AN INTEGRAL REPRESENTATION OF THE REMAINDER IN TAYLOR'S EXPANSION FORMULA FOR ANALYTIC FUNCTIONS ON BANACH ALGEBRAS

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ABSTRACT. In this paper we establish an integral representation of the remainder in Taylor's expansion formula for analytic functions of elements in Banach algebras when the functions are defined on convex domains. Error bounds are provided and some examples for the complex exponential in Banach algebras are also given.

1. Introduction

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and y, $x \in D$, then we have the following Taylor's expansion with integral remainder is valid

$$f(y) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x)(y-x)^k + \frac{1}{n!}(y-x)^{n+1} \int_0^1 f^{(n+1)}[(1-s)x + sy](1-s)^n ds \tag{1.1}$$

for $n \geq 0$, see for instance [24].

Consider the function $f(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi < \operatorname{Arg}(z) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C} \backslash \{x + iy : x \leq 0,\ y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{z^k},\ k \geq 1,\ z \in \mathbb{C}_\ell.$$

Using the representation (1.1) we then have

$$\operatorname{Log}(z) = \operatorname{Log}(x) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{z-x}{x}\right)^k + (-1)^n (z-x)^{n+1} \int_0^1 \frac{(1-s)^n ds}{[(1-s)x + sz]^{n+1}} \tag{1.2}$$

for all z, $x \in \mathbb{C}_\ell$ with $(1-s)x + sz \in \mathbb{C}_\ell$ for $s \in [0,1]$.

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Consider the complex exponential function $f(z) = \exp(z)$, then by (1.1) we get

$$\exp(z) = \exp(x) \sum_{k=0}^{n} \frac{1}{k!} (z-x)^k + \frac{1}{n!} (z-x)^{n+1} \int_0^1 (1-s)^n \exp[(1-s)x + sz]\, ds \tag{1.3}$$

for all $z, x \in \mathbb{C}$.

For various inequalities related to Taylor's expansions for real functions see [1]-[3], [8] and [16]-[23].

In order to extend Taylor's formula for function defined on Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \le \|a\| \|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a, b \in \operatorname{Inv}(\mathcal{B})$ then $ab \in \operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \operatorname{Inv}(\mathcal{B})$;
(ii) $\{a \in \mathcal{B}\colon \ \|1 - b\| < 1\} \subset \operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} : \ z - a \in \operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \operatorname{Inv}(\mathcal{B})$, $R_a(z) := (z-a)^{-1}$. For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z-w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \le \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i} \int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_{\delta} f(\xi)(\xi - a)^{-1} d\xi, \tag{1.4}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [6, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.5}$$

holds.

Let $\mathfrak{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\mathfrak{Hol}(a)$ is an algebra where if f, $g \in \mathfrak{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f+g$ have domain $D(f) \cap D(g)$. $\mathfrak{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [6, p. 201-203]:

Theorem 1. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) *The map $f \mapsto f(a)$ of $\mathfrak{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.*
(b) *If $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ has radius of convergence $r > \nu(a)$, then $f \in \mathfrak{Hol}(a)$ and $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.*
(c) *If $f(z) \equiv 1$, then $f(a) = 1$.*
(d) *If $f(z) = z$ for all z, $f(a) = a$.*
(e) *If $f, f_1, ..., f_n...$ are analytic on G, $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\|f_n(a) - f(a)\| \to 0$ as $n \to \infty$.*
(f) *The Riesz Functional Calculus is unique and if a, b are commuting elements in $\mathcal{B}$ and $f \in \mathfrak{Hol}(a)$, then $f(a)b = bf(a)$.*

For some recent norm inequalities for functions on Banach algebras, see [4]-[5] and [9]-[15].

In this paper we establish an integral representation of the remainder in Taylor's expansion formula for analytic functions of elements in Banach algebras when the functions are defined on convex domains. Error bounds are provided and some examples for the complex exponential in Banach algebras are also given.

2. Some Identities

We have:

Theorem 2. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, then for all $\lambda \in G$ and $n \geq 0$ we have*

$$f(a) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\lambda)(a - \lambda)^k + \frac{1}{n!}(a - \lambda)^{n+1} \int_0^1 f^{(n+1)}((1 - s)\lambda + sa)(1 - s)^n ds. \tag{2.1}$$

Proof. Assume that $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$. By using the analytic functional calculus (1.4) and the representation (1.1) we have for all $\lambda \in G$ that

$$f(a) = \frac{1}{2\pi i}\int_{\delta}\left(\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(\lambda)(\xi-\lambda)^{k}\right)(\xi-a)^{-1}d\xi \tag{2.2}$$
$$+\frac{1}{n!}\frac{1}{2\pi i}\int_{\delta}(\xi-\lambda)^{n+1}\left(\int_{0}^{1}f^{(n+1)}[(1-s)\lambda+s\xi](1-s)^{n}ds\right)(\xi-a)^{-1}d\xi$$
$$=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(\lambda)\left(\frac{1}{2\pi i}\int_{\delta}(\xi-\lambda)^{k}(\xi-a)^{-1}d\xi\right)$$
$$+\frac{1}{n!}\int_{0}^{1}\left(\frac{1}{2\pi i}\int_{\delta}(\xi-\lambda)^{n+1}f^{(n+1)}[(1-s)\lambda+s\xi](\xi-a)^{-1}d\xi\right)(1-s)^{n}ds,$$

where for the last equality we used Fubini's theorem.

