NORM INEQUALITIES OF OSTROWSKI TYPE FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t \in [0,1]$ from $\lambda(0) = \alpha$ to $\lambda(1) = \beta$, with $\beta \neq \alpha$. If $f : G \to \mathbb{C}$ is analytic on G, then by using the analytic functional calculus we obtain among others the following result

$$\left\| f(a) - \frac{1}{\beta - \alpha} \int_\gamma f(\lambda)\, d\lambda \right\| \leq K \frac{1}{|\beta - \alpha|} \int_\gamma \|a - \lambda\| \, |d\lambda| ,$$

provided

$$K := \sup_{(\lambda,t) \in \gamma \times [0,1]} \left\| f'((1-t)\lambda + ta) \right\| < \infty.$$

1. Introduction

In 1938, A. Ostrowski [12], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1 (Ostrowski, 1938 [12]). *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

For a recent survey on Ostrowski's inequality for scalar functions and Lebesgue integral see [7].

In order to extend Ostrowski's inequality for function defined on Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \leq \|a\| \|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*. We assume that the Banach algebra is *unital*, this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of

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a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\text{Inv}(\mathcal{B})$. If $a, b \in \text{Inv}(\mathcal{B})$ then $ab \in \text{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \text{Inv}(\mathcal{B})$;
(ii) $\{a \in \mathcal{B}\colon \ \|1 - b\| < 1\} \subset \text{Inv}(\mathcal{B})$;
(iii) $\text{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\text{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \text{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} : \ z - a \in \text{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \text{Inv}(\mathcal{B})$, $R_a(z) := (z - a)^{-1}$. For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z - w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \le \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_\delta f(\xi)(\xi - a)^{-1} d\xi, \tag{1.2}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \text{ins}(\delta)$, the inside of δ.

It is well known (see for instance [3, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.3}$$

holds.

Let $\mathfrak{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\mathfrak{Hol}(a)$ is an algebra where if $f, g \in \mathfrak{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f + g$ have domain $D(f) \cap D(g)$. $\mathfrak{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [3, p. 201-203]:

Theorem 2. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) *The map $f \mapsto f(a)$ of $\mathfrak{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.*

(b) *If* $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ *has radius of convergence* $r > \nu(a)$, *then* $f \in \mathfrak{Hol}(a)$ *and* $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.
(c) *If* $f(z) \equiv 1$, *then* $f(a) = 1$.
(d) *If* $f(z) = z$ *for all* z, $f(a) = a$.
(e) *If* $f, f_1, ..., f_n ...$ *are analytic on* G, $\sigma(a) \subset G$ *and* $f_n(z) \to f(z)$ *uniformly on compact subsets of* G, *then* $\|f_n(a) - f(a)\| \to 0$ *as* $n \to \infty$.
(f) *The Riesz Functional Calculus is unique and if* a, b *are commuting elements in* $\mathcal{B}$ *and* $f \in \mathfrak{Hol}(a)$, *then* $f(a)b = bf(a)$.

For some recent norm inequalities for functions on Banach algebras, see [1]-[2] and [5]-[11].

In what follows we establish some simple approximations for the element $f(a)$ with integral remainders where, as above, $a \in \mathcal{B}$, G is a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and f is analytic on G. Norm estimates of these remainders in terms of p-norms and Lipschitz constants are also provided. Several Ostrowski and perturbed Ostrowski norm inequalities are given as well.

2. Some Identities

We have:

Theorem 3. *Let* $\mathcal{B}$ *be a unital Banach algebra,* $a \in \mathcal{B}$ *and* G *be a convex domain of* $\mathbb{C}$ *with* $\sigma(a) \subset G$. *If* $f : G \to \mathbb{C}$ *is analytic on* G, *then for all* $\lambda \in G$ *we have*

$$f(a) = f(\lambda) + (a - \lambda) \int_0^1 f'((1-t)\lambda + ta)\, dt. \tag{2.1}$$

Moreover, for any $b \in \mathcal{B}$ *we have the perturbed identity*

$$f(a) = f(\lambda) + (a - \lambda) b + (a - \lambda) \int_0^1 [f'((1-t)\lambda + ta) - b]\, dt. \tag{2.2}$$

Proof. Since $f : G \to \mathbb{C}$ is analytic on G, which is convex, then for all $\lambda, \xi \in G$ we have

$$(f((1-t)\lambda + t\xi))' = (\xi - \lambda) f'((1-t)\lambda + t\xi).$$

Therefore

$$f(\xi) - f(\lambda) = \int_0^1 (f((1-t)\lambda + t\xi))'\, dt = (\xi - \lambda) \int_0^1 f'((1-t)\lambda + t\xi)\, dt,$$

giving that

$$f(\xi) = f(\lambda) + (\xi - \lambda) \int_0^1 f'((1-t)\lambda + t\xi)\, dt$$

for all $\xi \in G$.

