Right Complex Caputo fractional inequalities

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Abstract

Here we establish several important right complex Caputo type fractional inequalities of the following kinds: Ostrowski’s, Poincare’s, Sobolev’s, Opial’s and Hilbert-Pachpatte’s.

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1 Introduction

Here we follow [3].

Suppose \( \gamma \) is a smooth path parametrized by \( z(t), t \in [a, b] \) and \( f \) is a complex function which is continuous on \( \gamma \). Put \( z(a) = u \) and \( z(b) = w \) with \( u, w \in \mathbb{C} \). We define the integral of \( f \) on \( \gamma_{u,w} = \gamma \) as

\[
\int_\gamma f(z) \, dz = \int_{\gamma_{u,w}} f(z) \, dz := \int_a^b f(z(t)) \, z'(t) \, dt.
\]

We observe that the actual choice of parametrization of \( \gamma \) does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose \( \gamma \) is parametrized by \( z(t), t \in [a, b] \), which is differentiable on the intervals \( [a, c] \) and \( [c, b] \), then assuming that \( f \) is continuous on \( \gamma \) we define

\[
\int_{\gamma_{u,w}} f(z) \, dz := \int_{\gamma_{u,v}} f(z) \, dz + \int_{\gamma_{v,w}} f(z) \, dz,
\]

where \( v := z(c) \). This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

\[
\int_{\gamma_{u,w}} f(z) \, |dz| := \int_a^b f(z(t)) \, |z'(t)| \, dt
\]
and the length of the curve $\gamma$ is then
\[ l(\gamma) = \int_{\gamma_{u,w}} |dz| := \int_{a}^{b} |z'(t)| \, dt, \]
where $|\cdot|$ is the complex absolute value.

Let $f$ and $g$ be holomorphic in $G$, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the integration by parts formula
\[ \int_{\gamma_{u,w}} f(z) g'(z) \, dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u,w}} f'(z) g(z) \, dz. \quad (1) \]

We recall also the triangle inequality for the complex integral, namely
\[ \left| \int_{\gamma} f(z) \, dz \right| \leq \int_{\gamma} |f(z)||dz| \leq \|f\|_{\gamma,\infty} l(\gamma), \quad (2) \]
where $\|f\|_{\gamma,\infty} := \sup_{z \in \gamma} |f(z)|$.

We also define the $p$-norm with $p \geq 1$ by
\[ \|f\|_{\gamma,p} := \left( \int_{\gamma} |f(z)|^p \, |dz| \right)^{\frac{1}{p}}. \]
For $p = 1$ we have
\[ \|f\|_{\gamma,1} := \int_{\gamma} |f(z)| \, |dz|. \]

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder’s inequality we have
\[ \|f\|_{\gamma,1} \leq [l(\gamma)]^{\frac{1}{q}} \|f\|_{\gamma,p}. \quad (3) \]

We are inspired by the following extensions of Stekloff and Almansi inequalities to the complex integral:

**Theorem 1** ([3]) Let $f$ be analytic in $G$, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ and $z'(t) \neq 0$ for $t \in (a,b)$.

(i) If $\int_{\gamma} f(z) |dz| = 0$, then
\[ \int_{\gamma} |f(z)|^2 |dz| \leq \frac{1}{\pi^2} l^2(\gamma) \int_{\gamma} |f'(z)|^2 |dz|. \quad (4) \]

(ii) In addition, if $f(u) = f(w) = 0$, then
\[ \int_{\gamma} |f(z)|^2 |dz| \leq \frac{1}{4\pi^2} l^2(\gamma) \int_{\gamma} |f'(z)|^2 |dz|. \quad (5) \]
We are also inspired by complex Ostrowski type results:

**Theorem 2 ([4])** Let \( f : D \subseteq \mathbb{C} \to \mathbb{C} \) be an analytic function on the convex domain \( D \) with \( z,x,y \in D \) and \( \lambda \in \mathbb{C} \). Suppose \( \gamma \subset D \) is a smooth path parametrized by \( z(t), t \in [a,b] \) with \( z(a) = u \) and \( z(b) = w \) where \( u,w \in D \). Then

\[
\int_\gamma f(z) \, dz = [(1 - \lambda) f(x) + \lambda f(y)] (w - u) + (6)
\]

\[
(1 - \lambda) \sum_{k=1}^{n} \frac{1}{(k+1)!} f^{(k)}(x) [(w - x)^{k+1} + (-1)^k (x - u)^{k+1}] + \lambda \sum_{k=1}^{n} \frac{1}{(k+1)!} f^{(k)}(y) [(w - y)^{k+1} + (-1)^k (y - u)^{k+1}] + T_{n,\lambda} (\gamma, x, y),
\]

