LIPSCHITZ TYPE INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. In this paper we provide some bounds for the quantity $\|f(y)-f(x)\|$ where $f: D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the domain D and $x, y \in \mathcal{B}$, a Banach algebra, with the spectra $\sigma(x), \sigma(y) \subset D$. Applications for the exponential and logarithmic function on the Banach algebra $\mathcal{B}$ are also given.

1. Introduction

Let $\mathcal{B}$ be an algebra over $\mathbb{C}$. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|: \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \le \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*. We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a, b \in \operatorname{Inv}(\mathcal{B})$ then $ab \in \operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \operatorname{Inv}(\mathcal{B})$;
(ii) $\{b \in \mathcal{B}: \|1-b\| < 1\} \subset \operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}\mathcal{B}$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}\mathcal{B} \ni a \longmapsto a^{-1} \in \operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by λ. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C}: \lambda - a \in \operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a: \rho(a) \to \operatorname{Inv}(\mathcal{B})$,

$$R_a(\lambda) := (\lambda - a)^{-1}.$$

For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma).$$

We also have that

$$\sigma(a) \subset \{\lambda \in \mathbb{C}: |\lambda| \le \|a\|\}.$$

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The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functional* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) We have

$$\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Let f be an analytic functions on the open disk $D(0,R)$ given by the *power series*

$$f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j \quad (|\lambda| < R).$$

If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a+b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if a is an element in $\mathcal{B}$ such that $\|1-a\| < 1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b = -\sum_{n=1}^{\infty} \frac{1}{n}(1-a)^n,$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b) = a$.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [13] and [16].

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_{\delta} f(\xi)(\xi - a)^{-1} d\xi, \tag{1.1}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [6, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.2}$$

holds.

Let $\mathfrak{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\mathfrak{Hol}(a)$ is an algebra where if f, $g \in \mathfrak{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f+g$ have domain $D(f) \cap D(g)$. $\mathfrak{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [6, p. 201-203]:

Theorem 1. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) *The map $f \mapsto f(a)$ of $\mathfrak{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.*
(b) *If $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ has radius of convergence $r > \nu(a)$, then $f \in \mathfrak{Hol}(a)$ and $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.*
(c) *If $f(z) \equiv 1$, then $f(a) = 1$.*
(d) *If $f(z) = z$ for all z, $f(a) = a$.*
(e) *If f, $f_1, ..., f_n...$ are analytic on G, $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\|f_n(a) - f(a)\| \to 0$ as $n \to \infty$.*
(f) *The Riesz Functional Calculus is unique and if a, b are commuting elements in $\mathcal{B}$ and $f \in \mathfrak{Hol}(a)$, then $f(a)b = bf(a)$.*

For some recent norm inequalities for functions on Banach algebras, see [3]-[5] and [7]-[12].

One of the central problems in *perturbation theory* is to find bounds for

$$\|f(A) - f(B)\|$$

in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operators A and B can be defined. For some results on this topic, see [4], [14] and the references therein.

In [2] the author obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a)\|A - B\| \tag{1.3}$$

where f is an *operator monotone function* on $(0, \infty)$ and A, B are bounded linear operators on an Hilbert space with A, $B \geq aI_H > 0$.

In this paper we provide some bounds for the quantity $\|f(y) - f(x)\|$ where $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the domain D and x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D$. Applications for the exponential and logarithmic function on the Banach algebra $\mathcal{B}$ are also given.

2. The Results

We start with the following lemma that is of interest in itself:

Lemma 1. *For any x, $y \in \mathcal{B}$ with $\|x\|$, $\|y\| < 1$ we have*

$$\left\|(1-y)^{-1} - (1-x)^{-1}\right\| \leq \frac{\|y - x\|}{(1 - \|y\|)(1 - \|x\|)}. \tag{2.1}$$

Proof. We use the identity (see for instance [3, p. 254])

$$a^n - b^n = \sum_{j=0}^{n-1} a^{n-1-j} (a-b) b^j \tag{2.2}$$

that holds for any $a, b \in \mathcal{B}$ and $n \geq 1$.