From (1.2) we get

$$
\begin{aligned}
f(a) &= \frac{1}{2\pi i}\int_\delta f(\xi)(\xi-a)^{-1}\,d\xi \\
&= \frac{1}{2\pi i}\int_\delta \left[f(\lambda)+(\xi-\lambda)\int_0^1 f'((1-t)\lambda+t\xi)\,dt\right](\xi-a)^{-1}\,d\xi \\
&= \frac{1}{2\pi i}\int_\delta f(\lambda)(\xi-a)^{-1}\,d\xi \\
&\quad + \frac{1}{2\pi i}\int_\delta (\xi-\lambda)\left(\int_0^1 f'((1-t)\lambda+t\xi)\,dt\right)(\xi-a)^{-1}\,d\xi \\
&= f(\lambda)+\int_0^1\left(\frac{1}{2\pi i}\int_\delta (\xi-\lambda) f'((1-t)\lambda+t\xi)(\xi-a)^{-1}\,d\xi\right)dt,
\end{aligned} \tag{2.3}
$$

where for the last equality we used Fubini's theorem.

Since the function $g : G \to \mathbb{C}$

$$g(\xi) := (\xi-\lambda) f'((1-t)\lambda+t\xi)$$

is analytic for each $t\in[0,1]$ and $\lambda\in G$, then

$$
\begin{aligned}
g(a) &= (a-\lambda) f'((1-t)\lambda+ta) \\
&= \frac{1}{2\pi i}\int_\delta (\xi-\lambda) f'((1-t)\lambda+t\xi)(\xi-a)^{-1}\,d\xi
\end{aligned}
$$

for all $t\in[0,1]$ and $\lambda\in G$.

Then

$$
\begin{aligned}
&\int_0^1\left(\frac{1}{2\pi i}\int_\delta (\xi-\lambda) f'((1-t)\lambda+t\xi)(\xi-a)^{-1}\,d\xi\right)dt \\
&\quad = \int_0^1 (a-\lambda) f'((1-t)\lambda+ta)\,dt = (a-\lambda)\int_0^1 f'((1-t)\lambda+ta)\,dt
\end{aligned}
$$

and by (2.3) we get the desired result (2.1). □

Remark 1. *With the assumptions of Theorem 3 we have, by taking various values for* b*, the following identities of interest*

$$f(a) = f(\lambda)+(a-\lambda)f'(a)+(a-\lambda)\int_0^1 \left[f'((1-t)\lambda+ta)-f'(a)\right]dt, \tag{2.4}$$

$$f(a) = f(\lambda)+f'(\lambda)(a-\lambda)+(a-\lambda)\int_0^1 \left[f'((1-t)\lambda+ta)-f'(\lambda)\right]dt, \tag{2.5}$$

$$
\begin{aligned}
f(a) &= f(\lambda)+(a-\lambda)\frac{f'(a)+f'(\lambda)}{2} \\
&\quad + (a-\lambda)\int_0^1\left[f'((1-t)\lambda+ta)-\frac{f'(a)+f'(\lambda)}{2}\right]dt,
\end{aligned} \tag{2.6}
$$

$$f(a)=f(\lambda)+(a-\lambda)f'\left(\frac{a+\lambda}{2}\right)+(a-\lambda)\int_0^1\left[f'((1-t)\lambda+ta)-f'\left(\frac{a+\lambda}{2}\right)\right]dt, \tag{2.7}$$

and

$$f(a)=f(\lambda)+(a-\lambda)\int_0^1 f'((1-s)\lambda+sa)\,ds+(a-\lambda)\int_0^1\left[f'((1-t)\lambda+ta)-\int_0^1 f'((1-s)\lambda+sa)\,ds\right]dt. \tag{2.8}$$

Corollary 1. *With the assumptions of Theorem 3 and if $\gamma\subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t\in[0,1]$ from $\lambda(0)=\alpha$ to $\lambda(1)=\beta$, with $\beta\neq\alpha$, then*

$$\begin{aligned}f(a)-\frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda&=\frac{1}{\beta-\alpha}\int_\gamma(a-\lambda)\left(\int_0^1 f'((1-t)\lambda+ta)\,dt\right)d\lambda\\&=\frac{1}{\beta-\alpha}\int_0^1\left(\int_\gamma(a-\lambda)f'((1-t)\lambda+ta)\,d\lambda\right)dt.\end{aligned}\tag{2.9}$$