Using the functional calculus for the analytic functions $G \ni \xi \mapsto (\xi-\lambda)^{k} \in \mathbb{C}$ and $G \ni \xi \mapsto (\xi-\lambda)^{n+1}f^{(n+1)}[(1-s)\lambda+s\xi] \in \mathbb{C}$ we have

$$\frac{1}{2\pi i}\int_{\delta}(\xi-\lambda)^{k}(\xi-a)^{-1}d\xi=(a-\lambda)^{k}$$

and

$$\frac{1}{2\pi i}\int_{\delta}(\xi-\lambda)^{n+1}f^{(n+1)}[(1-s)\lambda+s\xi](\xi-a)^{-1}d\xi$$
$$=(a-\lambda)^{n+1}f^{(n+1)}[(1-s)\lambda+sa],$$

then by (2.2) we get the desired result (2.1). □

Corollary 1. *With the assumptions of Theorem 2 and if $b \in \mathcal{B}$, then we have the perturbed formula*

$$f(a)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(\lambda)(a-\lambda)^{k}+\frac{1}{(n+1)!}(a-\lambda)^{n+1}b \tag{2.3}$$
$$+\frac{1}{n!}(a-\lambda)^{n+1}\int_{0}^{1}\left[f^{(n+1)}((1-s)\lambda+sa)-b\right](1-s)^{n}ds.$$

Proof. We have

$$\frac{1}{n!}(a-\lambda)^{n+1}\int_{0}^{1}\left[f^{(n+1)}((1-s)\lambda+sa)-b\right](1-s)^{n}ds$$
$$=\frac{1}{n!}(a-\lambda)^{n+1}\int_{0}^{1}f^{(n+1)}((1-s)\lambda+sa)(1-s)^{n}ds$$
$$-\frac{1}{n!}(a-\lambda)^{n+1}b\int_{0}^{1}(1-s)^{n}ds$$
$$=\frac{1}{n!}(a-\lambda)^{n+1}\int_{0}^{1}f^{(n+1)}((1-s)\lambda+sa)(1-s)^{n}ds-\frac{1}{(n+1)!}(a-\lambda)^{n+1}b,$$

and by (2.1) we get (2.3). □

Remark 1. *Under the assumptions of Theorem 2 and if we take various particular values for b we can get the following particular equalities of interest*

$$f(a)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(\lambda)(a-\lambda)^{k}+\frac{1}{(n+1)!}(a-\lambda)^{n+1}f^{(n+1)}(a)+\frac{1}{n!}(a-\lambda)^{n+1}\int_{0}^{1}\left[f^{(n+1)}((1-s)\lambda+sa)-f^{(n+1)}(a)\right](1-s)^{n}\,ds, \tag{2.4}$$

$$f(a)=\sum_{k=0}^{n+1}\frac{1}{k!}f^{(k)}(\lambda)(a-\lambda)^{k}+\frac{1}{n!}(a-\lambda)^{n+1}\int_{0}^{1}\left[f^{(n+1)}((1-s)\lambda+sa)-f^{(n+1)}(\lambda)\right](1-s)^{n}\,ds, \tag{2.5}$$

$$f(a)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(\lambda)(a-\lambda)^{k}+\frac{1}{(n+1)!}(a-\lambda)^{n+1}f^{(n+1)}\left(\frac{\lambda+a}{2}\right)+\frac{1}{n!}(a-\lambda)^{n+1}\int_{0}^{1}\left[f^{(n+1)}((1-s)\lambda+sa)-f^{(n+1)}\left(\frac{\lambda+a}{2}\right)\right](1-s)^{n}\,ds, \tag{2.6}$$

and

$$\begin{aligned} f(a)&=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(\lambda)(a-\lambda)^{k}+\frac{1}{(n+1)!}(a-\lambda)^{n+1}\int_{0}^{1}f^{(n+1)}((1-\tau)\lambda+\tau a)\,d\tau+\frac{1}{n!}(a-\lambda)^{n+1}\\ &\quad\times\int_{0}^{1}\left[f^{(n+1)}((1-s)\lambda+sa)-\int_{0}^{1}f^{(n+1)}((1-\tau)\lambda+\tau a)\,d\tau\right](1-s)^{n}\,ds. \end{aligned} \tag{2.7}$$

