Right Complex Caputo fractional inequalities

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Abstract

Here we establish several important right complex Caputo type fractional inequalities of the following kinds: Ostrowski's, Poincare's, Sobolev's, Opial's and Hilbert-Pachpatte's.

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1 Introduction

Here we follow [3].

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt.$$

We observe that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\, dz := \int_{\gamma_{u,v}} f(z)\, dz + \int_{\gamma_{v,w}} f(z)\, dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\, |dz| := \int_a^b f(z(t))\, |z'(t)|\, dt$$

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and the length of the curve γ is then

$$l(\gamma) = \int_{\gamma_{u,w}} |dz| := \int_a^b |z'(t)|\, dt,$$

where $|\cdot|$ is the complex absolute value.

Let f and g be holomorphic in G, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the integration by parts formula

$$\int_{\gamma_{u,w}} f(z)\, g'(z)\, dz = f(w)\, g(w) - f(u)\, g(u) - \int_{\gamma_{u,w}} f'(z)\, g(z)\, dz. \quad (1)$$

We recall also the triangle inequality for the complex integral, namely

$$\left| \int_\gamma f(z)\, dz \right| \le \int_\gamma |f(z)|\, |dz| \le \|f\|_{\gamma,\infty}\, l(\gamma), \quad (2)$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\, |dz| \right)^{\frac{1}{p}}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\, |dz|.$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [l(\gamma)]^{\frac{1}{q}}\, \|f\|_{\gamma,p}. \quad (3)$$

We are inspired by the following extensions of Stekloff and Almansi inequalities to the complex integral:

Theorem 1 *([3]) Let f be analytic in G, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ and $z'(t) \ne 0$ for $t \in (a, b)$.*

(i) If $\int_\gamma f(z)\, |dz| = 0$, then

$$\int_\gamma |f(z)|^2\, |dz| \le \frac{1}{\pi^2} l^2(\gamma) \int_\gamma |f'(z)|^2\, |dz|. \quad (4)$$

(ii) In addition, if $f(u) = f(w) = 0$, then

$$\int_\gamma |f(z)|^2\, |dz| \le \frac{1}{4\pi^2} l^2(\gamma) \int_\gamma |f'(z)|^2\, |dz|. \quad (5)$$

We are also inspired by complex Ostrowski type results:

Theorem 2 *([4]) Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D with $z, x, y \in D$ and $\lambda \in \mathbb{C}$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ with $z(a) = u$ and $z(b) = w$ where $u, w \in D$. Then*

$$\int_\gamma f(z)\,dz = [(1-\lambda) f(x) + \lambda f(y)](w-u) + \tag{6}$$

$$(1-\lambda)\sum_{k=1}^{n} \frac{1}{(k+1)!} f^{(k)}(x)\left[(w-x)^{k+1} + (-1)^k (x-u)^{k+1}\right] +$$

$$\lambda \sum_{k=1}^{n} \frac{1}{(k+1)!} f^{(k)}(y)\left[(w-y)^{k+1} + (-1)^k (y-u)^{k+1}\right] + T_{n,\lambda}(\gamma, x, y),$$

where the remainder $T_{n,\lambda}(\gamma, x, y)$ is given by

$$T_{n,\lambda}(\gamma, x, y) :=$$

$$\frac{1}{n!}\left[(1-\lambda)\int_\gamma (z-x)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)x + sz](1-s)^n\,ds\right)dz\right.$$

$$\left. + (-1)^{n+1}\lambda \int_\gamma (y-z)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)z + sy]\,s^n ds\right)dz\right] = \tag{7}$$

$$\frac{1}{n!}\left[(1-\lambda)\int_0^1 (1-s)^n\left(\int_\gamma (z-x)^{n+1} f^{(n+1)}[(1-s)x + sz]\,dz\right)ds + \right.$$

$$\left.(-1)^{n+1}\lambda \int_0^1 s^n \left(\int_\gamma (y-z)^{n+1} f^{(n+1)}[(1-s)z + sy]\,dz\right)ds\right].$$