Moreover, for any $b\in\mathcal{B}$ we have the perturbed identity

$$\begin{aligned}f(a)-\frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda-\left(a-\frac{\alpha+\beta}{2}\right)b&=\frac{1}{\beta-\alpha}\int_\gamma(a-\lambda)\left(\int_0^1[f'((1-t)\lambda+ta)-b]\,dt\right)d\lambda\\&=\frac{1}{\beta-\alpha}\int_0^1\left(\int_\gamma(a-\lambda)[f'((1-t)\lambda+ta)-b]\,d\lambda\right)dt.\end{aligned}\tag{2.10}$$

Remark 2. *With the assumptions of Corollary 1 we have,*

$$\begin{aligned}f(a)-\frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda-\left(a-\frac{\alpha+\beta}{2}\right)f'(a)&=\frac{1}{\beta-\alpha}\int_\gamma(a-\lambda)\left(\int_0^1[f'((1-t)\lambda+ta)-f'(a)]\,dt\right)d\lambda\\&=\frac{1}{\beta-\alpha}\int_0^1\left(\int_\gamma(a-\lambda)[f'((1-t)\lambda+ta)-f'(a)]\,d\lambda\right)dt.\end{aligned}\tag{2.11}$$

3. Norm Inequalities

Let $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$ and $\lambda\in G$. We define $G_{\lambda,a}:=\{(1-t)\lambda+ta \mid \text{with } t\in[0,1]\}$. We observe that $G_{\lambda,a}$ is a convex subset in $\mathcal{B}$ for every $\lambda\in G$.

For two distinct elements u, v in the Banach algebra B we say that the function $g:G_{\lambda,a}\to\mathcal{B}$ belongs to the class $\Delta_{u,v}(G_{\lambda,a})$ if it satisfies the boundedness condition

$$\left\| g\left((1-t)\lambda + ta\right) - \frac{u+v}{2} \right\| \leq \frac{1}{2}\left\| v-u\right\| \tag{3.1}$$

for all $t \in [0,1]$. We write $g \in \Delta_{u,v}(G_{\lambda,a})$. This definition is an extension to Banach algebras valued functions of the scalar case, see [4].

We say that the function $g : G_{\lambda,a} \to B$ is Lipschitzian on $G_{\lambda,a}$ with the constant $L_{\lambda,a} > 0$, if for all x, $y \in G_{\lambda,a}$ we have

$$\left\| g(x) - g(y)\right\| \leq L_{\lambda,a}\left\| x-y\right\| .$$

This is equivalent to

$$\left\| g\left((1-t)\lambda + ta\right) - g\left((1-s)\lambda + sa\right) \right\| \leq L_{\lambda,a}\left| t-s\right| \left\| a-\lambda\right\| \tag{3.2}$$

for all t, $s \in [0,1]$. We write this by $g \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$.

Let $h : G \to \mathbb{C}$ be an analytic function on G. For $t \in [0,1]$ and $\lambda \in G$, the auxiliary function $h_{t,\lambda}$ defined on G by $h_{t,\lambda}(\xi) := h((1-t)\lambda + t\xi)$ is also analytic and using the analytic functional calculus (1.2) for the element $a \in \mathcal{B}$, we can define

$$\begin{aligned} \widetilde{h}\left((1-t)\lambda + ta\right) &:= h_{t,\lambda}(a) = \frac{1}{2\pi i}\int_{\gamma} h_{t,\lambda}(\xi)(\xi - a)^{-1} d\xi \\ &= \frac{1}{2\pi i}\int_{\gamma} h\left((1-t)\lambda + t\xi\right) (\xi - a)^{-1} d\xi. \end{aligned} \tag{3.3}$$

We say that the scalar function $h \in \Delta_{u,v}(G_{\lambda,a})$ if its extension $\widetilde{h} : G_{\lambda,a} \to B$ satisfies the boundedness condition (3.1). Also, we say that the scalar function $h \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$ if its extension $\widetilde{h} : G_{\lambda,a} \to B$ satisfies the Lipschitz condition (3.2).

