q}|dz|\right)^{\frac{1}{q}}\|f'\|_{\gamma_{u,v};p}+\left(\int_{\gamma_{v,w}}|z-w|^{q}|dz|\right)^{\frac{1}{q}}\|f'\|_{\gamma_{v,w};p}\leq$$

$$\left(\int_{\gamma_{u,v}}|z-u|^{q}|dz|+\int_{\gamma_{v,w}}|z-w|^{q}|dz|\right)^{\frac{1}{q}}\|f'\|_{\gamma_{u,w};p}.$$

Above $|\cdot|$ *is the complex absolute value.*

We are also motivated by the next complex Opial type inequality:

Theorem 2 *(see [2]) Let* $f:D\subseteq\mathbb{C}\to\mathbb{C}$ *be an analytic function on the domain* D *and let* $x,y,w\in D$. *Suppose* γ *is a smooth path parametrized by* $z(t)$, $t\in[a,b]$ *with* $z(a)=x$, $z(c)=y$, *and* $z(b)=w$, *where* $c\in[a,b]$ *is floating. Assume that* $f^{(k)}(x)=0$, $k=0,1,...,n$, $n\in\mathbb{Z}_{+}$, *and* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Then*

1)

$$\left|\int_{a}^{b}f(z(t))\,f^{(n+1)}(z(t))\,z'(t)\,dt\right|\leq\int_{a}^{b}|f(z(t))|\left|f^{(n+1)}(z(t))\right||z'(t)|\,dt\leq$$

$$\frac{1}{2^{\frac{1}{q}}n!}\left[\int_{a}^{b}\left(\int_{a}^{c}|z(c)-z(t)|^{pn}|z'(t)|\,dt\right)|z'(c)|\,dc\right]^{\frac{1}{p}}\cdot$$

$$\left(\int_{a}^{b}\left|f^{(n+1)}(z(t))\right|^{q}|z'(t)|\,dt\right)^{\frac{2}{q}},$$

equivalently it holds

2)

$$\left|\int_{\gamma_{x,w}}f(z)\,f^{(n+1)}(z)\,dz\right|\leq\int_{\gamma_{x,w}}|f(z)|\left|f^{(n+1)}(z)\right||dz|\leq$$

$$\frac{1}{2^{\frac{1}{q}}n!}\left[\int_{a}^{b}\left(\int_{\gamma_{x,y}}|z(c)-z|^{pn}|dz|\right)|z'(c)|\,dc\right]^{\frac{1}{p}}\left(\int_{\gamma_{x,w}}\left|f^{(n+1)}(z)\right|^{q}|dz|\right)^{\frac{2}{q}}.$$

Here we utilize on $\mathbb{C}$ the results of [1] which are for general Banach space valued functions.

Mainly we give different cases of the left fractional $\mathbb{C}$-Ostrowski type inequality and we continue with the left fractional: $\mathbb{C}$-Poincaré like and Sobolev like inequalities.

We present an Opial type left $\mathbb{C}$-fractional inequality, and we finish with the Hilbert-Pachpatte left $\mathbb{C}$-fractional inequalities.

2 Background

In this section all integrals are of Bochner type.

We need

Definition 3 *(see [4]) A definition of the Hausdorff measure h_α goes as follows: if (T,d) is a metric space, $A \subseteq T$ and $\delta > 0$, let $\Lambda(A,\delta)$ be the set of all arbitrary collections $(C)_i$ of subsets of T, such that $A \subseteq \cup_i C_i$ and $\operatorname{diam}(C_i) \leq \delta$ (diam =diameter) for every i. Now, for every $\alpha > 0$ define*

$$h_\alpha^\delta(A) := \inf\left\{\sum (diamC_i)^\alpha \mid (C_i)_i \in \Lambda(A,\delta)\right\}. \tag{1}$$

Then there exists $\lim\limits_{\delta\to 0} h_\alpha^\delta(A) = \sup\limits_{\delta>0} h_\alpha^\delta(A)$, and $h_\alpha(A) := \lim\limits_{\delta\to 0} h_\alpha^\delta(A)$ gives an outer measure on the power set $\mathcal{P}(T)$, which is countably additive on the σ-field of all Borel subsets of T. If $T = \mathbb{R}^n$, then the Hausdorff measure h_n, restricted to the σ-field of the Borel subsets of $\mathbb{R}^n$, equals the Lebesgue measure on $\mathbb{R}^n$ up to a constant multiple. In particular, $h_1(C) = \mu(C)$ for every Borel set $C \subseteq \mathbb{R}$, where μ is the Lebesgue measure.