Let $a\in\mathcal{B}$ with $\sigma(a)\subset G$ where G is a convex domain in

$$\mathbb{C}_{\ell}:=\mathbb{C}\backslash\{x+iy:x\leq 0,\ y=0\}.$$

The function $f(z)=\operatorname{Log}(z)$ is analytic in G and by using the functional calculus (1.4) we can define the element

$$\operatorname{Log}a:=\frac{1}{2\pi i}\int_{\delta}\operatorname{Log}(\xi)(\xi-a)^{-1}\,d\xi, \tag{2.8}$$

where $\delta\subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a)\subset\operatorname{ins}(\delta)$.

Now, by using some of the above identities for the Log function, we can state for $\lambda\in G$ and $n\geq 1$ that

$$\operatorname{Log}a=\operatorname{Log}\lambda+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k\lambda^{k}}(a-\lambda)^{k}+(-1)^{n}(a-\lambda)^{n+1}\int_{0}^{1}((1-s)\lambda+sa)^{-n-1}(1-s)^{n}\,ds, \tag{2.9}$$

$$\operatorname{Log} a=\operatorname{Log}\lambda+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k\lambda^{k}}(a-\lambda)^{k}+\frac{(-1)^{n}}{(n+1)}(a-\lambda)^{n+1}a^{-n-1}$$
$$+(-1)^{n}(a-\lambda)^{n+1}\int_{0}^{1}\left[((1-s)\lambda+sa)^{-n-1}-a^{-n-1}\right](1-s)^{n}ds, \tag{2.10}$$

and

$$\operatorname{Log} a=\operatorname{Log}\lambda+\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k\lambda^{k}}(a-\lambda)^{k}$$
$$+(-1)^{n}(a-\lambda)^{n+1}\int_{0}^{1}\left[((1-s)\lambda+sa)^{-n-1}-\lambda^{-n-1}\right](1-s)^{n}ds, \tag{2.11}$$

provided $(1-s)\lambda+sa$ is invertible for all $s\in[0,1]$.

For $n=0$ the sum-term above must be dropped.

If we use some of the general equalities above for the exponential function, we have

$$\exp a=\exp\lambda\sum_{k=0}^{n}\frac{1}{k!}(a-\lambda)^{k}$$
$$+\frac{1}{n!}(a-\lambda)^{n+1}\int_{0}^{1}\exp((1-s)\lambda+sa)(1-s)^{n}ds. \tag{2.12}$$

$$\exp a=\exp\lambda\sum_{k=0}^{n}\frac{1}{k!}(a-\lambda)^{k}+\frac{1}{(n+1)!}(a-\lambda)^{n+1}\exp a$$
$$+\frac{1}{n!}(a-\lambda)^{n+1}\int_{0}^{1}\left[\exp((1-s)\lambda+sa)-\exp(a)\right](1-s)^{n}ds, \tag{2.13}$$

and

$$\exp a=\exp\lambda\sum_{k=0}^{n+1}\frac{1}{k!}(a-\lambda)^{k}$$
$$+\frac{1}{n!}(a-\lambda)^{n+1}\int_{0}^{1}\left[\exp((1-s)\lambda+sa)-\exp\lambda\right](1-s)^{n}ds, \tag{2.14}$$

for all $a\in\mathcal{B}$, $\lambda\in\mathbb{C}$ and $n\geq 0$.

3. Norm Inequalities

We start with the following basic result:

Theorem 3. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic on G, then for all $\lambda\in G$ and $n\geq 0$*

we have

$$\left\| f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\lambda)(a-\lambda)^k \right\| \tag{3.1}$$
$$\leq \frac{1}{n!} \|a-\lambda\|^{n+1} \int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) \right\| (1-s)^n \, ds$$
$$\leq \frac{1}{n!} \|a-\lambda\|^{n+1} \begin{cases} \frac{1}{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)}((1-s)\lambda + sa) \right\| \\ \\ \frac{1}{(n^{qn}+1)^{1/q}} \left(\int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) \right\|^p ds \right)^{1/p} \\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) \right\| ds. \end{cases}$$

Proof. Using the equality (2.1) we have

$$\left\| f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\lambda)(a-\lambda)^k \right\| \tag{3.2}$$
$$= \frac{1}{n!} \left\| (a-\lambda)^{n+1} \int_0^1 f^{(n+1)}((1-s)\lambda + sa)(1-s)^n \, ds \right\|$$
$$\leq \frac{1}{n!} \left\| (a-\lambda)^{n+1} \right\| \left\| \int_0^1 f^{(n+1)}((1-s)\lambda + sa)(1-s)^n \, ds \right\|$$
$$\leq \frac{1}{n!} \|a-\lambda\|^{n+1} \int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) \right\| (1-s)^n \, ds =: A,$$

which proves the first inequality in (3.1).