From (2.1) we have the following fundamental inequalities

$$\begin{aligned} \left\| f(a) - f(\lambda)\right\| &\leq \left\| a-\lambda\right\| \left\| \int_0^1 f'\left((1-t)\lambda + ta\right) dt\right\| \\ &\leq \left\| a-\lambda\right\| \int_0^1 \left\| f'\left((1-t)\lambda + ta\right) \right\| dt \leq \left\| a-\lambda\right\| \sup_{t\in[0,1]} \left\| f'\left((1-t)\lambda + ta\right) \right\| , \end{aligned} \tag{3.4}$$

provided that $f : G \to \mathbb{C}$ is analytic on G and $\lambda \in G$.

We have:

Theorem 4. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. Assume also that $f : G \to \mathbb{C}$ is analytic on G and $\lambda \in G$. If there exists u, $v \in \mathcal{B}$ with $u \neq v$ such that $f' \in \Delta_{u,v}(G_{\lambda,a})$, then*

$$\left\| f(a) - f(\lambda) - (a-\lambda)\frac{u+v}{2}\right\| \leq \frac{1}{2}\left\| a-\lambda\right\| \left\| v-u\right\| . \tag{3.5}$$

Proof. Since $f' \in \Delta_{u,v}(G_{\lambda,a})$, then from (2.2) we have

$$\begin{aligned}\left\| f(a) - f(\lambda) - (a-\lambda)\frac{u+v}{2} \right\| &= \left\| (a-\lambda)\int_0^1 \left[f'((1-t)\lambda + ta) - \frac{u+v}{2} \right] dt \right\| \\ &\leq \|a-\lambda\| \left\| \int_0^1 \left[f'((1-t)\lambda + ta) - \frac{u+v}{2} \right] dt \right\| \\ &\leq \|a-\lambda\| \int_0^1 \left\| f'((1-t)\lambda + ta) - \frac{u+v}{2} \right\| dt \leq \frac{1}{2}\|a-\lambda\| \|v-u\|,\end{aligned}$$

which gives (3.5). □

We also have:

Theorem 5. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. Assume also that $f : G \to \mathbb{C}$ is analytic on G and $\lambda \in G$. If $f' \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$, then*

$$\|f(a) - f(\lambda) - (a-\lambda) f'(a)\| \leq \frac{1}{2}\|a-\lambda\|^2 L_{\lambda,a}, \tag{3.6}$$

$$\|f(a) - f(\lambda) - f'(\lambda)(a-\lambda)\| \leq \frac{1}{2}\|a-\lambda\|^2 L_{\lambda,a}, \tag{3.7}$$

$$\left\| f(a) - f(\lambda) - (a-\lambda) f'\left(\frac{a+\lambda}{2}\right) \right\| \leq \frac{1}{4}\|a-\lambda\|^2 L_{\lambda,a} \tag{3.8}$$

and

$$\left\| f(a) - f(\lambda) - (a-\lambda)\int_0^1 f'((1-s)\lambda + sa)\, ds \right\| \leq \frac{1}{3}\|a-\lambda\|^2 L_{\lambda,a}. \tag{3.9}$$

Proof. From the (2.4) and since $f' \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$, hence we have

$$\begin{aligned}&\|f(a) - f(\lambda) - (a-\lambda) f'(a)\| \\ &\qquad \leq \|a-\lambda\| \left\| \int_0^1 [f'((1-t)\lambda + ta) - f'(a)]\, dt \right\| \\ &\qquad \leq \|a-\lambda\| \int_0^1 \|f'((1-t)\lambda + ta) - f'(a)\|\, dt \\ &\qquad \leq \|a-\lambda\| L_{\lambda,a} \|a-\lambda\| \int_0^1 (1-t)\, dt = \frac{1}{2}\|a-\lambda\|^2 L_{\lambda,a},\end{aligned}$$

and the inequality (3.6) is obtained.

The inequality follows similarly from (2.5).

From (2.7) we have

$$
\begin{aligned}
(3.10)\quad & \left\| f(a) - f(\lambda) - (a-\lambda) f'\left(\frac{a+\lambda}{2}\right) \right\| \\
& \le \|a-\lambda\| \left\| \int_0^1 \left[f'((1-t)\lambda + ta) - f'\left(\frac{a+\lambda}{2}\right) \right] dt \right\| \\
& \le \|a-\lambda\| \int_0^1 \left\| f'((1-t)\lambda + ta) - f'\left(\frac{a+\lambda}{2}\right) \right\| dt \\
& \le \|a-\lambda\| L_{\lambda,a} \|a-\lambda\| \int_0^1 \left| t - \frac{1}{2} \right| dt = \frac{1}{4} \|a-\lambda\|^2 L_{\lambda,a},
\end{aligned}
$$

and the inequality (3.8) is proved.