Estimations of the above remainder follow:

Theorem 3 *([4]) Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D with $x, y \in D$ and $\lambda \in \mathbb{C}$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ with $z(a) = u$ and $z(b) = w$ where $u, w \in D$. Then we have the representation (6) and the remainder $T_{n,\lambda}(\gamma, x, y)$ satisfies the inequalities*

$$|T_{n,\lambda}(\gamma, x, y)| \le$$

$$\frac{1}{n!}\left[|1-\lambda|\left|\int_\gamma |z-x|^{n+1}\left(\int_0^1 \left|f^{(n+1)}[(1-s)x + sz]\right|(1-s)^n\,ds\right)|dz|\right|\right.$$

$$\left. + |\lambda|\left|\int_\gamma |y-z|^{n+1}\left(\int_0^1 \left|f^{(n+1)}[(1-s)z + sy]\right| s^n ds\right)|dz|\right|\right] \tag{8}$$

$$\leq \frac{1}{n!}\left|1-\lambda\right|\begin{cases}\frac{1}{n+1}\int_{\gamma}\left|z-x\right|^{n+1}\left(\max_{s\in[0,1]}\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|\right)\left|dz\right| \\ \\ \frac{1}{(qn+1)^{\frac{1}{p}}}\int_{\gamma}\left|z-x\right|^{n+1}\left(\int_0^1\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|^p ds\right)^{\frac{1}{p}}\left|dz\right| \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_{\gamma}\left|z-x\right|^{n+1}\left(\int_0^1\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|ds\right)\left|dz\right|\end{cases}$$

$$+\frac{1}{n!}\left|\lambda\right|\begin{cases}\frac{1}{n+1}\int_{\gamma}\left|y-z\right|^{n+1}\left(\max_{s\in[0,1]}\left|f^{(n+1)}\left[(1-s)z+sy\right]\right|\right)\left|dz\right| \\ \\ \frac{1}{(qn+1)^{\frac{1}{p}}}\int_{\gamma}\left|y-z\right|^{n+1}\left(\int_0^1\left|f^{(n+1)}\left[(1-s)z+sy\right]\right|^p ds\right)^{\frac{1}{p}}\left|dz\right| \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_{\gamma}\left|y-z\right|^{n+1}\left(\int_0^1\left|f^{(n+1)}\left[(1-s)z+sy\right]\right|ds\right)\left|dz\right|,\end{cases}$$

and

$$\left|T_{n,\lambda}(\gamma,x,y)\right|\leq$$

$$\frac{1}{n!}\left[\left|1-\lambda\right|\int_0^1(1-s)^n\left(\int_{\gamma}\left|z-x\right|^{n+1}\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|\left|dz\right|\right)ds+\right.$$

$$\left.\left|\lambda\right|\int_0^1 s^n\left(\int_{\gamma}\left|y-z\right|^{n+1}\left|f^{(n+1)}\left[(1-s)z+sy\right]\right|\left|dz\right|\right)ds\right]\leq$$

$$\frac{1}{n!}\left|1-\lambda\right|\begin{cases}\int_{\gamma}\left|z-x\right|^{n+1}\left|dz\right|\int_0^1(1-s)^n\left(\max_{z\in\gamma}\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|\right)ds \\ \\ \left(\int_{\gamma}\left|z-x\right|^{(n+1)q}\left|dz\right|\right)^{\frac{1}{q}}\int_0^1(1-s)^n\left(\int_{\gamma}\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|^p\left|dz\right|\right)^{\frac{1}{p}}ds \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \max_{z\in\gamma}\left(\left|z-x\right|^{n+1}\right)\int_0^1(1-s)^n\left(\int_{\gamma}\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|ds\right)\left|dz\right|\end{cases} \tag{9}$$