where the remainder \( T_{n,\lambda} (\gamma, x, y) \) is given by

\[
T_{n,\lambda} (\gamma, x, y) :=
\]

\[
\frac{1}{n!} \left[ (1 - \lambda) \int_\gamma (z - x)^{n+1} \left( \int_0^1 f^{(n+1)} [(1 - s) x + s z] (1 - s)^n \, ds \right) \, dz \right.
\]

\[
+ (1 - \lambda)^n \lambda \int_\gamma (y - z)^{n+1} \left( \int_0^1 f^{(n+1)} [(1 - s) z + s y] s^n \, ds \right) \, dz \right] = (7)
\]

\[
\frac{1}{n!} \left[ (1 - \lambda) \int_0^1 (1 - s)^n \left( \int_\gamma (z - x)^{n+1} f^{(n+1)} [(1 - s) x + s z] \, dz \right) \, ds +
\]

\[
(1 - \lambda)^n \lambda \int_0^1 s^n \left( \int_\gamma (y - z)^{n+1} f^{(n+1)} [(1 - s) z + s y] \, dz \right) \, ds \] .
\]

Estimations of the above remainder follow:

**Theorem 3 ([4])** Let \( f : D \subseteq \mathbb{C} \to \mathbb{C} \) be an analytic function on the convex domain \( D \) with \( x,y \in D \) and \( \lambda \in \mathbb{C} \). Suppose \( \gamma \subset D \) is a smooth path parametrized by \( z(t), t \in [a,b] \) with \( z(a) = u \) and \( z(b) = w \) where \( u,w \in D \). Then we have the representation (6) and the remainder \( T_{n,\lambda} (\gamma, x, y) \) satisfies the inequalities

\[
|T_{n,\lambda} (\gamma, x, y)| \leq
\]

\[
\frac{1}{n!} \left[ |1 - \lambda| \left| \int_\gamma |z - x|^{n+1} \left( \int_0^1 f^{(n+1)} [(1 - s) x + s z] (1 - s)^n \, ds \right) |dz| \right|
\]

\[
+ |\lambda| \left| \int_\gamma |y - z|^{n+1} \left( \int_0^1 f^{(n+1)} [(1 - s) z + s y] s^n \, ds \right) |dz| \right| \] .
\]
\[ \frac{1}{n!} \int_{\gamma} |z - x|^{n+1} \left( \max_{s \in [0,1]} |f^{(n+1)} \left[ (1 - s) x + s z \right]| \right) |dz| \]

\[ \leq \frac{1}{n!} |1 - \lambda| \left\{ \frac{1}{(q+1)!} \int_{\gamma} |z - x|^{n+1} \left( \int_0^1 |f^{(n+1)} \left[ (1 - s) x + s z \right]|^p \frac{1}{p} \right)^\frac{1}{p} |dz| \right\} \]

\[ \text{where } p, q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 \]

\[ \int_{\gamma} |z - x|^{n+1} \left( \int_0^1 |f^{(n+1)} \left[ (1 - s) x + s z \right]| \right) |dz| \]

\[ + \frac{1}{n!} \left\{ \frac{1}{(q+1)!} \int_{\gamma} |y - z|^{n+1} \left( \int_0^1 |f^{(n+1)} \left[ (1 - s) z + s y \right]|^p \right)^\frac{1}{p} |dz| \right\} \]

\[ \text{where } p, q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 \]

\[ \int_{\gamma} |y - z|^{n+1} \left( \int_0^1 |f^{(n+1)} \left[ (1 - s) z + s y \right]| \right) |dz|, \]

and

\[ |T_{n,\lambda} (\gamma, x, y)| \leq \frac{1}{n!} \left[ |1 - \lambda| \int_0^1 (1 - s)^n \left( \int_{\gamma} |z - x|^{n+1} \left| f^{(n+1)} \left[ (1 - s) x + s z \right] \right| \right) |dz| \right] ds + \]

\[ |\lambda| \int_0^1 s^n \left( \int_{\gamma} |y - z|^{n+1} \left| f^{(n+1)} \left[ (1 - s) z + s y \right] \right| \right) |dz| \right] ds \leq \]

\[ \frac{1}{n!} \left\{ \int_{\gamma} |z - x|^{(n+1)q} |dz| \right\}^\frac{1}{q} \int_0^1 (1 - s)^n \left( \int_{\gamma} |f^{(n+1)} \left[ (1 - s) x + s z \right]|^p \right)^\frac{1}{p} |dz| \right\} ds \]