Definition 4 *([1]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\nu > 0$; $n := \lceil \nu \rceil \in \mathbb{N}$, $\lceil \cdot \rceil$ is the ceiling of the number, $f : [a,b] \to X$. We assume that $f^{(n)} \in L_1([a,b],X)$. We call the Caputo-Bochner left fractional derivative of order ν:*

$$\left(D_{*a}^\nu f\right)(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \quad \forall\ x \in [a,b]. \tag{2}$$

If $\nu \in \mathbb{N}$, we set $D_{*a}^\nu f := f^{(\nu)}$ the ordinary X-valued derivative, defined similarly to the numerical one, and also set $D_{*a}^0 f := f$.

By [1] $\left(D_{*a}^\nu f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^\nu f \in L_1([a,b],X)$.

If $\left\|f^{(n)}\right\|_{L_\infty([a,b],X)} < \infty$, then by [1] $D_{*a}^\nu f \in C([a,b],X)$.

We need the left-fractional Taylor's formula:

Theorem 5 *([1]) Let $n \in \mathbb{N}$ and $f \in C^{n-1}([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space, and let $\nu \geq 0 : n = \lceil \nu \rceil$. Set*

$$F_x(t) := \sum_{i=0}^{n-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\ t \in [a,x], \tag{3}$$

where $x \in [a,b]$.

Assume that $f^{(n)}$ exists outside a μ-null Borel set $B_x \subseteq [a,x]$, such that

$$h_1(F_x(B_x)) = 0,\ \forall\ x \in [a,b]. \tag{4}$$

We also assume that $f^{(n)} \in L_1([a,b],X)$. Then

$$f(x) = \sum_{i=0}^{n-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \left(D_{*a}^\nu f\right)(z)\, dz, \tag{5}$$

$\forall\ x \in [a,b]$.

Next we mention an Ostrowski type inequality at left fractional level for Banach valued functions.

Theorem 6 *([1]) Let $\nu \geq 0$, $n = \lceil \nu \rceil$. Here all as in Theorem 5. Assume that $f^{(i)}(a) = 0$, $i = 1, ..., n-1$, and that $D_{*a}^{\nu} f \in L_{\infty}([a,b], X)$. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\,dx - f(a) \right\| \leq \frac{\| D_{*a}^{\nu} f \|_{L_{\infty}([a,b],X)}}{\Gamma(\nu+2)} (b-a)^{\nu}. \tag{6}$$

We mention an Ostrowski type L_p fractional inequality:

Theorem 7 *([1]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Here all as in Theorem 5. Assume that $f^{(k)}(a) = 0$, $k = 1, ..., n-1$, and $D_{*a}^{\nu} f \in L_q([a,b], X)$, where X is a Banach space. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\,dx - f(a) \right\| \leq \frac{\| D_{*a}^{\nu} f \|_{L_q([a,b],X)}}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (b-a)^{\nu - \frac{1}{q}}. \tag{7}$$

It follows

Corollary 8 *([1]) (to Theorem 7, case of $p = q = 2$). Let $\nu > \frac{1}{2}$, $n = \lceil \nu \rceil$. Here all as in Theorem 5. Assume that $f^{(k)}(a) = 0$, $k = 1, ..., n-1$, and $D_{*a}^{\nu} f \in L_2([a,b], X)$. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\,dx - f(a) \right\| \leq \frac{\| D_{*a}^{\nu} f \|_{L_2([a,b],X)}}{\Gamma(\nu) \left(\sqrt{2\nu - 1}\right) \left(\nu + \frac{1}{2}\right)} (b-a)^{\nu - \frac{1}{2}}. \tag{8}$$

Next comes the L_1 case of fractional Ostrowski inequality:

Theorem 9 *([1]) Let $\nu \geq 1$, $n = \lceil \nu \rceil$, and all as in Theorem 5. Assume that $f^{(k)}(a) = 0$, $k = 1, ..., n-1$, and $D_{*a}^{\nu} f \in L_1([a,b], X)$. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\,dx - f(a) \right\| \leq \frac{\| D_{*a}^{\nu} f \|_{L_1([a,b],X)}}{\Gamma(\nu+1)} (b-a)^{\nu - 1}. \tag{9}$$