$$+\frac{1}{n!}\left|\lambda\right|\begin{cases}\int_{\gamma}\left|z-y\right|^{n+1}\left|dz\right|\int_0^1 s^n\left(\max_{z\in\gamma}\left|f^{(n+1)}\left[(1-s)z+sy\right]\right|\right)ds \\ \\ \left(\int_{\gamma}\left|z-y\right|^{(n+1)q}\left|dz\right|\right)^{\frac{1}{q}}\int_0^1 s^n\left(\int_{\gamma}\left|f^{(n+1)}\left[(1-s)z+sy\right]\right|^p\left|dz\right|\right)^{\frac{1}{p}}ds \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \max_{z\in\gamma}\left(\left|z-y\right|^{n+1}\right)\int_0^1 s^n\left(\int_{\gamma}\left|f^{(n+1)}\left[(1-s)z+sy\right]\right|\left|dz\right|\right)ds.\end{cases}$$

In this article we utilize on $\mathbb{C}$ the results of [2] which are for general Banach space valued functions.

Mainly we give different cases of the right fractional $\mathbb{C}$-Ostrowski type inequality and we continue with the right fractional: $\mathbb{C}$-Poincaré like and Sobolev like inequalities.

We present an Opial type right $\mathbb{C}$-fractional inequality, and we finish with the Hilbert-Pachpatte right $\mathbb{C}$-fractional inequalities.

2 Background

In this section all integrals are of Bochner type.

We need

Definition 4 *(see [5]) A definition of the Hausdorff measure h_α goes as follows: if (T, d) is a metric space, $A \subseteq T$ and $\delta > 0$, let $\Lambda(A, \delta)$ be the set of all arbitrary collections $(C)_i$ of subsets of T, such that $A \subseteq \cup_i C_i$ and $diam(C_i) \leq \delta$ (diam =diameter) for every i. Now, for every $\alpha > 0$ define*

$$h_\alpha^\delta(A) := \inf\left\{\sum (diam C_i)^\alpha \mid (C_i)_i \in \Lambda(A, \delta)\right\}. \tag{10}$$

Then there exists $\lim_{\delta \to 0} h_\alpha^\delta(A) = \sup_{\delta > 0} h_\alpha^\delta(A)$, and $h_\alpha(A) := \lim_{\delta \to 0} h_\alpha^\delta(A)$ gives an outer measure on the power set $\mathcal{P}(T)$, which is countably additive on the σ-field of all Borel subsets of T. If $T = \mathbb{R}^n$, then the Hausdorff measure h_n, restricted to the σ-field of the Borel subsets of $\mathbb{R}^n$, equals the Lebesgue measure on $\mathbb{R}^n$ up to a constant multiple. In particular, $h_1(C) = \mu(C)$ for every Borel set $C \subseteq \mathbb{R}$, where μ is the Lebesgue measure.

Definition 5 *([2]) Let $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$, ($\lceil \cdot \rceil$ the ceiling of the number). We assume that $f^{(m)} \in L_1([a, b], X)$, where $f : [a, b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^\alpha f\right)(x) := \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^b (J - x)^{m - \alpha - 1} f^{(m)}(J)\, dJ, \quad \forall\, x \in [a, b], \tag{11}$$

where $f^{(m)}$ is the ordinary X-valued derivative, defined similarly to the numerical one.

We observe that $D_{b-}^m f(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $D_{b-}^0 f(x) = f(x)$.

By [2] $\left(D_{b-}^\alpha f\right)(x)$ exists almost everywhere on $[a, b]$ and $\left(D_{b-}^\alpha f\right) \in L_1([a, b], X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, then by [2], $D_{b-}^\alpha f \in C([a, b], X)$, hence $\left\|D_{b-}^\alpha f\right\| \in C([a, b])$.

We need the right-fractional Taylor's formula:

Theorem 6 *([2]) Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a, b], X)$. Set*

$$F_x(t) := \sum_{i=0}^{m-1} \frac{(x - t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [x, b], \tag{12}$$

where $x \in [a,b]$.