Using Hölder's integral inequality we have

$$\int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) \right\| (1-s)^n \, ds$$
$$\leq \begin{cases} \sup_{s\in[0,1]} \left\| f^{(n+1)}((1-s)\lambda + sa) \right\| \int_0^1 (1-s)^n ds \\ \\ \left(\int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) \right\|^p ds \right)^{1/p} \left(\int_0^1 (1-s)^{qn} ds \right)^{1/q} \\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \max_{s\in[0,1]} \{(1-s)^n\} \int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) \right\| ds \end{cases}$$
$$= \begin{cases} \frac{1}{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)}((1-s)\lambda + sa) \right\| \\ \\ \frac{1}{(n^{qn}+1)^{1/q}} \left(\int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) \right\|^p ds \right)^{1/p} \\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) \right\| ds, \end{cases}$$

which together with (3.2) gives the desired result. □

Let $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$ and $\lambda\in G$. We define $G_{\lambda,a}:=\{(1-t)\lambda+ta \mid \text{with } t\in[0,1]\}$. We observe that $G_{\lambda,a}$ is a convex subset in $\mathcal{B}$ for every $\lambda\in G$.

For two distinct elements u, v in the Banach algebra B we say that the function $g:G_{\lambda,a}\to\mathcal{B}$ belongs to the class $\Delta_{u,v}(G_{\lambda,a})$ if it satisfies the boundedness condition

$$\left\|g((1-t)\lambda+ta)-\frac{u+v}{2}\right\|\leq\frac{1}{2}\|v-u\| \tag{3.3}$$

for all $t\in[0,1]$. We write $g\in\Delta_{u,v}(G_{\lambda,a})$. This definition is an extension to Banach algebras valued functions of the scalar case, see [7].

We say that the function $g:G_{\lambda,a}\to B$ is Lipschitzian on $G_{\lambda,a}$ with the constant $L_{\lambda,a}>0$, if for all x, $y\in G_{\lambda,a}$ we have

$$\|g(x)-g(y)\|\leq L_{\lambda,a}\|x-y\|.$$

This is equivalent to

$$\|g((1-t)\lambda+ta)-g((1-s)\lambda+sa)\|\leq L_{\lambda,a}|t-s|\,\|a-\lambda\| \tag{3.4}$$

for all t, $s\in[0,1]$. We write this by $g\in\mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$.

Assume that $h:G\to\mathbb{C}$ is an analytic function on G. For $t\in[0,1]$ and $\lambda\in G$, the auxiliary function $h_{t,\lambda}$ defined on G by $h_{t,\lambda}(\xi):=h((1-t)\lambda+t\xi)$ is also analytic and using the analytic functional calculus (1.1) for the element $a\in\mathcal{B}$, we can define

$$\begin{aligned}\widetilde{h}((1-t)\lambda+ta)&:=h_{t,\lambda}(a)=\frac{1}{2\pi i}\int_\gamma h_{t,\lambda}(\xi)(\xi-a)^{-1}d\xi \\ &=\frac{1}{2\pi i}\int_\gamma h((1-t)\lambda+t\xi)(\xi-a)^{-1}d\xi.\end{aligned} \tag{3.5}$$

We say that the scalar function $h\in\Delta_{u,v}(G_{\lambda,a})$ if its extension $\widetilde{h}:G_{\lambda,a}\to B$ satisfies the boundedness condition (3.3). Also, we say that the scalar function $h\in\mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$ if its extension $\widetilde{h}:G_{\lambda,a}\to B$ satisfies the Lipschitz condition (3.4).

Theorem 4. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic on G, $\lambda\in G$, $n\geq 0$ and there exists two distinct elements u_n, v_n in the Banach algebra B such that $f^{(n+1)}\in\Delta_{u_n,v_n}(G_{\lambda,a})$, then*

$$\begin{aligned}&\left\|f(a)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(\lambda)(a-\lambda)^k-\frac{1}{(n+1)!}(a-\lambda)^{n+1}\frac{u_n+v_n}{2}\right\| \\ &\qquad\leq\frac{1}{2(n+1)!}\|a-\lambda\|^{n+1}\|u_n-v_n\|.\end{aligned} \tag{3.6}$$

Proof. Using the equality (2.3) we have

$$\begin{aligned}&f(a)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(\lambda)(a-\lambda)^k-\frac{1}{(n+1)!}(a-\lambda)^{n+1}\frac{u_n+v_n}{2} \\ &\quad=\frac{1}{n!}(a-\lambda)^{n+1}\int_0^1\left[f^{(n+1)}((1-s)\lambda+sa)-\frac{u_n+v_n}{2}\right](1-s)^n ds.\end{aligned}$$