We continue with a Poincaré like fractional inequality:

Theorem 10 *([1]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Here all as in Theorem 5. Assume that $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$, and $D_{*a}^{\nu} f \in L_q([a,b], X)$, where X is a Banach space. Then*

$$\| f \|_{L_q([a,b],X)} \leq \frac{(b-a)^{\nu}}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}} (q\nu)^{\frac{1}{q}}} \| D_{*a}^{\nu} f \|_{L_q([a,b],X)}. \tag{10}$$

Next comes a Sobolev like fractional inequality.

Theorem 11 *([1]) All as in the last Theorem 10. Let $r > 0$. Then*

$$\|f\|_{L_r([a,b],X)} \leq \frac{(b-a)^{\nu - \frac{1}{q} + \frac{1}{r}}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(r\left(\nu - \frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \left\|D_{*a}^{\nu} f\right\|_{L_q([a,b],X)}. \tag{11}$$

We mention the following Opial type fractional inequality:

Theorem 12 *([1]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n := \lceil \nu \rceil$. Let $[a,b] \subset \mathbb{R}$, X a Banach space, and $f \in C^{n-1}([a,b], X)$. Set*

$$F_x(t) := \sum_{i=0}^{n-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [a,x], \ \text{where } x \in [a,b]. \tag{12}$$

Assume that $f^{(n)}$ exists outside a μ-null Borel set $B_x \subseteq [a,x]$, such that

$$h_1\left(F_x(B_x)\right) = 0, \quad \forall\, x \in [a,b]. \tag{13}$$

We also assume that $f^{(n)} \in L_\infty([a,b], X)$. Assume also that $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$. Then

$$\int_a^x \|f(w)\| \left\|\left(D_{*a}^{\nu} f\right)(w)\right\| dw \leq$$

$$\frac{(x-a)^{\nu - 1 + \frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\nu)\left(\left(p(\nu-1)+1\right)\left(p(\nu-1)+2\right)\right)^{\frac{1}{p}}} \left(\int_a^x \left\|\left(D_{*a}^{\nu} f\right)(z)\right\|^q dz\right)^{\frac{2}{q}}, \tag{14}$$

$\forall\, x \in [a,b]$.

We finish this section with a Hilbert-Pachpatte left fractional inequality:

Theorem 13 *([1]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu_1 > \frac{1}{q}$, $\nu_2 > \frac{1}{p}$, $n_i := \lceil \nu_i \rceil$, $i = 1, 2$. Here $[a_i, b_i] \subset \mathbb{R}$, $i = 1, 2$; X is a Banach space. Let $f_i \in C^{n_i - 1}([a_i, b_i], X)$, $i = 1, 2$. Set*

$$F_{x_i}(t_i) := \sum_{j_i=0}^{n_i-1} \frac{(x_i - t_i)^{j_i}}{j_i!} f_i^{(j_i)}(t_i), \tag{15}$$

$\forall\, t_i \in [a_i, x_i]$, where $x_i \in [a_i, b_i]$; $i = 1, 2$. Assume that $f_i^{(n_i)}$ exists outside a μ-null Borel set $B_{x_i} \subseteq [a_i, x_i]$, such that

$$h_1\left(F_{x_i}(B_{x_i})\right) = 0, \quad \forall\, x_i \in [a_i, b_i]; \ i = 1, 2. \tag{16}$$

We also assume that $f_i^{(n_i)} \in L_1([a_i, b_i], X)$, and

$$f_i^{(k_i)}(a_i) = 0, \quad k_i = 0, 1, ..., n_i - 1; \ i = 1, 2, \tag{17}$$

and

$$\left(D_{*a_1}^{\nu_1} f_1\right) \in L_q\left(\left[a_1, b_1\right], X\right), \quad \left(D_{*a_2}^{\nu_2} f_2\right) \in L_p\left(\left[a_2, b_2\right], X\right).$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left\|f_1\left(x_1\right)\right\| \left\|f_2\left(x_2\right)\right\| dx_1 dx_2}{\left(\frac{\left(x_1-a_1\right)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)} + \frac{\left(x_2-a_2\right)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)} \leq$$

$$\frac{\left(b_1-a_1\right)\left(b_2-a_2\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)} \left\|D_{*a_1}^{\nu_1} f_1\right\|_{L_q([a_1,b_1],X)} \left\|D_{*a_2}^{\nu_2} f_2\right\|_{L_p([a_2,b_2],X)}. \tag{18}$$

3 Main Results

We need a special case of Definition 4 over $\mathbb{C}$.