Assume that $f^{(m)}$ *exists outside a* μ*-null Borel set* $B_x \subseteq [x,b]$, *such that*

$$h_1 \left(F_x \left(B_x \right) \right) = 0, \;\; \forall \; x \in [a,b] . \tag{13}$$

We also assume that $f^{(m)} \in L_1 \left([a,b], X \right)$. *Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-b)^i}{i!} f^{(i)}(b) + \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} \left(D_{b-}^{\alpha} f \right)(z)\, dz, \tag{14}$$

$\forall \; x \in [a,b]$.

Next we mention Ostrowski type inequalities at right fractional level for Banach valued functions. See also [1].

Theorem 7 *([2]) Let* $\alpha > 0$, $m = \lceil \alpha \rceil$. *Here all as in Theorem 6. Assume* $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, *and* $D_{b-}^{\alpha} f \in L_{\infty}\left([a,b], X \right)$. *Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right\| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_{\infty}([a,b],X)}}{\Gamma(\alpha+2)} (b-a)^{\alpha} . \tag{15}$$

We also give

Theorem 8 *([2]) Let* $\alpha \geq 1$, $m = \lceil \alpha \rceil$. *Here all as in Theorem 6. Assume that* $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, *and* $D_{b-}^{\alpha} f \in L_1 \left([a,b], X \right)$. *Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right\| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b],X)}}{\Gamma(\alpha+1)} (b-a)^{\alpha-1} . \tag{16}$$

We mention also an L_p abstract Ostrowski type inequality:

Theorem 9 *([2]) Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = \lceil \alpha \rceil$. *Here all as in Theorem 6. Assume that* $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, *and* $D_{b-}^{\alpha} f \in L_q \left([a,b], X \right)$. *Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right\| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b],X)}}{\Gamma(\alpha) \left(p(\alpha-1)+1 \right)^{\frac{1}{p}} \left(\alpha + \frac{1}{p} \right)} (b-a)^{\alpha - \frac{1}{q}} . \tag{17}$$

It follows

Corollary 10 *([2]) Let* $\alpha > \frac{1}{2}$, $m = \lceil \alpha \rceil$. *All as in Theorem 6. Assume* $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, $D_{b-}^{\alpha} f \in L_2 \left([a,b], X \right)$. *Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right\| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_2([a,b],X)}}{\Gamma(\alpha) \left(\sqrt{2\alpha-1} \right) \left(\alpha + \frac{1}{2} \right)} (b-a)^{\alpha - \frac{1}{2}} . \tag{18}$$

We continue with a Poincaré like right fractional inequality:

Theorem 11 *([2]) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $\alpha>\frac{1}{q}$, $m=\lceil\alpha\rceil$. Here all as in Theorem 6. Assume that $f^{(k)}(b)=0$, $k=0,1,...,m-1$, and $D_{b-}^{\alpha}f\in L_q([a,b],X)$, where X is a Banach space. Then*

$$\|f\|_{L_q([a,b],X)}\leq\frac{(b-a)^{\alpha}\left\|D_{b-}^{\alpha}f\right\|_{L_q([a,b],X)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}(q\alpha)^{\frac{1}{q}}}. \tag{19}$$

Next follows a right Sobolev like fractional inequality:

Theorem 12 *([2]) All as in the last Theorem 11. Let $r>0$. Then*

$$\|f\|_{L_r([a,b],X)}\leq\frac{(b-a)^{\alpha-\frac{1}{q}+\frac{1}{r}}\left\|D_{b-}^{\alpha}f\right\|_{L_q([a,b],X)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(r\left(\alpha-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}}. \tag{20}$$

We also mention the following Opial type right fractional inequality:

Theorem 13 *([2]) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $\alpha>\frac{1}{q}$, $m:=\lceil\alpha\rceil$. Let $[a,b]\subset\mathbb{R}$, X a Banach space, and $f\in C^{m-1}([a,b],X)$. Set*

$$F_x(t):=\sum_{i=0}^{m-1}\frac{(x-t)^i}{i!}f^{(i)}(t),\ \ \forall\ t\in[x,b],\ \text{where } x\in[a,b]. \tag{21}$$