From the identity (2.8) we also have

$$
\begin{aligned}
(3.11)\quad & \left\| f(a) - f(\lambda) - (a-\lambda) \int_0^1 f'((1-s)\lambda + sa)\, ds \right\| \\
& \le \|a-\lambda\| \left\| \int_0^1 \left[f'((1-t)\lambda + ta) - \int_0^1 f'((1-s)\lambda + sa)\, ds \right] dt \right\| \\
& = \|a-\lambda\| \left\| \int_0^1 \int_0^1 \left[f'((1-t)\lambda + ta) - f'((1-s)\lambda + sa) \right] dsdt \right\| \\
& \le \|a-\lambda\| \int_0^1 \int_0^1 \left\| f'((1-t)\lambda + ta) - f'((1-s)\lambda + sa) \right\| dsdt \\
& \le \|a-\lambda\| L_{\lambda,a} \|a-\lambda\| \int_0^1 \int_0^1 |t-s|\, dsdt.
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_0^1 \int_0^1 |t-s|\, dsdt &= \int_0^1 \left[\int_0^t (t-s)\, ds + \int_t^1 (s-t)\, ds \right] dt \\
&= \int_0^1 \frac{t^2 + (1-t)^2}{2} dt = \frac{1}{3},
\end{aligned}
$$

hence by (3.11) we get (3.9). □

4. Ostrowski Type Inequalities

We have the following Ostrowski type inequalities:

Theorem 6. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t \in [0,1]$ from $\lambda(0) = \alpha$ to $\lambda(1) = \beta$, with $\beta \neq \alpha$. If $f : G \to \mathbb{C}$ is analytic on G,* *then*

$$\left\| f(a) - \frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda \right\| \tag{4.1}$$
$$\leq \frac{1}{|\beta-\alpha|}\int_\gamma \|a-\lambda\| \left\| \int_0^1 f'((1-t)\lambda+ta)\,dt \right\| |d\lambda|$$
$$\leq \frac{1}{|\beta-\alpha|}\begin{cases} \sup_{\lambda\in\gamma}\|a-\lambda\| \int_\gamma \left\|\int_0^1 f'((1-t)\lambda+ta)\,dt\right\| |d\lambda|, \\ \\ \left(\int_\gamma \|a-\lambda\|^p |d\lambda|\right)^{1/p} \left(\int_\gamma \left\|\int_0^1 f'((1-t)\lambda+ta)\,dt\right\|^q |d\lambda|\right)^{1/q} \\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_\gamma \|a-\lambda\|\,|d\lambda| \sup_{\lambda\in\gamma}\left\|\int_0^1 f'((1-t)\lambda+ta)\,dt\right\| \end{cases}$$

and

$$\left\| f(a) - \frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda \right\| \tag{4.2}$$
$$\leq \frac{1}{|\beta-\alpha|}\int_\gamma \|a-\lambda\| \left\| \int_0^1 f'((1-t)\lambda+ta)\,dt \right\| |d\lambda|$$
$$\leq \frac{1}{|\beta-\alpha|}\int_\gamma\int_0^1 \|a-\lambda\| \|f'((1-t)\lambda+ta)\|\,dt\,|d\lambda|$$
$$\leq \frac{1}{|\beta-\alpha|}\begin{cases} \sup_{\lambda\in\gamma}\|a-\lambda\| \int_\gamma\int_0^1 \|f'((1-t)\lambda+ta)\|\,dt\,|d\lambda|, \\ \\ \left(\int_\gamma \|a-\lambda\|^p |d\lambda|\right)^{1/p} \left(\int_\gamma\int_0^1 \|f'((1-t)\lambda+ta)\|^q\,dt\,|d\lambda|\right)^{1/q} \\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_\gamma \|a-\lambda\|\,|d\lambda| \sup_{\lambda\in\gamma,t\in[0,1]}\|f'((1-t)\lambda+ta)\|. \end{cases}$$

Proof. From the first equality in (2.9) we have

$$\left\| f(a) - \frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda \right\| \tag{4.3}$$
$$= \left\| \frac{1}{\beta-\alpha}\int_\gamma (a-\lambda)\left(\int_0^1 f'((1-t)\lambda+ta)\,dt\right) d\lambda \right\|$$
$$\leq \frac{1}{|\beta-\alpha|}\int_\gamma \left\|(a-\lambda)\left(\int_0^1 f'((1-t)\lambda+ta)\,dt\right)\right\| |d\lambda|$$
$$\leq \frac{1}{|\beta-\alpha|}\int_\gamma \|a-\lambda\| \left\| \int_0^1 f'((1-t)\lambda+ta)\,dt \right\| |d\lambda|.$$