For $x, y \in \mathcal{B}$ we consider the function $\varphi : [0,1] \to \mathcal{B}$ defined by $\varphi(t) = [(1-t)x + ty]^n$. For $t \in (0,1)$ and $\varepsilon \neq 0$ with $t + \varepsilon \in (0,1)$ we have from (2.2) that

$$\begin{aligned}\varphi(t+\varepsilon) - \varphi(t) &= [(1-t-\varepsilon)x + (t+\varepsilon)y]^n - [(1-t)x + ty]^n \\ &= \varepsilon \sum_{j=0}^{n-1} [(1-t-\varepsilon)x + (t+\varepsilon)y]^{n-1-j} (y-x) [(1-t)x + ty]^j .\end{aligned}$$

Dividing with $\varepsilon \neq 0$ and taking the limit over $\varepsilon \to 0$ we have in the norm topology of $\mathcal{B}$ that

$$\begin{aligned}\varphi'(t) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\varphi(t+\varepsilon) - \varphi(t)] \\ &= \sum_{j=0}^{n-1} [(1-t)x + ty]^{n-1-j} (y-x) [(1-t)x + ty]^j .\end{aligned} \tag{2.3}$$

Integrating on $[0,1]$ we get from (2.3) that

$$\int_0^1 \varphi'(t)\, dt = \sum_{j=0}^{n-1} \int_0^1 [(1-t)x + ty]^{n-1-j} (y-x) [(1-t)x + ty]^j \, dt$$

and since

$$\int_0^1 \varphi'(t)\, dt = \varphi(1) - \varphi(0) = y^n - x^n$$

then we get the following *equality of interest*

$$y^n - x^n = \sum_{j=0}^{n-1} \int_0^1 [(1-t)x + ty]^{n-1-j} (y-x) [(1-t)x + ty]^j \, dt$$

for any $x, y \in \mathcal{B}$ and $n \geq 1$.

Taking the norm and utilizing the properties of Bochner integral for vector valued functions (see for instance [15, p. 21]) we have

$$\begin{aligned}\|y^n - x^n\| &\leq \sum_{j=0}^{n-1} \left\| \int_0^1 [(1-t)x + ty]^{n-1-j} (y-x) [(1-t)x + ty]^j \, dt \right\| \\ &\leq \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)x + ty]^{n-1-j} (y-x) [(1-t)x + ty]^j \right\| dt \\ &\leq \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)x + ty]^{n-1-j} \right\| \|y-x\| \left\| [(1-t)x + ty]^j \right\| dt \\ &\leq \sum_{j=0}^{n-1} \int_0^1 \|(1-t)x + ty\|^{n-1-j} \|y-x\| \|(1-t)x + ty\|^j \, dt \\ &= n \|y-x\| \int_0^1 \|(1-t)x + ty\|^{n-1} \, dt\end{aligned} \tag{2.4}$$

for any $x, y \in \mathcal{B}$ and $n \geq 1$.

Now, for any $m \geq 1$, by making use of the inequality (2.4) we have

$$\begin{aligned}(2.5)\qquad \left\| \sum_{n=0}^{m} y^n - \sum_{n=0}^{m} x^n \right\| &= \left\| \sum_{n=1}^{m} (y^n - x^n) \right\| \\ &\leq \sum_{n=1}^{m} \|y^n - x^n\| \\ &\leq \|y - x\| \sum_{n=1}^{m} n \int_0^1 \|(1-t)x + ty\|^{n-1} dt \\ &= \|y - x\| \int_0^1 \left(\sum_{n=1}^{m} n \|(1-t)x + ty\|^{n-1} \right) dt.\end{aligned}$$

Moreover, since $\|x\|, \|y\| < 1$, then the series $\sum_{n=0}^{\infty} y^n$, $\sum_{n=0}^{\infty} x^n$ and

$$\sum_{n=1}^{\infty} n \|(1-t)x + ty\|^{n-1}$$

are convergent and

$$\sum_{n=0}^{\infty} y^n = (1-y)^{-1}, \sum_{n=0}^{\infty} x^n = (1-x)^{-1}$$

while

$$\sum_{n=1}^{\infty} n \|(1-t)x + ty\|^{n-1} = (1 - \|(1-t)x + ty\|)^{-2}.$$

Therefore, by taking the limit over $m \to \infty$ in the inequality (2.5) we deduce the following inequality that is of interest in itself

$$(2.6)\qquad \left\| (1-y)^{-1} - (1-x)^{-1} \right\| \leq \|y - x\| \int_0^1 (1 - \|(1-t)x + ty\|)^{-2} dt$$

for all $\|x\|, \|y\| < 1$.