Assume that $f^{(m)}$ exists outside a μ-null Borel set $B_x\subseteq[x,b]$, such that

$$h_1(F_x(B_x))=0,\ \ \forall\ x\in[a,b]. \tag{22}$$

We assume that $f^{(m)}\in L_{\infty}([a,b],X)$. Assume also that $f^{(k)}(b)=0$, $k=0,1,...,m-1$. Then

$$\int_x^b\|f(w)\|\left\|\left(D_{b-}^{\alpha}f\right)(w)\right\|dw\leq$$

$$\frac{(b-x)^{\alpha-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\alpha)((p(\alpha-1)+1)(p(\alpha-1)+2))^{\frac{1}{p}}}\left(\int_x^b\left\|\left(D_{b-}^{\alpha}f\right)(z)\right\|^q dz\right)^{\frac{2}{q}}, \tag{23}$$

$\forall\ x\in[a,b]$.

Next we describe an abstract Hilbert-Pachpatte right fractional inequality:

Theorem 14 *([2]) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $\alpha_1>\frac{1}{q}$, $\alpha_2>\frac{1}{p}$, $m_i:=\lceil\alpha_i\rceil$, $i=1,2$. Here $[a_i,b_i]\subset\mathbb{R}$, $i=1,2$; X is a Banach space. Let $f_i\in C^{m_i-1}([a_i,b_i],X)$, $i=1,2$. Set*

$$F_{x_i}(t_i):=\sum_{j_i=0}^{m_i-1}\frac{(x_i-t_i)^{j_i}}{j_i!}f_i^{(j_i)}(t_i), \tag{24}$$

$\forall\ t_i \in [x_i, b_i]$, *where* $x_i \in [a_i, b_i]$; $i = 1, 2$. *Assume that* $f_i^{(m_i)}$ *exists outside a* μ*-null Borel set* $B_{x_i} \subseteq [x_i, b_i]$, *such that*

$$h_1 \left(F_{x_i} \left(B_{x_i}\right)\right) = 0, \ \ \forall\ x_i \in [a_i, b_i]\ ;\ i = 1, 2. \tag{25}$$

We also assume that $f_i^{(m_i)} \in L_1 \left([a_i, b_i], X\right)$, *and*

$$f_i^{(k_i)} \left(b_i\right) = 0, \ \ k_i = 0, 1, ..., , m_i - 1;\ i = 1, 2, \tag{26}$$

and

$$\left(D_{b_1-}^{\alpha_1} f_1\right) \in L_q \left([a_1, b_1], X\right), \quad \left(D_{b_2-}^{\alpha_2} f_2\right) \in L_p \left([a_2, b_2], X\right). \tag{27}$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\|f_1(x_1)\| \|f_2(x_2)\| \, dx_1 dx_2}{\left(\frac{(b_1 - x_1)^{p(\alpha_1 - 1)+1}}{p(p(\alpha_1-1)+1)} + \frac{(b_2 - x_2)^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right)} \leq$$

$$\frac{(b_1 - a_1)(b_2 - a_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\|D_{b_1-}^{\alpha_1} f_1\right\|_{L_q([a_1,b_1],X)} \left\|D_{b_2-}^{\alpha_2} f_2\right\|_{L_p([a_2,b_2],X)}. \tag{28}$$

3 Main Results

We need a special case of Definition 5 over $\mathbb{C}$.

Definition 15 *Let* $[a, b] \subset \mathbb{R}$, $\nu > 0$; $n := \lceil \nu \rceil \in \mathbb{N}$, $\lceil \cdot \rceil$ *is the ceiling of the number and* $f \in C^n([a, b], \mathbb{C})$. *We call Caputo-Complex right fractional derivative of order* ν:

$$\left(D_{b-}^{\nu} f\right)(x) := \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (\lambda - x)^{n-\nu-1} f^{(n)}(\lambda) \, d\lambda, \ \ \forall\ x \in [a, b], \tag{29}$$

where the derivatives $f', ...f^{(n)}$ *are defined as the numerical derivative.*

If $\nu \in \mathbb{N}$, *we set* $D_{b-}^{\nu} f := (-1)^{\nu} f^{(\nu)}$ *the ordinary* $\mathbb{C}$*-valued derivative and also* $D_{b-}^0 f := f$.