If we use Hölder's integral inequality for the integral $\int_\gamma$ we have

$$\int_\gamma \|a-\lambda\| \left\| \int_0^1 f'((1-t)\lambda + ta)\,dt \right\| |d\lambda|$$
$$\leq \begin{cases} \sup_{\lambda\in\gamma} \|a-\lambda\| \int_\gamma \left\| \int_0^1 f'((1-t)\lambda+ta)\,dt \right\| |d\lambda| \\ \\ \left(\int_\gamma \|a-\lambda\|^p |d\lambda|\right)^{1/p} \left(\int_\gamma \left\| \int_0^1 f'((1-t)\lambda+ta)\,dt\right\|^q |d\lambda|\right)^{1/q} \\ \text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_\gamma \|a-\lambda\| |d\lambda| \sup_{\lambda\in\gamma} \left\| \int_0^1 f'((1-t)\lambda+ta)\,dt \right\|, \end{cases}$$

which together with (4.3) gives (4.1).

The first two inequalities in (4.2) are obvious from above. The rest follows by Hölder's inequality for the double integral $\int_\gamma \int_0^1$ and we omit the details. □

Remark 3. *Using the second identity in (2.9) we get*

$$\left\| f(a) - \frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda \right\| \tag{4.4}$$
$$\leq \frac{1}{|\beta-\alpha|}\int_0^1 \left\| \int_\gamma (a-\lambda) f'((1-t)\lambda+ta)\,d\lambda \right\| dt$$
$$\leq \frac{1}{|\beta-\alpha|}\int_0^1 \left(\int_\gamma \|(a-\lambda) f'((1-t)\lambda+ta)\| |d\lambda| \right) dt$$
$$\leq \frac{1}{|\beta-\alpha|}\int_0^1 \left(\int_\gamma \|a-\lambda\| \|f'((1-t)\lambda+ta)\| |d\lambda| \right) dt.$$

By Hölder's inequality for $\int_\gamma$ we have

$$\int_\gamma \|a-\lambda\| \|f'((1-t)\lambda+ta)\| |d\lambda|$$
$$\leq \begin{cases} \sup_{\lambda\in\gamma} \|a-\lambda\| \int_\gamma \|f'((1-t)\lambda+ta)\| |d\lambda|, \\ \\ \left(\int_\gamma \|a-\lambda\|^p |d\lambda|\right)^{1/p} \left(\int_\gamma \|f'((1-t)\lambda+ta)\|^q |d\lambda|\right)^{1/q} \\ \textit{where } p,\ q>1 \textit{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_\gamma \|a-\lambda\| |d\lambda| \sup_{\lambda\in\gamma} \|f'((1-t)\lambda+ta)\|, \end{cases}$$

which implies that

$$\frac{1}{|\beta-\alpha|}\int_0^1\left(\int_\gamma \|a-\lambda\|\,\|f'((1-t)\lambda+ta)\|\,|d\lambda|\right)dt \tag{4.5}$$

$$\leq \frac{1}{|\beta-\alpha|}\begin{cases}\sup_{\lambda\in\gamma}\|a-\lambda\|\int_0^1\int_\gamma\|f'((1-t)\lambda+ta)\|\,|d\lambda|\,dt,\\ \\ \left(\int_\gamma\|a-\lambda\|^p|d\lambda|\right)^{1/p}\int_0^1\left(\int_\gamma\|f'((1-t)\lambda+ta)\|^q|d\lambda|\right)^{1/q}dt\\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \int_\gamma\|a-\lambda\|\,|d\lambda|\int_0^1\sup_{\lambda\in\gamma}\|f'((1-t)\lambda+ta)\|\,dt.\end{cases}$$

Therefore, by (4.4) and (4.5) we get the bounds

$$\left\|f(a)-\frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda\right\| \tag{4.6}$$

$$\leq \frac{1}{|\beta-\alpha|}\begin{cases}\sup_{\lambda\in\gamma}\|a-\lambda\|\int_0^1\int_\gamma\|f'((1-t)\lambda+ta)\|\,|d\lambda|\,dt,\\ \\ \left(\int_\gamma\|a-\lambda\|^p|d\lambda|\right)^{1/p}\int_0^1\left(\int_\gamma\|f'((1-t)\lambda+ta)\|^q|d\lambda|\right)^{1/q}dt\\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \int_\gamma\|a-\lambda\|\,|d\lambda|\int_0^1\sup_{\lambda\in\gamma}\|f'((1-t)\lambda+ta)\|\,dt.\end{cases}$$

