Mixed Complex fractional inequalities

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Abstract

Here we present some important mixed generalized fractional complex analytic inequalities of the following kinds: Polya's, Ostrowski's and Poincaré's.

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1 Introduction

Here we follow [5].

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt.$$

We observe that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\, dz := \int_{\gamma_{u,v}} f(z)\, dz + \int_{\gamma_{v,w}} f(z)\, dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\, |dz| := \int_a^b f(z(t))\, |z'(t)|\, dt$$

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and the length of the curve γ is then

$$l(\gamma)=\int_{\gamma_{u,w}}|dz|:=\int_a^b|z'(t)|\,dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma\subset G$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$. Then we have the integration by parts formula

$$\int_{\gamma_{u,w}}f(z)\,g'(z)\,dz=f(w)\,g(w)-f(u)\,g(u)-\int_{\gamma_{u,w}}f'(z)\,g(z)\,dz. \tag{1}$$

We recall also the triangle inequality for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right|\le\int_\gamma|f(z)|\,|dz|\le\|f\|_{\gamma,\infty}\,l(\gamma), \tag{2}$$

where $\|f\|_{\gamma,\infty}:=\sup_{z\in\gamma}|f(z)|$.

We also define the p-norm with $p\ge 1$ by

$$\|f\|_{\gamma,p}:=\left(\int_\gamma|f(z)|^p\,|dz|\right)^{\frac{1}{p}}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1}:=\int_\gamma|f(z)|\,|dz|.$$

If $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1}\le[l(\gamma)]^{\frac{1}{q}}\,\|f\|_{\gamma,p}.$$

Motivations to our work follow:

We mention the following Wirtinger type inequality for complex functions:

Theorem 1 *([5]) Let f be analytic in G, a domain of complex numbers and suppose $\gamma\subset G$ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ from $z(a)=u$ to $z(b)=w$ and $z'(t)\ne 0$ for $t\in(a,b)$.*

(i) If $f(u)=f(w)=0$, then

$$\int_\gamma|f(z)|^2\,|dz|\le\frac{1}{\pi^2}l^2(\gamma)\int_\gamma|f'(z)|^2\,|dz|. \tag{3}$$

The equality holds in (3) iff

$$f(v)=K\sin\left[\frac{\pi l(\gamma_{u,v})}{l(\gamma)}\right],\quad K\in\mathbb{C}, \tag{4}$$

where $v=z(t)$, $t\in[a,b]$ and $l(\gamma_{u,v})=\int_a^t|z'(s)|\,ds$.

(ii) If $f(u) = 0$, then

$$\int_{\gamma} |f(z)|^2 |dz| \leq \frac{4}{\pi^2} l^2(\gamma) \int_{\gamma} |f'(z)|^2 |dz| . \tag{5}$$

The equality holds in (5) iff

$$f(v) = K \sin\left[\frac{\pi l(\gamma_{u,v})}{2l(\gamma)}\right], \quad K \in \mathbb{C}, \tag{6}$$

where $v = z(t)$, $t \in [a,b]$.

We mention some complex trapezoid type inequalities:

Proposition 2 *([5]) Let g be analytic in G, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$, $w \neq u$ and $z'(t) \neq 0$ for $t \in (a,b)$. Then*

$$\left| \frac{1}{w-u} \int_{\gamma} g(z)\, dz - \frac{g(u) + g(w)}{2} \right| \leq$$

$$\frac{1}{\pi} \frac{l(\gamma)}{|w-u|} \left(\frac{1}{l(\gamma)} \int_{\gamma} \left| g'(z) - \frac{g(w) - g(u)}{w-u} \right|^2 |dz| \right)^{\frac{1}{2}} . \tag{7}$$

Proposition 3 *([5]) Let g be analytic in G, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$, $w \neq u$ and $z'(t) \neq 0$ for $t \in (a,b)$. If $u + w - z \in G$ for $z \in \gamma$, then*

$$\left| \frac{1}{w-u} \int_{\gamma} \widehat{g(z)} dz - \frac{g(u) + g(w)}{2} \right| \leq$$

$$\frac{1}{2\pi} \frac{l(\gamma)}{|w-u|} \left(\frac{1}{l(\gamma)} \int_{\gamma} |g'(z) - g'(u+w-z)|^2 |dz| \right)^{\frac{1}{2}} , \tag{8}$$

where $\widehat{g(z)} := \frac{g(z)+g(u+w-z)}{2}$, $z \in \gamma$.