Now, by the triangle inequality and the fact that $\|x\|, \|y\| < 1$ we have

$$1 - \|(1-t)x + ty\| \geq 1 - (1-t)\|x\| - t\|y\| > 0$$

for all $t \in [0,1]$.

This implies that

$$(2.7)\qquad (1 - \|(1-t)x + ty\|)^{-2} \leq (1 - (1-t)\|x\| - t\|y\|)^{-2}$$

for all $t \in [0,1]$.

Integrating (2.7) over $t \in [0,1]$, we get

$$(2.8)\qquad \int_0^1 (1 - \|(1-t)x + ty\|)^{-2} dt \leq \int_0^1 (1 - (1-t)\|x\| - t\|y\|)^{-2} dt.$$

Observe that for $\|x\| \neq \|y\|$ we have

$$\begin{aligned}
&\int_0^1 (1-(1-t)\|x\| - t\|y\|)^{-2}\,dt \\
&= -\frac{1}{\|x\|-\|y\|}\int_0^1 d\left[(1-\|x\|+t(\|x\|-\|y\|))^{-1}\right] \\
&= -\frac{1}{\|x\|-\|y\|}\left[(1-\|y\|)^{-1}-(1-\|x\|)^{-1}\right] \\
&= \frac{1}{1-\|y\|}\frac{1}{1-\|x\|}.
\end{aligned}$$

Therefore

$$\int_0^1 (1-\|(1-t)x+ty\|)^{-2}\,dt \le \frac{1}{1-\|y\|}\frac{1}{1-\|x\|}. \tag{2.9}$$

We also observe that, by (2.8) for $\|y\| = \|x\|$ the inequality (2.9) also holds.

By employing (2.6), (2.8) and (2.9) we obtain the desired result (2.1). □

Our main result is as follows:

Theorem 2. *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D$ and γ a rectifiable path in D and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\delta)$. Then we have*

$$\|f(y)-f(x)\| \le \frac{1}{2\pi}\|y-x\|\int_\gamma \frac{|f(\xi)|\,|d\xi|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}. \tag{2.10}$$

Proof. Using the Riesz functional calculus, we have

$$\begin{aligned}
f(y)-f(x) &= \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-y)^{-1}\,d\xi - \int_\gamma f(\xi)(\xi-x)^{-1}\,d\xi \\
&= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi-y)^{-1}-(\xi-x)^{-1}\right]d\xi.
\end{aligned}$$

By taking the norm in this equality and using the integral's properties we get

$$\begin{aligned}
\|f(y)-f(x)\| &\le \frac{1}{2\pi}\int_\gamma |f(\xi)|\left[\left\|(\xi-y)^{-1}-(\xi-x)^{-1}\right\|\right]|d\xi| \\
&= \frac{1}{2\pi}\int_\gamma |f(\xi)|\,|\xi|^{-1}\left[\left\|\left(1-\frac{y}{\xi}\right)^{-1}-\left(1-\frac{x}{\xi}\right)^{-1}\right\|\right]|d\xi|.
\end{aligned} \tag{2.11}$$

Since $\left\|\frac{y}{\xi}\right\|, \left\|\frac{x}{\xi}\right\| < 1$ for $\xi \in \gamma$ then we can apply Lemma 1 to get

$$\begin{aligned}
\left\|\left(1-\frac{y}{\xi}\right)^{-1}-\left(1-\frac{x}{\xi}\right)^{-1}\right\| &\le \frac{\left\|\frac{y}{\xi}-\frac{x}{\xi}\right\|}{\left(1-\left\|\frac{y}{\xi}\right\|\right)\left(1-\left\|\frac{x}{\xi}\right\|\right)} \\
&= \frac{|\xi|\,\|y-x\|}{(|\xi|-\|y\|)(|\xi|-\|x\|)},
\end{aligned}$$

which gives by integration that

$$\begin{aligned}(2.12)\qquad &\frac{1}{2\pi}\int_{\gamma}|f(\xi)||\xi|^{-1}\left[\left\|\left(1-\frac{y}{\xi}\right)^{-1}-\left(1-\frac{x}{\xi}\right)^{-1}\right\|\right]|d\xi| \\ &\leq \frac{1}{2\pi}\int_{\gamma}|f(\xi)||\xi|^{-1}\frac{|\xi|\,\|y-x\|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}|d\xi| \\ &= \frac{1}{2\pi}\|y-x\|\int_{\gamma}\frac{|f(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}|d\xi|.\end{aligned}$$