Corollary 2. *With the assumptions of Theorem 2 and if*

$$\|f'\|_{a,\gamma\times[0,1],\infty}:=\sup_{(\lambda,t)\in\gamma\times[0,1]}\|f'((1-t)\lambda+ta)\|<\infty,$$

then from (4.2) we get

$$\left\|f(a)-\frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda\right\|\leq\frac{1}{|\beta-\alpha|}\|f'\|_{a,\gamma\times[0,1],\infty}\int_\gamma\|a-\lambda\|\,|d\lambda|. \tag{4.7}$$

5. Perturbed Ostrowski Type Inequalities

Let $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$ and $\gamma\subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t\in[0,1]$ from $\lambda(0)=\alpha$ to $\lambda(1)=\beta$, with $\beta\neq\alpha$. We define the following subset of the Banach algebra $\mathcal{B}$

$$G_{\gamma,a}:=\cup_{\lambda\in\gamma}G_{\lambda,a}=\cup_{\lambda\in\gamma}\{(1-t)\lambda+ta\mid\text{with }t\in[0,1]\}.$$

For two distinct elements u, v in the Banach algebra B we say that the function $g:G_{\gamma,a}\to\mathcal{B}$ belongs to the class $\Delta_{u,v}(G_{\gamma,a})$ if it satisfies the boundedness condition

$$\left\|g((1-t)\lambda+ta)-\frac{u+v}{2}\right\|\leq\frac{1}{2}\|v-u\| \tag{5.1}$$

for all $t\in[0,1]$ and $\lambda\in\gamma$. We write $g\in\Delta_{u,v}(G_{\gamma,a})$.

We say that the function $g:G_{\gamma,a}\to B$ is Lipschitzian on $G_{\gamma,a}$ with the constant $L_{\gamma,a}>0$, if for all x, $y\in G_{\gamma,a}$ we have

$$\|g(x)-g(y)\|\leq L_{\gamma,a}\|x-y\|.$$

This is equivalent to

$$\left\| g\left(\left(1-t\right) \lambda +ta\right) -g\left(\left(1-s\right) \lambda +sa\right) \right\| \leq L_{\gamma ,a}\left\vert t-s\right\vert \left\Vert a-\lambda \right\Vert \tag{5.2}$$

for all t, $s\in [0,1]$ and $\lambda \in \gamma$. We write this by $g\in \mathfrak{Lip}_{L_{\gamma ,a}}\left(G_{\gamma ,a}\right)$.

We say that the scalar function $h\in \Delta _{u,v}\left(G_{\gamma ,a}\right)$ if its extension $\widetilde{h}:G_{\gamma ,a}\rightarrow B$ defined by (3.3) satisfies the boundedness condition (5.1). Also, we say that the scalar function $h\in \mathfrak{Lip}_{L_{\gamma ,a}}\left(G_{\gamma ,a}\right)$ if its extension $\widetilde{h}:G_{\gamma ,a}\rightarrow B$ satisfies the Lipschitz condition (5.2).

Theorem 7. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma \left(a\right) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda \left(t\right)$, $t\in [0,1]$ from $\lambda \left(0\right) =\alpha$ to $\lambda \left(1\right) =\beta$, with $\beta \neq \alpha$. If $f:G\rightarrow \mathbb{C}$ is analytic on G and there exists u, $v\in \mathcal{B}$ with $u\neq v$ such that $f^{\prime }\in \Delta _{u,v}\left(G_{\gamma ,a}\right)$, then*

$$\begin{aligned} &\left\Vert f\left(a\right) -\frac{1}{\beta -\alpha }\int_{\gamma }f\left(\lambda \right) d\lambda -\left(a-\frac{\alpha +\beta }{2}\right) \frac{u+v}{2}\right\Vert \\ &\qquad \leq \frac{1}{2}\frac{\left\Vert v-u\right\Vert }{\left\vert \beta -\alpha \right\vert }\int_{\gamma }\left\Vert a-\lambda \right\Vert \left\vert d\lambda \right\vert . \end{aligned} \tag{5.3}$$