By taking the norm and using the fact that $f^{(n+1)} \in \Delta_{u_n,v_n}(G_{\lambda,a})$ we have

$$\begin{aligned}
&\left\| f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\lambda)(a-\lambda)^k - \frac{1}{(n+1)!}(a-\lambda)^{n+1}\frac{u_n+v_n}{2} \right\| \\
&\quad = \frac{1}{n!}\left\| (a-\lambda)^{n+1} \int_0^1 \left[f^{(n+1)}((1-s)\lambda + sa) - \frac{u_n+v_n}{2}\right](1-s)^n ds \right\| \\
&\quad \le \frac{1}{n!}\left\| (a-\lambda)^{n+1}\right\| \left\| \int_0^1 \left[f^{(n+1)}((1-s)\lambda + sa) - \frac{u_n+v_n}{2}\right](1-s)^n ds \right\| \\
&\quad \le \frac{1}{n!}\|a-\lambda\|^{n+1} \int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) - \frac{u_n+v_n}{2}\right\|(1-s)^n ds \\
&\le \frac{1}{2n!}\|a-\lambda\|^{n+1}\|u_n-v_n\| \int_0^1 (1-s)^n ds = \frac{1}{2(n+1)!}\|a-\lambda\|^{n+1}\|u_n-v_n\|,
\end{aligned}$$

which proves the desired inequality (3.6). □

We also have:

Theorem 5. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, $\lambda \in G$, $n \ge 0$ and there exists $L_{\lambda,a,n} > 0$ such that $f^{(n+1)} \in \mathfrak{Lip}_{L_{\lambda,a,n}}(G_{\lambda,a})$, then*

$$\begin{aligned}
&\left\| f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\lambda)(a-\lambda)^k - \frac{1}{(n+1)!}(a-\lambda)^{n+1} f^{(n+1)}(a) \right\| \\
&\qquad \le \frac{1}{(n+2)\,n!}\|a-\lambda\|^{n+2} L_{\lambda,a,n},
\end{aligned} \tag{3.7}$$

$$\left\| f(a) - \sum_{k=0}^{n+1} \frac{1}{k!} f^{(k)}(\lambda)(a-\lambda)^k \right\| \le \frac{1}{(n+2)!}\|a-\lambda\|^{n+2} L_{\lambda,a,n}, \tag{3.8}$$

$$\begin{aligned}
&\left\| f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\lambda)(a-\lambda)^k - \frac{1}{(n+1)!}(a-\lambda)^{n+1} f^{(n+1)}\left(\frac{\lambda+a}{2}\right) \right\| \\
&\qquad \le \frac{1}{n!}\|a-\lambda\|^{n+2} K_n L_{\lambda,a,n}
\end{aligned} \tag{3.9}$$

where

$$K_n = \frac{1}{n+2}\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] - \frac{1}{2(n+1)}\left[1 - \left(\frac{1}{2}\right)^{n}\right], \ n \ge 0,$$

and

$$\begin{aligned}
&\Bigg\| f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\lambda)(a-\lambda)^k \\
&\qquad - \frac{1}{(n+1)!}(a-\lambda)^{n+1} \int_0^1 f^{(n+1)}((1-\tau)\lambda + \tau a)\, d\tau \Bigg\| \\
&\qquad\qquad \le \frac{n}{2(n+2)!}\|a-\lambda\|^{n+2} L_{\lambda,a,n}, \ n \ge 1.
\end{aligned} \tag{3.10}$$

Proof. Using equality (2.4) and the fact that $f^{(n+1)} \in \mathfrak{Lip}_{L_{\lambda,a,n}}(G_{\lambda,a})$ we have

$$\begin{aligned}
&\left\| f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\lambda)(a-\lambda)^k - \frac{1}{(n+1)!}(a-\lambda)^{n+1} f^{(n+1)}(a) \right\| \\
&\quad \leq \frac{1}{n!} \|a-\lambda\|^{n+1} \int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) - f^{(n+1)}(a) \right\| (1-s)^n ds \\
&\quad \leq \frac{1}{n!} \|a-\lambda\|^{n+1} L_{\lambda,a,n} \int_0^1 \|(1-s)\lambda + sa - a\| (1-s)^n ds \\
&\quad = \frac{1}{n!} \|a-\lambda\|^{n+2} L_{\lambda,a,n} \int_0^1 (1-s)^{n+1} ds = \frac{1}{(n+2)\,n!} \|a-\lambda\|^{n+2} L_{\lambda,a,n},
\end{aligned}$$

which proves (3.7).