Definition 14 *Let $[a,b] \subset \mathbb{R}$, $\nu > 0$; $n := \lceil \nu \rceil \in \mathbb{N}$, $\lceil \cdot \rceil$ is the ceiling of the number and $f \in C^n([a,b], \mathbb{C})$. We call Caputo-Complex left fractional derivative of order ν:*

$$\left(D_{*a}^{\nu} f\right)(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \quad \forall\, x \in [a,b], \tag{19}$$

where the derivatives $f', \ldots f^{(n)}$ are defined as the numerical derivative.

If $\nu \in \mathbb{N}$, we set $D_{*a}^{\nu} f := f^{(\nu)}$ the ordinary $\mathbb{C}$-valued derivative and also set $D_{*a}^{0} f := f$.

Notice here (by [1]) that $D_{*a}^{\nu} f \in C([a,b], \mathbb{C})$.

We make

Remark 15 *Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ (i.e. there exists $z'(t)$ and is continuous) and from now on f is a complex function which is continuous on γ.*

Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt = \int_a^b h(t)\, dt, \tag{20}$$

where $h(t) := f(z(t))\, z'(t)$, $t \in [a,b]$.

We notice that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\, dz := \int_{\gamma_{u,v}} f(z)\, dz + \int_{\gamma_{v,w}} f(z)\, dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ *is then*

$$l(\gamma) = \int_{\gamma_{u,w}} |dz| := \int_a^b |z'(t)|\,dt.$$

We mention also the triangle inequality for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,l(\gamma), \tag{21}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We give the following left-fractional $\mathbb{C}$-Taylor's formula:

Theorem 16 *Let* $h \in C^n([a,b],\mathbb{C})$, $n = \lceil \nu \rceil$, $\nu \ge 0$. *Then*

$$h(t) = \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!} h^{(i)}(a) + \frac{1}{\Gamma(\nu)} \int_a^t (t-\lambda)^{\nu-1} (D_{*a}^\nu h)(\lambda)\,d\lambda, \tag{22}$$

$\forall\ t \in [a,b]$,

in particular it holds,

$$f(z(t))\,z'(t) = \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!} \left(f(z(a))\,z'(a)\right)^{(i)} +$$

$$\frac{1}{\Gamma(\nu)} \int_a^t (t-\lambda)^{\nu-1} \left(D_{*a}^\nu f(z(\cdot))\,z'(\cdot)\right)(\lambda)\,d\lambda, \tag{23}$$

$\forall\ t \in [a,b]$.

Proof. By Theorem 5. ■

It follows a left fractional $\mathbb{C}$-Ostroswski type inequality

Theorem 17 *Let* $n \in \mathbb{N}$ *and* $h \in C^n([a,b],\mathbb{C})$, *where* $[a,b] \subset \mathbb{R}$, *and let* $\nu \ge 0 : n = \lceil \nu \rceil$. *Assume that* $h^{(i)}(a) = 0$, $i = 1,...,n-1$. *Then*

$$\left|\frac{1}{b-a}\int_a^b h(t)\,dt - f(a)\right| \le \frac{\|D_{*a}^\nu h\|_{\infty,[a,b]}}{\Gamma(\nu+2)} (b-a)^\nu, \tag{24}$$

in particular when $h(t) := f(z(t))\,z'(t)$ *and* $\left(f(z(t))\,z'(t)\right)^{(i)}|_{t=a} = 0$, $i = 1,...n-1$, *we get*

$$\left|\frac{1}{b-a}\int_{\gamma_{u,w}} f(z)\,dz - f(u)\,z'(a)\right| = \left|\frac{1}{b-a}\int_a^b f(z(t))\,z'(t)\,dt - f(z(a))\,z'(a)\right|$$

$$\le \frac{\|D_{*a}^\nu f(z(t))\,z'(t)\|_{\infty,[a,b]}}{\Gamma(\nu+2)} (b-a)^\nu. \tag{25}$$