By making use of (2.11) and (2.12) we get (2.10). □

Corollary 1. *With the assumptions of Theorem 2 and if*

$$\|f\|_{\gamma,\infty} := \sup_{\xi\in\gamma}|f(\xi)| < \infty,$$

then

$$\|f(y)-f(x)\| \leq \frac{1}{2\pi}\|y-x\|\,\|f\|_{\gamma,\infty}\int_{\gamma}\frac{|d\xi|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}. \tag{2.13}$$

Remark 1. *If we assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then by taking γ parametrized by $\xi(t) = Re^{2\pi it}$ where $t \in [0,1]$, then $d\xi(t) = 2\pi iRe^{2\pi it}dt$, $|d\xi(t)| = 2\pi Rdt$, $|\xi| = R$ and by (2.10) we get*

$$\|f(y)-f(x)\| \leq \frac{R\|y-x\|}{(R-\|y\|)(R-\|x\|)}\int_0^1\left|f\left(Re^{2\pi it}\right)\right|dt. \tag{2.14}$$

Moreover, if

$$\|f\|_{R,\infty} := \sup_{t\in[0,1]}\left|f\left(Re^{2\pi it}\right)\right| < \infty,$$

then we have the simpler inequality

$$\|f(y)-f(x)\| \leq \frac{R\|f\|_{R,\infty}\|y-x\|}{(R-\|y\|)(R-\|x\|)}. \tag{2.15}$$

3. Some Examples

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right| = |\exp[R(\cos(2\pi t) + i\sin(2\pi t))]| = \exp[R\cos(2\pi t)]$$

and then by (2.14) we get

$$\|\exp y - \exp x\| \leq \frac{R\|y-x\|}{(R-\|y\|)(R-\|x\|)}\int_0^1\exp[R\cos(2\pi t)]dt. \tag{3.1}$$

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function*. For $n=0$ we have $I_0(z)$ given by

$$I_0(z)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta-\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

If we change the variable $\theta=2\pi t$, then $dt=\frac{1}{2\pi}d\theta$ and

$$\begin{aligned}\int_0^1\exp[R\cos(2\pi t)]\,dt&=\frac{1}{2\pi}\int_0^{2\pi}\exp[R\cos\theta]\,d\theta\\&=\frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi\exp[R\cos\theta]\,d\theta+\frac{1}{\pi}\int_\pi^{2\pi}\exp[R\cos\theta]\,d\theta\right)\\&=\frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi\exp[R\cos\theta]\,d\theta+\frac{1}{\pi}\int_0^\pi\exp[-R\cos\theta]\,d\theta\right)\\&=\frac{1}{2}\left(I_0(R)+I_0(-R)\right)=I_0(R).\end{aligned}$$

From (3.1) we then get

$$\|\exp y-\exp x\|\le\frac{RI_0(R)\|y-x\|}{(R-\|y\|)(R-\|x\|)},\tag{3.2}$$

for $x,\,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$.

By using the power series

$$f(z):=\ln(1-z)^{-1}=\sum_{n=1}^{\infty}\frac{1}{n}z^n$$

that is convergent on open disk $D(0,1)$, we can define

$$\ln(1-a)^{-1}:=\sum_{n=1}^{\infty}\frac{1}{n}a^n$$

for all elements a in $\mathcal{B}$ with $\|a\|<1$.

We observe that

$$\left|\ln(1-z)^{-1}\right|\le\sum_{n=1}^{\infty}\frac{1}{n}|z|^n=\ln(1-|z|)^{-1}$$

for $|z|<1$.

Now if we assume that $x,\,y\in\mathcal{B}$ and $\|x\|,\|y\|<R<1$, then by (2.14) we get

$$\left\|\ln(1-y)^{-1}-\ln(1-x)^{-1}\right\|\le\frac{R\ln\left[(1-R)^{-1}\right]\|y-x\|}{(R-\|y\|)(R-\|x\|)}.\tag{3.3}$$

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