In this article we utilize on $\mathbb{C}$ the results of [4] which are for general Banach space valued functions.

We give mixed fractional: $\mathbb{C}$-Polya type integral inequality and $\mathbb{C}$-Ostrowski type integral inequality. We finish with right and left fractional $\mathbb{C}$-Poincaré like inequalities.

2 Background

Here $C([a,b], X)$ stands for the space of continuous functions from $[a,b]$ into X, where $(X, \|\cdot\|)$ is a Banach space.

All integrals here are of Bochner type ([6]). By [2], we have that: if $f \in C([a,b],X)$, then $f \in L_\infty([a,b],X)$ and $f \in L_1([a,b],X)$. Derivatives for vector valued functions are defined according to [7], p. 83, similar to numerical ones.

We need

Definition 4 *([4]) Let $f \in C([a,b],X)$, where X is a Banach space. Let $\nu > 0$, we define the right Riemann-Liouville fractional Bochner integral operator*

$$\left(J_{b-}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} f(z)\, dz, \quad \forall\ x \in [a,b], \tag{9}$$

where Γ is the gamma function.

In [3], we have proved that

$$\left(J_{b-}^{\nu} f\right) \in C([a,b],X).$$

Furthermore in [3], we have proved that

$$J_{b-}^{\nu} J_{b-}^{\mu} f = J_{b-}^{\nu+\mu} f = J_{b-}^{\mu} J_{b-}^{\nu} f,$$

for any $\mu, \nu > 0$; any $f \in C([a,b],X)$.

We need

Definition 5 *([4]) Let $\nu > 0$, $n := [\nu]$, where $[\cdot]$ is the integral part, $\alpha = \nu - n$, $0 < \alpha < 1$, $\nu \notin \mathbb{N}$. Define the subspace of functions*

$$C_{b-}^{\nu}([a,b],X) := \left\{ f \in C^n([a,b],X) : J_{b-}^{1-\alpha} f^{(n)} \in C^1([a,b],X) \right\}. \tag{10}$$

Define the Banach space valued right generalized ν-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\nu} f := (-1)^{n-1} \left(J_{b-}^{1-\alpha} f^{(n)} \right)'. \tag{11}$$

Notice that

$$J_{b-}^{1-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^b (z-x)^{-\alpha} f^{(n)}(z)\, dz \tag{12}$$

exists for $f \in C_{b-}^{\nu}([a,b],X)$, and

$$\left(D_{b-}^{\nu} f\right)(x) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (z-x)^{-\alpha} f^{(n)}(z)\, dz. \tag{13}$$

I.e.

$$\left(D_{b-}^{\nu} f\right)(x) = \frac{(-1)^{n-1}}{\Gamma(n-\nu+1)} \frac{d}{dx} \int_x^b (z-x)^{n-\nu} f^{(n)}(z)\, dz. \tag{14}$$

If $\nu \in \mathbb{N}$, then $\alpha = 0$, $n = \nu$, and

$$\left(D_{b-}^{\nu} f\right)(x) = \left(D_{b-}^{n} f\right)(x) = (-1)^n f^{(n)}(x). \tag{15}$$

Notice that $D_{b-}^{\nu} f \in C([a,b],X)$.

We mention the following right fractional Taylor's formula.

Theorem 6 *([4]) Let* $f \in C_{b-}^{\nu}([a,b],X)$, $\nu > 0$, $n := [\nu]$. *Then*
1) If $\nu \geq 1$, *we get*

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \left(J_{b-}^{\nu} D_{b-}^{\nu} f\right)(x), \quad \forall\, x \in [a,b]. \tag{16}$$

2) If $0 < \nu < 1$, *we get*

$$f(x) = J_{b-}^{\nu} D_{b-}^{\nu} f(x), \quad \forall\, x \in [a,b].$$

We have that

$$J_{b-}^{\nu} D_{b-}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} \left(D_{b-}^{\nu} f\right)(z)\, dz, \quad \forall\, x \in [a,b].$$