Notice here (by [2]) that $D_{b-}^{\nu} f \in C([a, b], \mathbb{C})$.

We give the following right-fractional $\mathbb{C}$-Taylor's formula:

Theorem 16 *Let* $h \in C^n([a, b], \mathbb{C})$, $n = \lceil \nu \rceil$, $\nu \geq 0$. *Then*

$$h(t) = \sum_{i=0}^{n-1} \frac{(t-b)^i}{i!} h^{(i)}(b) + \frac{1}{\Gamma(\nu)} \int_t^b (\lambda - t)^{\nu-1} \left(D_{b-}^{\nu} h\right)(\lambda) \, d\lambda, \tag{30}$$

$\forall\ t \in [a, b]$,

in particular when $h(t) := f(z(t)) z'(t) \in C^n([a, b], \mathbb{C})$, *where* $f(z)$, $z(t)$, $t \in [a, b]$ *are as in 1. Introduction, it holds,*

$$f(z(t)) z'(t) = \sum_{i=0}^{n-1} \frac{(t-b)^i}{i!} \left(f(z(b)) z'(b)\right)^{(i)} +$$

$$\frac{1}{\Gamma(\nu)}\int_t^b (\lambda - t)^{\nu-1}\left(D_{b-}^{\nu} f(z(\cdot))\, z'(\cdot)\right)(\lambda)\, d\lambda, \tag{31}$$

$\forall\ t \in [a,b]$.

Proof. By Theorem 6. ■

It follows a right fractional $\mathbb{C}$-Ostroswski type inequality

Theorem 17 *Let $n \in \mathbb{N}$ and $h \in C^n([a,b],\mathbb{C})$, where $[a,b] \subset \mathbb{R}$, and let $\nu > 0 : n = \lceil \nu \rceil$. Assume that $h^{(i)}(b) = 0$, $i = 1,...,n-1$. Then*

$$\left|\frac{1}{b-a}\int_a^b h(t)\, dt - h(b)\right| \leq \frac{\left\|D_{b-}^{\nu} h\right\|_{\infty,[a,b]}}{\Gamma(\nu+2)}(b-a)^{\nu}, \tag{32}$$

in particular when $h(t) := f(z(t))\, z'(t) \in C^n([a,b],\mathbb{C})$, where $f(z)$, $z(t)$, $t \in [a,b]$ are as in 1. Introduction, and $(f(z(t))\, z'(t))^{(i)}|_{t=b} = 0$, $i = 1,...n-1$, we get

$$\left|\frac{1}{b-a}\int_{\gamma_{u,w}} f(z)\, dz - f(w)\, z'(b)\right| = \left|\frac{1}{b-a}\int_a^b f(z(t))\, z'(t)\, dt - f(z(b))\, z'(b)\right|$$

$$\leq \frac{\left\|D_{b-}^{\nu} f(z(t))\, z'(t)\right\|_{\infty,[a,b]}}{\Gamma(\nu+2)}(b-a)^{\nu}. \tag{33}$$

Proof. By Theorem 7. ■

The corresponding $\mathbb{C}$-Ostrowski type L_p inequality follows:

Theorem 18 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Here $h \in C^n([a,b],\mathbb{C})$. Assume that $h^{(i)}(b) = 0$, $i = 1,...,n-1$. Then*

$$\left|\frac{1}{b-a}\int_a^b h(t)\, dt - h(b)\right| \leq \frac{\left\|D_{b-}^{\nu} h\right\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(b-a)^{\nu-\frac{1}{q}}, \tag{34}$$

in particular when $h(t) := f(z(t))\, z'(t) \in C^n([a,b],\mathbb{C})$, where $f(z)$, $z(t)$, $t \in [a,b]$ are as in 1. Introduction, and $(f(z(t))\, z'(t))^{(i)}|_{t=b} = 0$, $i = 1,...n-1$, we get:

$$\left|\frac{1}{b-a}\int_{\gamma_{u,w}} f(z)\, dz - f(w)\, z'(b)\right| = \left|\frac{1}{b-a}\int_a^b f(z(t))\, z'(t)\, dt - f(z(b))\, z'(b)\right|$$

$$\leq \frac{\left\|D_{b-}^{\nu}(f(z(t))\, z'(t))\right\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(b-a)^{\nu-\frac{1}{q}}. \tag{35}$$

Proof. By Theorem 9. ■

It follows

Corollary 19 *(to Theorem 18, case of $p = q = 2$). We have that*

$$\left| \frac{1}{b-a} \int_{\gamma_{u,w}} f(z)\, dz - f(w)\, z'(b) \right| \leq \frac{\left\| D_{b-}^{\nu} \left(f(z(t))\, z'(t) \right) \right\|_{L_2([a,b],\mathbb{C})}}{\Gamma(\nu) \sqrt{2\nu - 1} \left(\nu + \frac{1}{2} \right)} (b-a)^{\nu - \frac{1}{2}}. \tag{36}$$

We continue with an L_1 fractional $\mathbb{C}$-Ostrowski type inequality:

Theorem 20 *Let $\nu \geq 1$, $n = \lceil \nu \rceil$. Assume that $h(t) := f(z(t))\, z'(t) \in C^n([a,b], \mathbb{C})$, where $f(z)$, $z(t)$, $t \in [a,b]$ are as in 1. Introduction, and such that $h^{(i)}(b) = 0$, $i = 1, ..., n-1$. Then*

$$\left| \frac{1}{b-a} \int_{\gamma_{u,w}} f(z)\, dz - f(w)\, z'(b) \right| \leq \frac{\left\| D_{b-}^{\nu} \left(f(z(t))\, z'(t) \right) \right\|_{L_1([a,b],\mathbb{C})}}{\Gamma(\nu + 1)} (b-a)^{\nu - 1}. \tag{37}$$

Proof. By Theorem 8. ■

It follows a Poincaré like $\mathbb{C}$-fractional inequality:

Theorem 21 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Let $h \in C^n([a,b], \mathbb{C})$. Assume that $h^{(i)}(b) = 0$, $i = 1, ..., n-1$. Then*

$$\|h\|_{L_q([a,b],\mathbb{C})} \leq \frac{(b-a)^{\nu} \left\| D_{b-}^{\nu} h \right\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu) (p(\nu - 1) + 1)^{\frac{1}{p}} (q\nu)^{\frac{1}{q}}}, \tag{38}$$

in particular when $h(t) := f(z(t))\, z'(t) \in C^n([a,b], \mathbb{C})$, where $f(z)$, $z(t)$, $t \in [a,b]$ are as in 1. Introduction, and $\left(f(z(t))\, z'(t) \right)^{(i)} |_{t=b} = 0$, $i = 1, ...n-1$, we get:

$$\|f(z(t))\, z'(t)\|_{L_q([a,b],\mathbb{C})} \leq$$
$$\frac{(b-a)^{\nu}}{\Gamma(\nu) (p(\nu - 1) + 1)^{\frac{1}{p}} (q\nu)^{\frac{1}{q}}} \left\| D_{b-}^{\nu} \left(f(z(t))\, z'(t) \right) \right\|_{L_q([a,b],\mathbb{C})}. \tag{39}$$

Proof. By Theorem 11. ■

The corresponding Sobolev like inequality follows:

Theorem 22 *All as in Theorem 21. Let $r > 0$. Then*

$$\|f(z(t))\, z'(t)\|_{L_r([a,b],\mathbb{C})} \leq$$
$$\frac{(b-a)^{\nu - \frac{1}{q} + \frac{1}{r}}}{\Gamma(\nu) (p(\nu - 1) + 1)^{\frac{1}{p}} \left(r \left(\nu - \frac{1}{q} \right) + 1 \right)^{\frac{1}{r}}} \left\| D_{b-}^{\nu} \left(f(z(t))\, z'(t) \right) \right\|_{L_q([a,b],\mathbb{C})}. \tag{40}$$