Proof. By Theorem 6. ■

The corresponding $\mathbb{C}$-Ostrowski type L_p inequality follows:

Theorem 18 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Here $h \in C^n([a,b], \mathbb{C})$. Assume that $h^{(i)}(a) = 0$, $i = 1, ..., n-1$. Then*

$$\left| \frac{1}{b-a} \int_a^b h(t)\, dt - h(a) \right| \leq \frac{\|D_{*a}^{\nu} h\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (b-a)^{\nu - \frac{1}{q}}, \tag{26}$$

in particular when $h(t) := f(z(t))\, z'(t)$ and $(f(z(t))\, z'(t))^{(i)}|_{t=a} = 0$, $i = 1, ...n-1$, we get:

$$\left| \frac{1}{b-a} \int_{\gamma_{u,w}} f(z)\, dz - f(u)\, z'(a) \right| = \left| \frac{1}{b-a} \int_a^b f(z(t))\, z'(t)\, dt - f(z(a))\, z'(a) \right|$$

$$\leq \frac{\|D_{*a}^{\nu}(f(z(t))\, z'(t))\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (b-a)^{\nu - \frac{1}{q}}. \tag{27}$$

Proof. By Theorem 7. ■

It follows

Corollary 19 *(to Theorem 18, case of $p = q = 2$). We have that*

$$\left| \frac{1}{b-a} \int_{\gamma_{u,w}} f(z)\, dz - f(u)\, z'(a) \right| \leq \frac{\|D_{*a}^{\nu}(f(z(t))\, z'(t))\|_{L_2([a,b],\mathbb{C})}}{\Gamma(\nu)\,\sqrt{2\nu-1}\left(\nu + \frac{1}{2}\right)} (b-a)^{\nu - \frac{1}{2}}. \tag{28}$$

We continue with an L_1 fractional $\mathbb{C}$-Ostrowski type inequality:

Theorem 20 *Let $\nu \geq 1$, $n = \lceil \nu \rceil$. Assume that $h \in C^n([a,b], \mathbb{C})$, where $h(t) := f(z(t))\, z'(t)$, and such that $h^{(i)}(a) = 0$, $i = 1, ..., n-1$. Then*

$$\left| \frac{1}{b-a} \int_{\gamma_{u,w}} f(z)\, dz - f(u)\, z'(a) \right| \leq \frac{\|D_{*a}^{\nu}(f(z(t))\, z'(t))\|_{L_1([a,b],\mathbb{C})}}{\Gamma(\nu+1)} (b-a)^{\nu-1}. \tag{29}$$

Proof. By Theorem 9. ■

It follows a Poincaré like $\mathbb{C}$-fractional inequality:

Theorem 21 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Let $h \in C^n([a,b], \mathbb{C})$. Assume that $h^{(i)}(a) = 0$, $i = 1, ..., n-1$. Then*

$$\|h\|_{L_q([a,b],\mathbb{C})} \leq \frac{(b-a)^{\nu}\, \|D_{*a}^{\nu} h\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}}\,(q\nu)^{\frac{1}{q}}}, \tag{30}$$

in particular when $h(t) := f(z(t))\, z'(t)$ *and* $\left(f(z(t))\, z'(t)\right)^{(i)}|_{t=a} = 0$, $i = 1, \ldots n-1$, *we get:*

$$\left\| f(z(t))\, z'(t) \right\|_{L_q([a,b],\mathbb{C})} \leq$$

$$\frac{(b-a)^{\nu}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}}\,(q\nu)^{\frac{1}{q}}} \left\| D_{*a}^{\nu} \left(f(z(t))\, z'(t)\right) \right\|_{L_q([a,b],\mathbb{C})}. \tag{31}$$

Proof. By Theorem 10. ■

The corresponding Sobolev like inequality follows:

Theorem 22 *All as in Theorem 21. Let* $r > 0$. *Then*

$$\left\| f(z(t))\, z'(t) \right\|_{L_r([a,b],\mathbb{C})} \leq$$

$$\frac{(b-a)^{\nu-\frac{1}{q}+\frac{1}{r}}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(r\left(\nu-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \left\| D_{*a}^{\nu} \left(f(z(t))\, z'(t)\right) \right\|_{L_q([a,b],\mathbb{C})}. \tag{32}$$