Definition 7 *([4]) Let* $f \in C([a,b],X)$. *Let* $\nu > 0$, *we define the left Riemann-Liouville fractional Bochner integral operator*

$$\left(J_a^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} f(z)\, dz, \quad \forall\, x \in [a,b]. \tag{17}$$

In [2], we have proved that

$$\left(J_a^{\nu} f\right) \in C([a,b],X).$$

Furthermore in [2], we have proved that

$$J_a^{\nu} J_a^{\mu} f = J_a^{\nu+\mu} f = J_a^{\mu} J_a^{\nu} f, \tag{18}$$

$\forall\, \mu, \nu > 0$, $\forall\, f \in C([a,b],X)$.

Definition 8 *([4]) Let* $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $\nu \notin \mathbb{N}$. *Define the subspace of functions*

$$C_a^{\nu}([a,b],X) := \left\{ f \in C^n([a,b],X) : J_a^{1-\alpha} f^{(n)} \in C^1([a,b],X) \right\}. \tag{19}$$

Define the Banach space valued left generalized ν*-fractional derivative of* f *over* $[a,b]$ *as*

$$\left(D_a^{\nu} f\right) := \left(J_a^{1-\alpha} f^{(n)}\right)'. \tag{20}$$

Notice that

$$J_a^{1-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-z)^{-\alpha} f^{(n)}(z)\, dz \tag{21}$$

exists for $f \in C_a^{\nu}([a,b],X)$, *and*

$$\left(D_a^{\nu} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-z)^{-\alpha} f^{(n)}(z)\, dz. \tag{22}$$

I.e.

$$\left(D_a^{\nu} f\right)(x) = \frac{1}{\Gamma(n-\nu+1)} \frac{d}{dx} \int_a^x (x-z)^{n-\nu} f^{(n)}(z)\, dz. \tag{23}$$

If $\nu \in \mathbb{N}$, *then* $\alpha = 0$, $n = \nu$, *and*

$$\left(D_a^{\nu} f\right)(x) = \left(D_a^{n} f\right)(x) = f^{(n)}(x). \tag{24}$$

Notice that $D_a^{\nu} f \in C([a,b], X)$.

We mention the following left fractional Taylor's formula.

Theorem 9 *([4]) Let* $f \in C_a^{\nu}([a,b], X)$, $\nu > 0$, $n := [\nu]$. *Then*
1) If $\nu \geq 1$, *we get*

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + \left(J_a^{\nu} D_a^{\nu} f\right)(x), \quad \forall\, x \in [a,b]. \tag{25}$$

2) If $0 < \nu < 1$, *we get*

$$f(x) = J_a^{\nu} D_a^{\nu} f(x), \quad \forall\, x \in [a,b]. \tag{26}$$

We have that

$$J_a^{\nu} D_a^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \left(D_a^{\nu} f\right)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{27}$$

We mention the following fractional Polya type integral inequality without any boundary conditions, see also [1], p. 4.

Theorem 10 *([4]) Let* $0 < \nu < 1$, $f \in C([a,b], X)$. *Assume that* $f \in C_a^{\nu}\left(\left[a, \frac{a+b}{2}\right], X\right)$ *and* $f \in C_{b-}^{\nu}\left(\left[\frac{a+b}{2}, b\right], X\right)$. *Set*

$$M(f) = \max\left\{ \left\|\left\|D_a^{\nu} f\right\|\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\|\left\|D_{b-}^{\nu} f\right\|\right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\}. \tag{28}$$

Then

$$\left\| \int_a^b f(x)\, dx \right\| \leq \int_a^b \|f(x)\|\, dx \leq M(f) \frac{(b-a)^{\nu+1}}{\Gamma(\nu+2)\, 2^{\nu}}. \tag{29}$$

Inequality (29) is sharp, namely it is attained by

$$f_*(x) = \left\{ \begin{array}{l} (x-a)^{\nu} \overrightarrow{i}, \ \ x \in \left[a, \frac{a+b}{2}\right], \\ (b-x)^{\nu} \overrightarrow{i}, \ \ x \in \left[\frac{a+b}{2}, b\right] \end{array} \right\}, \quad 0 < \nu < 1, \tag{30}$$

$\overrightarrow{i} \in X : \left\| \overrightarrow{i} \right\| = 1$.
Clearly here non zero constant vector function f *are excluded.*

We also mention the following fractional Ostrowski type inequality, see also [1], pp. 379-381.