From (2.5) we have

$$\begin{aligned}
&\left\| f(a) - \sum_{k=0}^{n+1} \frac{1}{k!} f^{(k)}(\lambda)(a-\lambda)^k \right\| \\
&\quad \leq \frac{1}{n!} \|a-\lambda\|^{n+1} \int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) - f^{(n+1)}(\lambda) \right\| (1-s)^n ds \\
&\quad \leq \frac{1}{n!} \|a-\lambda\|^{n+1} L_{\lambda,a,n} \int_0^1 \|(1-s)\lambda + sa - \lambda\| (1-s)^n ds \\
&\quad = \frac{1}{n!} \|a-\lambda\|^{n+2} L_{\lambda,a,n} \int_0^1 s(1-s)^n ds = \frac{1}{n!} \|a-\lambda\|^{n+2} L_{\lambda,a,n} \int_0^1 (1-s) s^n ds \\
&\quad = \frac{1}{n!} \|a-\lambda\|^{n+2} L_{\lambda,a,n} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{1}{(n+2)!} \|a-\lambda\|^{n+2} L_{\lambda,a,n},
\end{aligned}$$

which proves (3.8).

Using (2.6) we have

$$\begin{aligned}
&\left\| f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\lambda)(a-\lambda)^k - \frac{1}{(n+1)!}(a-\lambda)^{n+1} f^{(n+1)}\left(\frac{\lambda+a}{2}\right) \right\| \\
&\quad \leq \frac{1}{n!} \|a-\lambda\|^{n+1} \int_0^1 \left\| f^{(n+1)}((1-s)\lambda + sa) - f^{(n+1)}\left(\frac{\lambda+a}{2}\right) \right\| (1-s)^n ds \\
&\quad \leq \frac{1}{n!} \|a-\lambda\|^{n+1} L_{\lambda,a,n} \int_0^1 \left\| (1-s)\lambda + sa - \frac{\lambda+a}{2} \right\| (1-s)^n ds \\
&\quad = \frac{1}{n!} \|a-\lambda\|^{n+2} L_{\lambda,a,n} \int_0^1 \left| s - \frac{1}{2} \right| (1-s)^n ds.
\end{aligned} \tag{3.11}$$

Now, observe that

$$\begin{aligned}
\int_0^1 \left|s-\frac{1}{2}\right|(1-s)^n\,ds &= \int_0^1 \left|s-\frac{1}{2}\right| s^n ds \\
&= \int_0^{1/2}\left(\frac{1}{2}-s\right)s^n ds + \int_{1/2}^1 \left(s-\frac{1}{2}\right)s^n ds \\
&= \frac{\left(\frac{1}{2}\right)^{n+1}}{n+1} - 2\frac{\left(\frac{1}{2}\right)^{n+2}}{n+2} + \frac{1}{n+2} - \frac{1}{2}\frac{1}{n+1} \\
&= \frac{1}{n+1}\left[\left(\frac{1}{2}\right)^{n+1} - \frac{1}{2}\right] + \frac{1}{n+2}\left[1-2\left(\frac{1}{2}\right)^{n+2}\right] \\
&= \frac{1}{2(n+1)}\left[\left(\frac{1}{2}\right)^{n} - 1\right] + \frac{1}{n+2}\left[1-\left(\frac{1}{2}\right)^{n+1}\right] \\
&= \frac{1}{n+2}\left[1-\left(\frac{1}{2}\right)^{n+1}\right] - \frac{1}{2(n+1)}\left[1-\left(\frac{1}{2}\right)^{n}\right] = K_n
\end{aligned}$$

and by (3.11) we get (3.9).

Using the representation (2.7) we have

$$\begin{aligned}
&\left\| f(a) - \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(\lambda)(a-\lambda)^k \right. \\
&\qquad \left. -\frac{1}{(n+1)!}(a-\lambda)^{n+1}\int_0^1 f^{(n+1)}((1-\tau)\lambda+\tau a)\,d\tau \right\| \\
&\le \frac{1}{n!}\|a-\lambda\|^{n+1} \\
&\quad\times \int_0^1 \left\| f^{(n+1)}((1-s)\lambda+sa) - \int_0^1 f^{(n+1)}((1-\tau)\lambda+\tau a)\,d\tau \right\| (1-s)^n ds \\
&= \frac{1}{n!}\|a-\lambda\|^{n+1} \\
&\quad\times \int_0^1 \left\| \int_0^1 \left[f^{(n+1)}((1-s)\lambda+sa) - f^{(n+1)}((1-\tau)\lambda+\tau a)\right] d\tau \right\| (1-s)^n ds \\
&\le \frac{1}{n!}\|a-\lambda\|^{n+1} \\
&\quad\times \int_0^1\int_0^1 \left\| f^{(n+1)}((1-s)\lambda+sa) - f^{(n+1)}((1-\tau)\lambda+\tau a)\right\| (1-s)^n dsd\tau \\
&\le \frac{1}{n!}\|a-\lambda\|^{n+1} L_{\lambda,a,n}\int_0^1\int_0^1 \|(1-s)\lambda+sa-((1-\tau)\lambda+\tau a)\|(1-s)^n dsd\tau \\
&= \frac{1}{n!}\|a-\lambda\|^{n+2} L_{\lambda,a,n}\int_0^1\int_0^1 |\tau-s|(1-s)^n dsd\tau =: B
\end{aligned} \tag{3.12}$$