where \( p, q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 \)

\[ \max_{s \in \gamma} \left( |z - x|^{n+1} \right) \int_0^1 (1 - s)^n \left( \int_{\gamma} |f^{(n+1)} \left[ (1 - s) x + s z \right]| \right) |dz| \]

\[ \left( \int_{\gamma} |y - z|^{(n+1)q} |dz| \right)^\frac{1}{q} \int_0^1 s^n \left( \int_{\gamma} |f^{(n+1)} \left[ (1 - s) z + s y \right]|^p |dz| \right)^\frac{1}{p} ds \]

where \( p, q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 \)

\[ \max_{s \in \gamma} \left( |y - z|^{n+1} \right) \int_0^1 s^n \left( \int_{\gamma} |f^{(n+1)} \left[ (1 - s) z + s y \right]| |dz| \right) ds. \]

In this article we utilize on \( \mathbb{C} \) the results of [2] which are for general Banach space valued functions.

Mainly we give different cases of the right fractional \( \mathbb{C} \)-Ostrowski type inequality and we continue with the right fractional; \( \mathbb{C} \)-Poincaré like and Sobolev like inequalities.
We present an Opial type right \( C \)-fractional inequality, and we finish with the Hilbert-Pachpatte right \( C \)-fractional inequalities.

## 2 Background

In this section all integrals are of Bochner type.

We need

**Definition 4** (see [5]) A definition of the Hausdorff measure \( h_\alpha \) goes as follows: if \( (T, d) \) is a metric space, \( A \subseteq T \) and \( \delta > 0 \), let \( \Lambda (A, \delta) \) be the set of all arbitrary collections \( (C_i) \) of subsets of \( T \), such that \( A \subseteq \bigcup C_i \) and diam \( (C_i) \leq \delta \) (diam = diameter) for every \( i \). Now, for every \( \alpha > 0 \) define

\[
h_\alpha^\delta (A) := \inf \left\{ \sum (\text{diam}C_i)^\alpha \mid (C_i) \in \Lambda (A, \delta) \right\}.
\]

Then there exists \( \lim_{\delta \to 0} h_\alpha^\delta (A) = \sup_{\delta > 0} h_\alpha^\delta (A) \), and \( h_\alpha (A) := \lim_{\delta \to 0} h_\alpha^\delta (A) \) gives an outer measure on the power set \( P (T) \), which is countably additive on the \( \sigma \)-field of all Borel subsets of \( T \). If \( T = \mathbb{R}^n \), then the Hausdorff measure \( h_\alpha \), restricted to the \( \sigma \)-field of the Borel subsets of \( \mathbb{R}^n \), equals the Lebesgue measure up to a constant multiple. In particular, \( h_1 (C) = \mu (C) \) for every Borel set \( C \subseteq \mathbb{R} \), where \( \mu \) is the Lebesgue measure.

**Definition 5** ([2]) Let \([a, b] \subset \mathbb{R}, (X, \|\cdot\|)\) be a Banach space, \( \alpha > 0 \), \( m := [\alpha] \), (\( [\cdot] \) the ceiling of the number). We assume that \( f^{(m)} \in L_1 ([a, b], X) \), where \( f : [a, b] \to X \). We call the Caputo-Bochner right fractional derivative of order \( \alpha \):

\[
(D_b^\alpha f) (x) := \frac{(-1)^m}{\Gamma (m - \alpha)} \int_x^b (J - x)^{m-\alpha} f^{(m)} (J) \, dJ, \quad \forall \, x \in [a, b],
\]

where \( f^{(m)} \) is the ordinary \( X \)-valued derivative, defined similarly to the numerical one.

We observe that \( D_b^m f (x) = (-1)^m f^{(m)} (x) \), for \( m \in \mathbb{N} \), and \( D_b^0 f (x) = f (x) \).

By [2] \( (D_b^\alpha f) (x) \) exists almost everywhere on \([a, b]\) and \( (D_b^\alpha f) \in L_1 ([a, b], X) \). If \( \| f^{(m)} \|_{L_\infty ([a, b], X)} < \infty \), an \( \alpha \notin \mathbb{N} \), then by [2], \( D_b^\alpha f \in C ([a, b], X) \), hence \( \| D_b^\alpha f \| \in C ([a, b]) \).