Proof. By Theorem 11. ■

We continue with an Opial type $\mathbb{C}$-fractional inequality

Theorem 23 *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $\nu > \frac{1}{q}$, $n := \lceil \nu \rceil$, $h \in C^n([a,b], \mathbb{C})$. *Assume* $h^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$. *Then*

$$\int_a^x |h(t)|\, \left|\left(D_{*a}^{\nu} h\right)(t)\right| dt \leq$$

$$\frac{(x-a)^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}}\,\Gamma(\nu)\,\left((p(\nu-1)+1)(p(\nu-1)+2)\right)^{\frac{1}{p}}} \left(\int_a^x \left|\left(D_{*a}^{\nu} h\right)(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{33}$$

$\forall\, x \in [a,b]$,

in particular when $h(t) := f(z(t))\, z'(t)$ *and* $\left(f(z(t))\, z'(t)\right)^{(i)}|_{t=a} = 0$, $i = 1, \ldots n-1$, *we get:*

$$\int_a^x |f(z(t))|\, \left|\left(D_{*a}^{\nu} \left(f(z(t))\, z'(t)\right)\right)\right| |z'(t)|\, dt \leq$$

$$\frac{(x-a)^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}}\,\Gamma(\nu)\,\left((p(\nu-1)+1)(p(\nu-1)+2)\right)^{\frac{1}{p}}} \left(\int_a^x \left|D_{*a}^{\nu} \left(f(z(t))\, z'(t)\right)\right|^q dt\right)^{\frac{2}{q}}, \tag{34}$$

$\forall\, x \in [a,b]$.

Proof. By Theorem 12. ■

We finish with Hilbert-Pachpatte left $\mathbb{C}$-fractional inequalities:

Theorem 24 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu_1 > \frac{1}{q}$, $\nu_2 > \frac{1}{p}$, $n_i := \lceil \nu_i \rceil$, $i = 1, 2$. Let $h_i \in C^{n_i}([a_i, b_i], \mathbb{C})$, $i = 1, 2$. Assume $h_i^{(k_i)}(a_i) = 0$, $k_i = 0, 1, ..., n_i - 1$; $i = 1, 2$. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|h_1(t_1)| |h_2(t_2)| \, dt_1 dt_2}{\left(\frac{(t_1 - a_1)^{p(\nu_1 - 1) + 1}}{p(p(\nu_1 - 1) + 1)} + \frac{(t_2 - a_2)^{q(\nu_2 - 1) + 1}}{q(q(\nu_2 - 1) + 1)} \right)} \leq$$

$$\frac{(b_1 - a_1)(b_2 - a_2)}{\Gamma(\nu_1) \Gamma(\nu_2)} \left\| D_{*a_1}^{\nu_1} h_1 \right\|_{L_q([a_1, b_1], \mathbb{C})} \left\| D_{*a_2}^{\nu_2} h_2 \right\|_{L_p([a_2, b_2], \mathbb{C})}, \tag{35}$$

in particular when $h_1(t_1) := f_1(z_1(t_1)) z_1'(t_1)$ and $h_2(t_2) := f_2(z_2(t_2)) z_2'(t_2)$, with $h_i^{(k_i)}(a_i) = 0$, $k_i = 0, 1, ... n_i - 1$; $i = 1, 2$, we get:

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(z_1(t_1)) z_1'(t_1)| |f_2(z_2(t_2)) z_2'(t_2)| \, dt_1 dt_2}{\left(\frac{(t_1 - a_1)^{p(\nu_1 - 1) + 1}}{p(p(\nu_1 - 1) + 1)} + \frac{(t_2 - a_2)^{q(\nu_2 - 1) + 1}}{q(q(\nu_2 - 1) + 1)} \right)} \leq \frac{(b_1 - a_1)(b_2 - a_2)}{\Gamma(\nu_1) \Gamma(\nu_2)} \cdot$$

$$\left\| D_{*a_1}^{\nu_1} (f_1(z_1(t_1)) z_1'(t_1)) \right\|_{L_q([a_1, b_1], \mathbb{C})} \left\| D_{*a_2}^{\nu_2} (f_2(z_2(t_2)) z_2'(t_2)) \right\|_{L_p([a_2, b_2], \mathbb{C})}. \tag{36}$$

Proof. By Theorem 13. ■

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