Theorem 11 *([4]) Let $\nu \geq 1$, $n = [\nu]$, $f \in C([a,b], X)$, $x_0 \in [a,b]$. Assume that $f|_{[a,x_0]} \in C^{\nu}_{x_0-}([a,x_0], X)$, $f|_{[x_0,b]} \in C^{\nu}_{x_0}([x_0,b], X)$, and $f^{(i)}(x_0) = 0$, for $i = 1, ..., n-1$, which is void when $1 \leq \nu < 2$. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\,dx - f(x_0) \right\| \leq \frac{1}{(b-a)\Gamma(\nu+2)} \cdot$$

$$\left\{ \left\| \left\| D^{\nu}_{x_0-} f \right\| \right\|_{\infty,[a,x_0]} (x_0-a)^{\nu+1} + \left\| \left\| D^{\nu}_{x_0} f \right\| \right\|_{\infty,[x_0,b]} (b-x_0)^{\nu+1} \right\} \leq$$

$$\frac{1}{(b-a)\Gamma(\nu+2)} \max\left(\left\| \left\| D^{\nu}_{x_0-} f \right\| \right\|_{\infty,[a,x_0]}, \left\| \left\| D^{\nu}_{x_0} f \right\| \right\|_{\infty,[x_0,b]} \right) \cdot$$

$$\left[(b-x_0)^{\nu+1} + (x_0-a)^{\nu+1} \right] \leq \tag{31}$$

$$\max\left(\left\| \left\| D^{\nu}_{x_0-} f \right\| \right\|_{\infty,[a,x_0]}, \left\| \left\| D^{\nu}_{x_0} f \right\| \right\|_{\infty,[x_0,b]} \right) \frac{(b-a)^{\nu}}{\Gamma(\nu+2)}.$$

We continue with a right fractional Poincaré like inequality:

Theorem 12 *([4]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = [\alpha]$. Let $f \in C^{\alpha}_{b-}([a,b], X)$. Assume that $f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$, when $\alpha \geq 1$. Then*

$$\|f\|_{L_q([a,b],X)} \leq \frac{(b-a)^{\alpha} \left\| D^{\alpha}_{b-} f \right\|_{L_q([a,b],X)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} (q\alpha)^{\frac{1}{q}}}. \tag{32}$$

We finally mention a Poincaré like left fractional inequality:

Theorem 13 *([4]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = [\nu]$. Let $f \in C^{\nu}_{a}([a,b], X)$. Assume that $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$, if $\nu \geq 1$. Then*

$$\|f\|_{L_q([a,b],X)} \leq \frac{(b-a)^{\nu}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} (q\nu)^{\frac{1}{q}}} \left\| D^{\nu}_{a} f \right\|_{L_q([a,b],X)}. \tag{33}$$

All this background next is applied for $X = \mathbb{C}$, the complex numbers with $\|\cdot\| = |\cdot|$ the absolute value.

3 Main Results

From now on here $f(z)$ and $z(t)$, $t \in [a,b]$, are as in section 1. Introduction.

We give a fractional $\mathbb{C}$-Polya inequality

Theorem 14 *Let $0 < \nu < 1$, $h \in C([a,b], \mathbb{C})$. Assume that $h \in C^{\nu}_{a}\left(\left[a, \frac{a+b}{2}\right], \mathbb{C}\right)$ and $h \in C^{\nu}_{b-}\left(\left[\frac{a+b}{2}, b\right], \mathbb{C}\right)$. Set*

$$M(h) = \max\left\{ \left\| \left| D^{\nu}_{a} h \right| \right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\| \left| D^{\nu}_{b-} h \right| \right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\}. \tag{34}$$

Then

$$\left|\int_a^b h(x)\,dx\right| \leq \int_a^b |h(x)|\,dx \leq M(h)\,\frac{(b-a)^{\nu+1}}{\Gamma(\nu+2)\,2^{\nu}}. \tag{35}$$

Inequality (35) is sharp, namely it is attained by

$$h_*(x) = \left\{ \begin{array}{ll} (x-a)^{\nu}\,\widetilde{c}, & x \in \left[a, \frac{a+b}{2}\right], \\ (b-x)^{\nu}\,\widetilde{c}, & x \in \left[\frac{a+b}{2}, b\right] \end{array} \right\}, \quad 0 < \nu < 1, \tag{36}$$

where $\widetilde{c} \in \mathbb{C} : |\widetilde{c}| = 1$.