We need the right-fractional Taylor’s formula:

**Theorem 6** ([2]) Let \([a, b] \subset \mathbb{R}, X \) be a Banach space, \( \alpha > 0 \), \( m = [\alpha] \), \( f \in C^{m-1} ([a, b], X) \). Set

\[
F_x (t) := \sum_{i=0}^{m-1} \frac{(x - t)^i}{i!} f^{(i)} (t), \quad \forall \, t \in [x, b],
\]

\[\text{(12)}\]
where $x \in [a, b]$.

Assume that $f^{(m)}$ exists outside a $\mu$-null Borel set $B_x \subseteq [x, b]$, such that

$$h_1 (F_x (B_x)) = 0, \quad \forall x \in [a, b].$$

(13)

We also assume that $f^{(m)} \in L_1 ([a, b], X)$. Then

$$f (x) = \sum_{i=0}^{m-1} \frac{(x-b)^i}{i!} f^{(i)} (b) + \frac{1}{\Gamma (\alpha)} \int_x^b (z-x)^{\alpha-1} (D^\alpha_{b-} f) (z) \, dz,$$

(14)

$\forall x \in [a, b]$.

Next we mention Ostrowski type inequalities at right fractional level for Banach valued functions. See also [1].

**Theorem 7** ([2]) Let $\alpha > 0$, $m = [\alpha]$. Here all as in Theorem 6. Assume $f^{(k)} (b) = 0$, $k = 1, \ldots, m - 1$, and $D^\alpha_{b-} f \in L_\infty ([a, b], X)$. Then

$$\left\| \frac{1}{b-a} \int_a^b f (x) \, dx - f (b) \right\| \leq \frac{\| D^\alpha_{b-} f \|_{L_\infty ([a, b], X)}}{\Gamma (\alpha + 2)} (b-a)^\alpha.$$  

(15)

We also give

**Theorem 8** ([2]) Let $\alpha \geq 1$, $m = [\alpha]$. Here all as in Theorem 6. Assume that $f^{(k)} (b) = 0$, $k = 1, \ldots, m - 1$, and $D^\alpha_{b-} f \in L_1 ([a, b], X)$. Then

$$\left\| \frac{1}{b-a} \int_a^b f (x) \, dx - f (b) \right\| \leq \frac{\| D^\alpha_{b-} f \|_{L_1 ([a, b], X)}}{\Gamma (\alpha + 1)} (b-a)^{\alpha-1}.$$  

(16)

We mention also an $L_p$ abstract Ostrowski type inequality:

**Theorem 9** ([2]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = [\alpha]$. Here all as in Theorem 6. Assume that $f^{(k)} (b) = 0$, $k = 1, \ldots, m - 1$, and $D^\alpha_{b-} f \in L_q ([a, b], X)$. Then

$$\left\| \frac{1}{b-a} \int_a^b f (x) \, dx - f (b) \right\| \leq \frac{\| D^\alpha_{b-} f \|_{L_q ([a, b], X)}}{\Gamma (\alpha) (p (\alpha - 1) + 1)^{\frac{1}{p}}} (b-a)^{\alpha - \frac{1}{q}}.$$  

(17)

It follows

**Corollary 10** ([2]) Let $\alpha > \frac{1}{2}$, $m = [\alpha]$. All as in Theorem 6. Assume $f^{(k)} (b) = 0$, $k = 1, \ldots, m - 1$, $D^\alpha_{b-} f \in L_2 ([a, b], X)$. Then

$$\left\| \frac{1}{b-a} \int_a^b f (x) \, dx - f (b) \right\| \leq \frac{\| D^\alpha_{b-} f \|_{L_2 ([a, b], X)}}{\Gamma (\alpha) (\sqrt{2\alpha} - 1) (\alpha + \frac{1}{2})} (b-a)^{\alpha - \frac{1}{2}}.$$  

(18)
We continue with a Poincaré like right fractional inequality:

**Theorem 11** ([2]) Let \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \), and \( \alpha > \frac{1}{q} \), \( m = \lfloor \alpha \rfloor \). Here all as in Theorem 6. Assume that \( f^{(k)} (b) = 0 \), \( k = 0, 1, ..., m - 1 \), and \( D_{b-}^\alpha f \in L_q ([a, b], X) \), where \( X \) is a Banach space. Then

\[
\|f\|_{L_q([a, b], X)} \leq \frac{(b-a)^{\alpha} \|D_{b-}^\alpha f\|_{L_q([a, b], X)}}{\Gamma (\alpha) (p (\alpha - 1) + 1)^{\frac{1}{q}} (qa)^{\frac{1}{q}}}. \tag{19}
\]

Next follows a right Sobolev like fractional inequality:

**Theorem 12** ([2]) All as in the last Theorem 11. Let \( r > 0 \). Then

\[
\|f\|_{L_r([a, b], X)} \leq \frac{(b-a)^{\alpha - \frac{1}{r} + \frac{1}{q}} \|D_{b-}^\alpha f\|_{L_q([a, b], X)}}{\Gamma (\alpha) (p (\alpha - 1) + 1)^{\frac{1}{q}} (r (\alpha - \frac{1}{q}) + 1)^{\frac{1}{r}}}. \tag{20}
\]

We also mention the following Opial type right fractional inequality:

**Theorem 13** ([2]) Let \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \), and \( \alpha > \frac{1}{q} \), \( m := \lfloor \alpha \rfloor \). Let \([a, b] \subset \mathbb{R} \), \( X \) a Banach space, and \( f \in C^{m-1} ([a, b], X) \). Set

\[
F_x (t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)} (t), \forall t \in [x, b], \text{ where } x \in [a, b]. \tag{21}
\]

Assume that \( f^{(m)} \) exists outside a \( \mu \)-null Borel set \( B_x \subseteq [x, b] \), such that

\[
h_1 (F_x (B_x)) = 0, \forall x \in [a, b]. \tag{22}
\]

We assume that \( f^{(m)} \in L_\infty ([a, b], X) \). Assume also that \( f^{(k)} (b) = 0 \), \( k = 0, 1, ..., m - 1 \). Then

\[
\int_x^b \|f (w)\| \| (D_{b-}^\alpha f) (w)\| \, dw \leq \frac{(b-x)^{\alpha - \frac{1}{p} + \frac{1}{q}}}{2^{\frac{1}{p}} \Gamma (\alpha) (p (\alpha - 1) + 1) (p (\alpha - 1) + 2)^{\frac{1}{p}}} \left( \int_x^b \| (D_{b-}^\alpha f) (z)\|^q \, dz \right)^{\frac{2}{q}} \tag{23}
\]

\[\forall x \in [a, b].\]

Next we describe an abstract Hilbert-Pachpatte right fractional inequality:

**Theorem 14** ([2]) Let \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \), and \( \alpha_1 > \frac{1}{q} \), \( \alpha_2 > \frac{1}{r} \). \( m_i := \lceil \alpha_i \rceil \), \( i = 1, 2 \), \( \alpha_i \) denotes the integer part of \( \alpha_i \). Here \([a_i, b_i] \subset \mathbb{R}, i = 1, 2 \), \( X \) is a Banach space. Let \( f_i \in C^{m_i-1} ([a_i, b_i], X), i = 1, 2 \). Set

\[
F_{x_i} (t_i) := \sum_{j_i=0}^{m_i-1} \frac{(x_i - t_i)^{j_i}}{j_i!} f_i^{(j_i)} (t_i), \tag{24}
\]

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\[ h_1 (F_{x_i} (B_{x_i})) = 0, \quad \forall x_i \in [a_i, b_i]; \quad i = 1, 2. \]  

We also assume that \( f_i^{(m_i)} \) exists outside a \( \mu \)-null Borel set \( B_{x_i} \subseteq [x_i, b_i] \), such that

\[ f_i^{(k_i)} (b_i) = 0, \quad k_i = 0, 1, \ldots, m_i - 1; \quad i = 1, 2. \]

and

\[ (D_{b_1}^{p_1} f_1) \in L_q ([a_1, b_1], X), \quad (D_{b_2}^{p_2} f_2) \in L_p ([a_2, b_2], X). \]

Then

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\| f_1 (x_1) \| \| f_2 (x_2) \| \| dx_1 dx_2 \|}{(b_1 - a_1)^{s_1} (a_1 - 1) + 1} + \frac{(b_2 - a_2)^{s_2} (a_2 - 1) + 1}{q \Gamma (n_1) \Gamma (a_2)} \| D_{b_1}^{p_1} f_1 \|_{L_q ([a_1, b_1], X)} \| D_{b_2}^{p_2} f_2 \|_{L_p ([a_2, b_2], X)}. \]

\section{3 Main Results}

We need a special case of Definition 5 over \( \mathbb{C} \).

**Definition 15** Let \([a, b] \subset \mathbb{R}, \nu > 0; n := \lfloor \nu \rfloor \in \mathbb{N}, \lfloor \cdot \rfloor \) is the ceiling of the number and \( f \in C^n ([a, b], \mathbb{C}) \). We call Caputo-Complex right fractional derivative of order \( \nu \):

\[
(D_{b}^\nu f) (x) := \frac{(-1)^n}{\Gamma (n - \nu)} \int_{a}^{b} (x - \lambda)^{n-\nu-1} f^{(n)} (\lambda) d\lambda, \quad \forall x \in [a, b],
\]

where the derivatives \( f', \ldots, f^{(n)} \) are defined as the numerical derivative.