Proof. By Theorem 12. ■

We continue with an Opial type $\mathbb{C}$-fractional inequality

Theorem 23 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $\nu>\frac{1}{q}$, $n:=\lceil\nu\rceil$, $h\in C^{n}([a,b],\mathbb{C})$. Assume $h^{(k)}(b)=0$, $k=0,1,...,n-1$. Then*

$$\int_{x}^{b}|h(t)|\left|\left(D_{b-}^{\nu}h\right)(t)\right|dt\leq$$

$$\frac{(b-x)^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\nu)\left((p(\nu-1)+1)(p(\nu-1)+2)\right)^{\frac{1}{p}}}\left(\int_{x}^{b}\left|\left(D_{b-}^{\nu}h\right)(t)\right|^{q}dt\right)^{\frac{2}{q}}, \tag{41}$$

$\forall\ x\in[a,b]$,

in particular when $h(t):=f(z(t))z'(t)\in C^{n}([a,b],\mathbb{C})$, where $f(z)$, $z(t)$, $t\in[a,b]$ are as in 1. Introduction, and $\left(f(z(t))z'(t)\right)^{(i)}|_{t=b}=0$, $i=1,...n-1$, we get:

$$\int_{x}^{b}|f(z(t))|\left|\left(D_{b-}^{\nu}(f(z(t))z'(t))\right)\right||z'(t)|dt\leq$$

$$\frac{(b-x)^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\nu)\left((p(\nu-1)+1)(p(\nu-1)+2)\right)^{\frac{1}{p}}}\left(\int_{x}^{b}\left|D_{b-}^{\nu}(f(z(t))z'(t))\right|^{q}dt\right)^{\frac{2}{q}}, \tag{42}$$

$\forall\ x\in[a,b]$.

Proof. By Theorem 13. ■

We finish with Hilbert-Pachpatte left $\mathbb{C}$-fractional inequalities:

Theorem 24 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $\nu_1>\frac{1}{q}$, $\nu_2>\frac{1}{p}$, $n_i:=\lceil\nu_i\rceil$, $i=1,2$. Let $h_i\in C^{n_i}([a_i,b_i],\mathbb{C})$, $i=1,2$. Assume $h_i^{(k_i)}(b_i)=0$, $k_i=0,1,...,n_i-1$; $i=1,2$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|h_1(t_1)||h_2(t_2)|dt_1dt_2}{\left(\frac{(b_1-t_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(b_2-t_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\left\|D_{b_1-}^{\nu_1}h_1\right\|_{L_q([a_1,b_1],\mathbb{C})}\left\|D_{b_2-}^{\nu_2}h_2\right\|_{L_p([a_2,b_2],\mathbb{C})}, \tag{43}$$

in particular when $h_1(t_1):=f_1(z_1(t_1))z_1'(t_1)$ and $h_2(t_2):=f_2(z_2(t_2))z_2'(t_2)$ as in 1. Introduction, with $h_i^{(k_i)}(b_i)=0$, $k_i=0,1,...n_i-1$; $i=1,2$, we get:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(z_1(t_1))z_1'(t_1)||f_2(z_2(t_2))z_2'(t_2)|dt_1dt_2}{\left(\frac{(b_1-t_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(b_2-t_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\cdot$$

$$\left\|D_{b_1-}^{\nu_1}(f_1(z_1(t_1))z_1'(t_1))\right\|_{L_q([a_1,b_1],\mathbb{C})}\left\|D_{b_2-}^{\nu_2}(f_2(z_2(t_2))z_2'(t_2))\right\|_{L_p([a_2,b_2],\mathbb{C})}. \tag{44}$$

Proof. By Theorem 14. ■

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