Clearly here non-zero constant functions h *are excluded.*

Proof. By Theorem 10 for $X = \mathbb{C}$. ■

Next we apply Theorem 14 for $h(t) = f(z(t))\,z'(t)$, $t \in [a,b]$, to derive the following complex fractional Polya inequality:

Theorem 15 *Let* $0 < \nu < 1$, $f(z(\cdot))\,z'(\cdot) \in C([a,b],\mathbb{C})$. *Assume that* $f(z(\cdot))\,z'(\cdot) \in C_a^{\nu}\left(\left[a, \frac{a+b}{2}\right], \mathbb{C}\right)$ *and* $f(z(\cdot))\,z'(\cdot) \in C_{b-}^{\nu}\left(\left[\frac{a+b}{2}, b\right], \mathbb{C}\right)$. *Set*

$$M(f(z(\cdot))\,z'(\cdot)) =$$

$$\max\left\{ \left\|\left|D_a^{\nu}(f(z(\cdot))\,z'(\cdot))\right|\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\|\left|D_{b-}^{\nu}(f(z(\cdot))\,z'(\cdot))\right|\right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\}. \tag{37}$$

Then

$$\left|\int_{\gamma} f(z)\,dz\right| = \left|\int_{\gamma_{u,w}} f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|$$

$$\leq \int_a^b |f(z(t))|\,|z'(t)|\,dt = \int_{\gamma_{u,w}} |f(z)|\,|dz| \leq M(f(z(\cdot))\,z'(\cdot))\,\frac{(b-a)^{\nu+1}}{\Gamma(\nu+2)\,2^{\nu}}. \tag{38}$$

Proof. By Theorem 14. ■

Note: No boundary conditions are needed inTheorems 14, 15.

We continue with a fractional $\mathbb{C}$-Ostrowski type inequality:

Theorem 16 *Let* $\nu \geq 1$, $n = [\nu]$, $h \in C([a,b],\mathbb{C})$, $x_0 \in [a,b]$. *Assume that* $h|_{[a,x_0]} \in C_{x_0-}^{\nu}([a,x_0],\mathbb{C})$, $h|_{[x_0,b]} \in C_{x_0}^{\nu}([x_0,b],\mathbb{C})$, *and* $h^{(i)}(x_0) = 0$, *for* $i = 1, ..., n-1$, *which is void when* $1 \leq \nu < 2$. *Then*

$$\left|\frac{1}{b-a}\int_a^b h(x)\,dx - h(x_0)\right| \leq \frac{1}{(b-a)\,\Gamma(\nu+2)}\cdot$$

$$\left\{ \left\|\left|D_{x_0-}^{\nu}h\right|\right\|_{\infty,[a,x_0]}(x_0-a)^{\nu+1} + \left\|\left|D_{x_0}^{\nu}h\right|\right\|_{\infty,[x_0,b]}(b-x_0)^{\nu+1} \right\} \leq$$

$$\frac{1}{(b-a)\,\Gamma(\nu+2)}\cdot$$

$$\max\left(\left\|\left|D_{x_0-}^{\nu}h\right|\right\|_{\infty,[a,x_0]}, \left\|\left|D_{x_0}^{\nu}h\right|\right\|_{\infty,[x_0,b]} \right)\left[(b-x_0)^{\nu+1} + (x_0-a)^{\nu+1}\right] \leq \tag{39}$$

$$\max\left(\left\|\left|D_{x_0-}^{\nu}h\right|\right\|_{\infty,[a,x_0]}, \left\|\left|D_{x_0}^{\nu}h\right|\right\|_{\infty,[x_0,b]} \right)\frac{(b-a)^{\nu}}{\Gamma(\nu+2)}.$$