If \( \nu \in \mathbb{N} \), we set \( D_{b}^\nu f := (-1)^{\nu} f^{(\nu)} \) the ordinary \( \mathbb{C} \)-valued derivative and also \( D_{b}^0 f := f \).

Notice here (by [2]) that \( D_{b}^\nu f \in C ([a, b], \mathbb{C}) \).

We give the following right-fractional C-Taylor’s formula:

**Theorem 16** Let \( h \in C^n ([a, b], \mathbb{C}), n = \lfloor \nu \rfloor, \nu \geq 0. \) Then

\[
 h (t) = \sum_{i=0}^{n-1} \frac{(t - b)^i}{i!} h^{(i)} (b) + \frac{1}{\Gamma (\nu)} \int_{t}^{b} (\lambda - t)^{\nu-1} D_{b}^\nu h (\lambda) d\lambda, \quad \forall t \in [a, b],
\]

in particular when \( h (t) := f (z (t)) z' (t) \in C^n ([a, b], \mathbb{C}), \) where \( f (z), z (t), t \in [a, b] \) are as in 1. Introduction, it holds,

\[
f (z (t)) z' (t) = \sum_{i=0}^{n-1} \frac{(t - b)^i}{i!} (f (z (b)) z' (b))^{(i)} + \]
\[
\frac{1}{\Gamma(\nu)} \int_{t}^{b} (\lambda - t)^{\nu - 1} \left( D_{b-}^{\nu} f (z (\cdot)) z' (\cdot) \right) (\lambda) d\lambda, \quad (31)
\]
\[\forall t \in [a, b].\]

**Proof.** By Theorem 6. ■

It follows a right fractional \( \mathbb{C} \)-Ostrowski type inequality

**Theorem 17** Let \( n \in \mathbb{N} \) and \( h \in C^n ([a, b], \mathbb{C}) \), where \( [a, b] \subset \mathbb{R} \), and let \( \nu > 0 : n = [\nu] \). Assume that \( h^{(i)} (b) = 0, i = 1, \ldots, n - 1 \). Then

\[
\left| \frac{1}{b - a} \int_{a}^{b} h (t) dt - h (b) \right| \leq \frac{\| D_{b-}^{\nu} h \|_{\infty, [a, b]}}{\Gamma (\nu + 2)} (b - a)^{\nu}, \quad (32)
\]
in particular when \( h (t) := f (z (t)) z' (t) \in C^n ([a, b], \mathbb{C}) \), where \( f (z), z (t), t \in [a, b] \) are as in 1. Introduction, and \( (f (z (t)) z' (t))^{(i)} \big|_{t=b=0, i = 1, \ldots, n-1} \) we get:

\[
\left| \frac{1}{b - a} \int_{a}^{b} \gamma_{a, w} f (z) dz - f (w) z' (b) \right| = \frac{1}{b - a} \int_{a}^{b} f (z (t)) z' (t) dt - f (z (b)) z' (b) \right| \leq \frac{\| D_{b-}^{\nu} f (z (t)) z' (t) \|_{\infty, [a, b]}}{\Gamma (\nu + 2)} (b - a)^{\nu}. \quad (33)
\]

**Proof.** By Theorem 7. ■

The corresponding \( \mathbb{C} \)-Ostrowski type \( L^p \) inequality follows:

**Theorem 18** Let \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \), and \( \nu > \frac{1}{q} \). \( n = [\nu] \). Here \( h \in C^n ([a, b], \mathbb{C}) \). Assume that \( h^{(i)} (b) = 0, i = 1, \ldots, n - 1 \). Then

\[
\left| \frac{1}{b - a} \int_{a}^{b} h (t) dt - h (b) \right| \leq \frac{\| D_{b-}^{\nu} h \|_{L^q ([a, b], \mathbb{C})}}{\Gamma (\nu) (p (\nu - 1) + 1)^{\frac{1}{p}} \left( \nu + \frac{1}{p} \right)} (b - a)^{\nu - \frac{1}{q}}, \quad (34)
\]
in particular when \( h (t) := f (z (t)) z' (t) \in C^n ([a, b], \mathbb{C}) \), where \( f (z), z (t), t \in [a, b] \) are as in 1. Introduction, and \( (f (z (t)) z' (t))^{(i)} \big|_{t=b=0, i = 1, \ldots, n-1} \) we get:

\[
\left| \frac{1}{b - a} \int_{a}^{b} \gamma_{a, w} f (z) dz - f (w) z' (b) \right| = \frac{1}{b - a} \int_{a}^{b} f (z (t)) z' (t) dt - f (z (b)) z' (b) \right| \leq \frac{\| D_{b-}^{\nu} (f (z (t)) z' (t)) \|_{L^q ([a, b], \mathbb{C})}}{\Gamma (\nu) (p (\nu - 1) + 1)^{\frac{1}{p}} \left( \nu + \frac{1}{p} \right)} (b - a)^{\nu - \frac{1}{q}}. \quad (35)
\]