Now, observe that

$$\begin{aligned}
&\int_0^1 \int_0^1 |\tau - s| (1-s)^n \, ds d\tau \\
&= \int_0^1 \left(\int_0^\tau (\tau - s)(1-s)^n \, ds \right) d\tau + \int_0^1 \int_\tau^1 (s-\tau)(1-s)^n \, ds d\tau = T_n
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^\tau (\tau - s)(1-s)^n \, ds &= -\frac{1}{n+1} \int_0^\tau (\tau - s) \, d\left[(1-s)^{n+1}\right] \\
&= -\frac{1}{n+1} \left[(\tau - s)(1-s)^{n+1} \Big|_0^\tau + \int_0^\tau (1-s)^{n+1} \, ds \right] \\
&= -\frac{1}{n+1} \left[-\tau - \left.\frac{(1-s)^{n+2}}{n+2} \right|_0^\tau \right] \\
&= -\frac{1}{n+1} \left[-\tau - \left.\frac{(1-\tau)^{n+2} - 1}{n+2} \right|_0^\tau \right] \\
&= -\frac{1}{n+1} \left[\frac{-(n+2)\tau - (1-\tau)^{n+2} + 1}{n+2} \right]
\end{aligned}$$

and

$$\begin{aligned}
\int_\tau^1 (s-\tau)(1-s)^n \, ds &= -\frac{1}{n+1} \int_\tau^1 (\tau - s) \, d\left[(1-s)^{n+1}\right] \\
&= -\frac{1}{n+1} \left[(\tau - s)(1-s)^{n+1} \Big|_\tau^1 + \int_\tau^1 (1-s)^{n+1} \, ds \right] \\
&= -\frac{1}{n+1} \left[- \left.\frac{(1-s)^{n+2}}{n+2} \right|_\tau^1 \right] = -\frac{1}{n+1} \frac{(1-\tau)^{n+2}}{n+2},
\end{aligned}$$

then

$$\begin{aligned}
&\int_0^\tau (\tau - s)(1-s)^n \, ds + \int_\tau^1 (s-\tau)(1-s)^n \, ds \\
&= -\frac{1}{n+1} \left[\frac{-(n+2)\tau - (1-\tau)^{n+2} + 1}{n+2} \right] - \frac{1}{n+1} \frac{(1-\tau)^{n+2}}{n+2} \\
&= \frac{1}{(n+1)(n+2)} \left[(n+2)\tau + (1-\tau)^{n+2} - 1 - (1-\tau)^{n+2} \right] \\
&= \frac{1}{(n+1)(n+2)} \left[(n+2)\tau - 1 \right],
\end{aligned}$$

which implies that

$$\begin{aligned}
T_n &= \frac{1}{(n+1)(n+2)} \int_0^1 \left[(n+2)\tau - 1 \right] dt = \frac{1}{(n+1)(n+2)} \left[\left(\frac{n+2}{2} - 1 \right) \right] \\
&= \frac{n}{2(n+1)(n+2)},
\end{aligned}$$

and by (3.12) we get the desired result (3.10). $\square$

4. Examples for the Exponential Function

Using the inequality (3.1) for the exponential function we get

$$\left\|\exp a-\exp\lambda\sum_{k=0}^{n}\frac{1}{k!}(a-\lambda)^{k}\right\| \tag{4.1}$$
$$\leq\frac{1}{n!}\|a-\lambda\|^{n+1}\int_{0}^{1}\|\exp((1-s)\lambda+sa)\|(1-s)^{n}ds$$
$$\leq\frac{1}{n!}\|a-\lambda\|^{n+1}\begin{cases}\frac{1}{n+1}\sup_{s\in[0,1]}\|\exp((1-s)\lambda+sa)\|\\ \\ \frac{1}{(n^{qn}+1)^{1/q}}\left(\int_0^1\|\exp((1-s)\lambda+sa)\|^{p}ds\right)^{1/p}\\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1\\ \\ \int_0^1\|\exp((1-s)\lambda+sa)\|ds,\end{cases}$$

for all $a\in\mathcal{B}$ and $\lambda\in\mathbb{C}$.