Proof. By Theorem 11. ■

Next we apply Theorem 16 for $h(t) = f(z(t))z'(t)$, $t \in [a,b]$, to derive the following complex fractional Ostrowski type inequality:

Theorem 17 *Let $\nu \geq 1$, $n = [\nu]$, $f(z(\cdot))z'(\cdot) \in C([a,b],\mathbb{C})$, $c \in [a,b]$; $v := z(c)$. Assume that $f(z(\cdot))z'(\cdot)|_{[a,c]} \in C_{c-}^{\nu}([a,c],\mathbb{C})$, $f(z(\cdot))z'(\cdot)|_{[c,b]} \in C_{c}^{\nu}([c,b],\mathbb{C})$, and $(f(z(\cdot))z'(\cdot))^{(i)}(c) = 0$, for $i = 1,...,n-1$, which is void when $1 \leq \nu < 2$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(z(t))z'(t)\,dt - f(z(c))z'(c)\right| = \left|\frac{1}{b-a}\int_{\gamma_{u,w}} f(z)\,dz - f(v)z'(c)\right|$$

$$= \left|\frac{1}{b-a}\int_{\gamma} f(z)\,dz - f(v)z'(c)\right| \leq \frac{1}{(b-a)\Gamma(\nu+2)}\cdot$$

$$\left\{\left\|\left|D_{c-}^{\nu}(f(z(\cdot))z'(\cdot))\right|\right\|_{\infty,[a,c]}(c-a)^{\nu+1} + \left\||D_{c}^{\nu}(f(z(\cdot))z'(\cdot))|\right\|_{\infty,[c,b]}(b-c)^{\nu+1}\right\} \leq$$

$$\frac{1}{(b-a)\Gamma(\nu+2)}\max\left(\left\|\left|D_{c-}^{\nu}(f(z(\cdot))z'(\cdot))\right|\right\|_{\infty,[a,c]}, \left\||D_{c}^{\nu}(f(z(\cdot))z'(\cdot))|\right\|_{\infty,[c,b]}\right)$$

$$\cdot\left[(b-c)^{\nu+1} + (c-a)^{\nu+1}\right] \leq \tag{40}$$

$$\max\left(\left\|\left|D_{c-}^{\nu}(f(z(\cdot))z'(\cdot))\right|\right\|_{\infty,[a,c]}, \left\||D_{c}^{\nu}(f(z(\cdot))z'(\cdot))|\right\|_{\infty,[c,b]}\right)\frac{(b-a)^{\nu}}{\Gamma(\nu+2)}.$$

Proof. By Theorem 16. ■

Next comes a right fractional $\mathbb{C}$-Poincaré like inequality:

Theorem 18 *Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = [\alpha]$. Let $f(z(\cdot))z'(\cdot) \in C_{b-}^{\alpha}([a,b],\mathbb{C})$. Assume that $(f(z(\cdot))z'(\cdot))^{(k)}(b) = 0$, $k = 0,1,...,m-1$, when $\alpha \geq 1$. Then*

$$\|f(z(\cdot))z'(\cdot)\|_{L_q([a,b],\mathbb{C})} \leq \frac{(b-a)^{\alpha}\left\|D_{b-}^{\alpha}(f(z(\cdot))z'(\cdot))\right\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}(q\alpha)^{\frac{1}{q}}}. \tag{41}$$

Proof. By Theorem 12. ■

We finish with a left fractional $\mathbb{C}$-Poincaré like inequality:

Theorem 19 *Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = [\nu]$. Let $f(z(\cdot))z'(\cdot) \in C_{a}^{\nu}([a,b],\mathbb{C})$. Assume that $(f(z(\cdot))z'(\cdot))^{(k)}(a) = 0$, $k = 0,1,...,n-1$, if $\nu \geq 1$. Then*

$$\|f(z(\cdot))z'(\cdot)\|_{L_q([a,b],\mathbb{C})} \leq \frac{(b-a)^{\nu}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}(q\nu)^{\frac{1}{q}}}\left\|D_{a}^{\nu}(f(z(\cdot))z'(\cdot))\right\|_{L_q([a,b],\mathbb{C})}. \tag{42}$$

Proof. By Theorem 13. ■

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