**Proof.** By Theorem 9. ■

It follows
Corollary 19 (to Theorem 18, case of \( p = q = 2 \)). We have that
\[
\left| \frac{1}{b-a} \int_{\gamma_{a,w}} f(z) \, dz - f(w) \right| \leq \frac{\|D_{b-}^\nu (f(z(t)) z'(t))\|_{L^2([a,b],\mathbb{C})}}{\Gamma(\nu) \sqrt{2\nu - \Gamma(\nu + \frac{1}{2})}} (b-a)^{\nu - \frac{1}{2}}.
\]

(36)

We continue with an \( L_1 \) fractional \( \mathbb{C} \)-Ostrowski type inequality:

**Theorem 20** Let \( \nu \geq 1, n = \lfloor \nu \rfloor \). Assume that \( h(t) := f(z(t)) z'(t) \in C^n([a,b],\mathbb{C}) \), where \( f(z), z(t), t \in [a,b] \) are as in 1. Introduction, and such that \( h^{(i)}(b) = 0, i = 1, \ldots, n-1 \). Then
\[
\left| \frac{1}{b-a} \int_{\gamma_{a,w}} f(z) \, dz - f(w) \right| \leq \frac{\|D_{b-}^\nu (f(z(t)) z'(t))\|_{L^1([a,b],\mathbb{C})}}{\Gamma(\nu + 1)} (b-a)^{\nu - 1}.
\]

(37)

**Proof.** By Theorem 8. \( \blacksquare \)

It follows a Poincaré like \( \mathbb{C} \)-fractional inequality:

**Theorem 21** Let \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \) and \( \nu > \frac{1}{q}, n = \lfloor \nu \rfloor \). Let \( h \in C^n([a,b],\mathbb{C}) \). Assume that \( h^{(i)}(b) = 0, i = 1, \ldots, n-1 \). Then
\[
\|h\|_{L^q([a,b],\mathbb{C})} \leq \frac{(b-a)^\nu \|D_{b-}^\nu h\|_{L^q([a,b],\mathbb{C})}}{\Gamma(\nu) (p(\nu - 1) + 1)^\frac{1}{p} (q\nu)^\frac{1}{q}},
\]

(38)
in particular when \( h(t) := f(z(t)) z'(t) \in C^n([a,b],\mathbb{C}) \), where \( f(z), z(t), t \in [a,b] \) are as in 1. Introduction, and \( (f(z(t)) z'(t))^{(i)} |_{t=b} = 0, i = 1, \ldots, n-1 \), we get:
\[
\|f(z(t)) z'(t)\|_{L^q([a,b],\mathbb{C})} \leq \frac{(b-a)^\nu}{\Gamma(\nu) (p(\nu - 1) + 1)^\frac{1}{p} (q\nu)^\frac{1}{q}} \|D_{b-}^\nu (f(z(t)) z'(t))\|_{L^q([a,b],\mathbb{C})}.
\]

(39)

**Proof.** By Theorem 11. \( \blacksquare \)

The corresponding Sobolev like inequality follows:

**Theorem 22** All as in Theorem 21. Let \( r > 0 \). Then
\[
\|f(z(t)) z'(t)\|_{L^r([a,b],\mathbb{C})} \leq \frac{(b-a)^{\nu - \frac{1}{q} + \frac{1}{r}}}{\Gamma(\nu) (p(\nu - 1) + 1)^\frac{1}{p} \left(r \left(\nu - \frac{1}{q}\right) + 1\right)^\frac{1}{r}} \|D_{b-}^\nu (f(z(t)) z'(t))\|_{L^q([a,b],\mathbb{C})}.
\]

(40)