Observe that for all $a\in\mathcal{B}$ and $\lambda\in\mathbb{C}$ and $s\in[0,1]$

$$\exp((1-s)\lambda+sa)=\exp[(1-s)\lambda]\exp(sa),$$

which gives

$$\|\exp((1-s)\lambda+sa)\| \tag{4.2}$$
$$=|\exp[(1-s)\lambda]|\,\|\exp(sa)\|=\exp[(1-s)\operatorname{Re}\lambda]\,\|\exp(sa)\|$$
$$\leq\exp[(1-s)\operatorname{Re}\lambda]\exp(s\|a\|)=\exp[(1-s)\operatorname{Re}\lambda+s\|a\|].$$

Using the first inequality in (4.1) and (4.2) we get

$$\left\|\exp a-\exp\lambda\sum_{k=0}^{n}\frac{1}{k!}(a-\lambda)^{k}\right\| \tag{4.3}$$
$$\leq\frac{1}{n!}\|a-\lambda\|^{n+1}\int_{0}^{1}\exp[(1-s)\operatorname{Re}\lambda+s\|a\|](1-s)^{n}ds.$$

If we put

$$E_{n}(\lambda,a):=\int_{0}^{1}\exp[(1-s)\operatorname{Re}\lambda+s\|a\|](1-s)^{n}ds$$
$$=\int_{0}^{1}\exp[s\operatorname{Re}\lambda+(1-s)\|a\|]s^{n}ds=\int_{0}^{1}\exp[\|a\|+s(\operatorname{Re}\lambda-\|a\|)]s^{n}ds$$

then by using the integration by parts and assuming that $\operatorname{Re}\lambda \neq \|a\|$ we have

$$\begin{aligned}
&\int_0^1 \exp\left[\|a\| + s\left(\operatorname{Re}\lambda - \|a\|\right)\right] s^n ds \\
&= \frac{1}{\operatorname{Re}\lambda - \|a\|} \int_0^1 s^n d\left(\exp\left[\|a\| + s\left(\operatorname{Re}\lambda - \|a\|\right)\right]\right) \\
&= \frac{1}{\operatorname{Re}\lambda - \|a\|} \\
&\times \left[s^n \exp\left[\|a\| + s\left(\operatorname{Re}\lambda - \|a\|\right)\right]\Big|_0^1 - n \int_0^1 s^{n-1} \exp\left[\|a\| + s\left(\operatorname{Re}\lambda - \|a\|\right)\right] ds \right] \\
&= \frac{1}{\operatorname{Re}\lambda - \|a\|} \left[\exp\left(\operatorname{Re}\lambda\right) - n \int_0^1 s^{n-1} \exp\left[\|a\| + s\left(\operatorname{Re}\lambda - \|a\|\right)\right] ds \right],
\end{aligned}$$

which gives the recursive relation

$$E_n(\lambda, a) = \frac{1}{\operatorname{Re}\lambda - \|a\|} \left[\exp\left(\operatorname{Re}\lambda\right) - nE_{n-1}(\lambda, a)\right], \ n \geq 1 \tag{4.4}$$

with

$$E_0(\lambda, a) = \frac{\exp\left(\operatorname{Re}\lambda\right) - \exp\left(\|a\|\right)}{\operatorname{Re}\lambda - \|a\|}. \tag{4.5}$$

If $\operatorname{Re}\lambda = \|a\|$, then

$$E_n(\lambda, a) = \frac{1}{n+1} \exp\|a\|.$$

Therefore, for any $a \in \mathcal{B}$ and $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
&\left\| \exp a - \exp\lambda \sum_{k=0}^{n} \frac{1}{k!} (a - \lambda)^k \right\| \\
&\leq \frac{1}{n!} \|a - \lambda\|^{n+1} \begin{cases} E_n(\lambda, a) & \text{if } \operatorname{Re}\lambda \neq \|a\|, \\ \\ \frac{1}{n+1}\exp\|a\| & \text{if } \operatorname{Re}\lambda = \|a\|, \end{cases}
\end{aligned} \tag{4.6}$$

where $E_n(\lambda, a)$ is defined by (4.4) and (4.5).

Since

$$\sup_{s\in[0,1]} \exp\left[(1-s)\operatorname{Re}\lambda + s\|a\|\right] \leq \exp\left[\max\left\{\operatorname{Re}\lambda, \|a\|\right\}\right],$$

hence by the first branch in the second inequality in (4.1) we get for $n \geq 0$ that

$$\left\| \exp a - \exp\lambda \sum_{k=0}^{n} \frac{1}{k!} (a - \lambda)^k \right\| \leq \frac{1}{(n+1)!} \|a - \lambda\|^{n+1} \exp\left[\max\left\{\operatorname{Re}\lambda, \|a\|\right\}\right] \tag{4.7}$$

for any $a \in \mathcal{B}$ and $\lambda \in \mathbb{C}$.

Similar inequalities can be also stated by employing the other general bounds established above for analytic functions. The details are omitted.

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