**Proof.** By Theorem 12. \( \blacksquare \)

We continue with an Opial type \( \mathbb{C} \)-fractional inequality.
Theorem 23 Let \( p, q > 1 \) : \( \frac{1}{p} + \frac{1}{q} = 1 \), and \( \nu > \frac{1}{q} \), \( n := [\nu] \), \( h \in C^n ([a, b], \mathbb{C}) \).
Assume \( h^{(k)} (b) = 0 \), \( k = 0, 1, ..., n - 1 \). Then
\[
\int_{a}^{b} |h(t)| |(D_{b-}^\nu h) (t)| \, dt \leq \frac{2\Gamma (\nu) ((p (\nu - 1) + 1) (p (\nu - 1) + 2))^{\frac{1}{p}}}{(b - x)^{\nu - 1 + \frac{2}{p}}} \left( \int_{a}^{b} |(D_{b-}^\nu h) (t)|^q \, dt \right)^{\frac{1}{q}}, \quad \text{(41)}
\]
for all \( x \in [a, b] \),
in particular when \( h(t) := f(z(t)) z'(t) \in C^n ([a, b], \mathbb{C}) \), where \( f(z), z(t), t \in [a, b] \) are as in 1. Introduction, and \( (f(z(t)) z'(t)) |_{t=b} = 0 \), \( i = 1, ..., n - 1 \), we get:
\[
\int_{a}^{b} |f(z(t))|(D_{b-}^\nu (f(z(t)) z'(t)))| z'(t) | \, dt \leq \frac{2\Gamma (\nu) ((p (\nu - 1) + 1) (p (\nu - 1) + 2))^{\frac{1}{p}}}{(b - x)^{\nu - 1 + \frac{2}{p}}} \left( \int_{a}^{b} |D_{b-}^\nu (f(z(t)) z'(t))|^q \, dt \right)^{\frac{1}{q}}, \quad \text{(42)}
\]
for all \( x \in [a, b] \).

Proof. By Theorem 13. \( \square \)

We finish with Hilbert-Pachpatte left C-fractional inequalities:

Theorem 24 Let \( p, q > 1 \) : \( \frac{1}{p} + \frac{1}{q} = 1 \), and \( \nu_1 > \frac{1}{q} \), \( \nu_2 > \frac{1}{p} \), \( n_i := [\nu_i] \), \( i = 1, 2 \).
Let \( h_i \in C^n ([a_i, b_i], \mathbb{C}) \), \( i = 1, 2 \). Assume \( h_i^{(k)} (b_i) = 0 \), \( k_i = 0, 1, ..., n_i - 1 \);
\( i = 1, 2 \). Then
\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|h_1(t_1)| |h_2(t_2)| dt_1 dt_2}{\left( \frac{(b_1 - t_1)^{(\nu_1 - 1) + 1}}{p (p (\nu_1 - 1) + 1)} + \frac{(b_2 - t_2)^{(\nu_2 - 1) + 1}}{q (q (\nu_2 - 1) + 1)} \right)} \leq \frac{(b_1 - a_1) (b_2 - a_2)}{\Gamma (\nu_1) \Gamma (\nu_2)} \left\| D_{b_1-}^{\nu_1} h_1 \right\|_{L_q([a_1, b_1], \mathbb{C})} \left\| D_{b_2-}^{\nu_2} h_2 \right\|_{L_q([a_2, b_2], \mathbb{C})}, \quad \text{(43)}
\]
in particular when \( h_1(t_1) := f_1(z_1(t_1)) z'_1(t_1) \) and \( h_2(t_2) := f_2(z_2(t_2)) z'_2(t_2) \) as in 1. Introduction, with \( h_i^{(k)} (b_i) = 0 \), \( k_i = 0, 1, ..., n_i - 1 \); \( i = 1, 2 \), we get:
\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(z_1(t_1)) z'_1(t_1)| |f_2(z_2(t_2)) z'_2(t_2)| dt_1 dt_2}{\left( \frac{(b_1 - t_1)^{(\nu_1 - 1) + 1}}{p (p (\nu_1 - 1) + 1)} + \frac{(b_2 - t_2)^{(\nu_2 - 1) + 1}}{q (q (\nu_2 - 1) + 1)} \right)} \leq \frac{(b_1 - a_1) (b_2 - a_2)}{\Gamma (\nu_1) \Gamma (\nu_2)} \right\| D_{b_1-}^{\nu_1} (f_1(z_1(t_1)) z'_1(t_1)) \right\|_{L_q([a_1, b_1], \mathbb{C})} \left\| D_{b_2-}^{\nu_2} (f_2(z_2(t_2)) z'_2(t_2)) \right\|_{L_q([a_2, b_2], \mathbb{C})}, \quad \text{(44)}
\]

Proof. By Theorem 14. \( \square \)
References