Proof. Using the identity (2.10) and taking into account that $f^{\prime }\in \Delta _{u,v}\left(G_{\gamma ,a}\right)$, we have

$$\begin{aligned} &\left\Vert f\left(a\right) -\frac{1}{\beta -\alpha }\int_{\gamma }f\left(\lambda \right) d\lambda -\left(a-\frac{\alpha +\beta }{2}\right) \frac{u+v}{2}\right\Vert \\ &\quad \leq \frac{1}{\left\vert \beta -\alpha \right\vert }\int_{\gamma }\left\Vert \left(a-\lambda \right) \left(\int_{0}^{1}\left[f^{\prime }\left(\left(1-t\right) \lambda +ta\right) -\frac{u+v}{2}\right] dt\right) \right\Vert \left\vert d\lambda \right\vert \\ &\quad \leq \frac{1}{\left\vert \beta -\alpha \right\vert }\int_{\gamma }\left\Vert a-\lambda \right\Vert \left(\int_{0}^{1}\left\Vert f^{\prime }\left(\left(1-t\right) \lambda +ta\right) -\frac{u+v}{2}\right\Vert dt\right) \left\vert d\lambda \right\vert \\ &\qquad \leq \frac{1}{2}\frac{\left\Vert v-u\right\Vert }{\left\vert \beta -\alpha \right\vert }\int_{\gamma }\left\Vert a-\lambda \right\Vert \left\vert d\lambda \right\vert , \end{aligned}$$

which proves the desired result (5.3). □

We also have:

Theorem 8. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma \left(a\right) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda \left(t\right)$, $t\in [0,1]$ from $\lambda \left(0\right) =\alpha$ to $\lambda \left(1\right) =\beta$, with $\beta \neq \alpha$. If $f:G\rightarrow \mathbb{C}$ is analytic on G and there exists $L_{\gamma ,a}>0$ so that $f^{\prime }\in \mathfrak{Lip}_{L_{\gamma ,a}}\left(G_{\gamma ,a}\right)$, then*

$$\begin{aligned} &\left\Vert f\left(a\right) -\frac{1}{\beta -\alpha }\int_{\gamma }f\left(\lambda \right) d\lambda -\left(a-\frac{\alpha +\beta }{2}\right) f^{\prime }\left(a\right) \right\Vert \\ &\qquad \leq \frac{L_{\gamma ,a}}{2\left\vert \beta -\alpha \right\vert }\int_{\gamma }\left\Vert a-\lambda \right\Vert ^{2}\left\vert d\lambda \right\vert . \end{aligned} \tag{5.4}$$

Proof. From the identity (2.11) and since $f' \in \mathfrak{Lip}_{L_{\gamma,a}}(G_{\gamma,a})$, hence

$$\begin{aligned}
&\left\| f(a) - \frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda - \left(a - \frac{\alpha+\beta}{2}\right) f'(a) \right\| \\
&\quad \le \frac{1}{|\beta-\alpha|}\int_\gamma \left\| (a-\lambda)\left(\int_0^1 \left[f'((1-t)\lambda + ta) - f'(a)\right] dt\right) \right\| |d\lambda| \\
&\quad \le \frac{1}{|\beta-\alpha|}\int_\gamma \|a-\lambda\| \left\| \int_0^1 \left[f'((1-t)\lambda + ta) - f'(a)\right] dt \right\| |d\lambda| \\
&\quad \le \frac{1}{|\beta-\alpha|}\int_\gamma \|a-\lambda\| \left(\int_0^1 \left\|f'((1-t)\lambda + ta) - f'(a)\right\| dt \right) |d\lambda| \\
&\quad \le L_{\gamma,a}\frac{1}{|\beta-\alpha|}\int_\gamma \|a-\lambda\|^2 \left(\int_0^1 (1-t)\,dt\right) |d\lambda| = \frac{L_{\gamma,a}}{2|\beta-\alpha|}\int_\gamma \|a-\lambda\|^2 |d\lambda|,
\end{aligned}$$

which proves the desired result (5.4). □

We also have:

Theorem 9. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t \in [0,1]$ from $\lambda(0) = \alpha$ to $\lambda(1) = \beta$, with $\beta \neq \alpha$. If $f : G \to \mathbb{C}$ is analytic on G and there exists $L_{\gamma,a} > 0$ so that $f' \in \mathfrak{Lip}_{L_{\gamma,a}}(G_{\gamma,a})$, then*

$$\begin{aligned}
(5.5)\quad &\left\| \frac{1}{2}\left[f(a) + \frac{f(\beta)(\beta-a) + f(\alpha)(a-\alpha)}{\beta-\alpha}\right] - \frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda \right\| \\
&\qquad\qquad \le \frac{L_{\gamma,a}}{4|\beta-\alpha|}\int_\gamma \|a-\lambda\|^2 d\lambda.
\end{aligned}$$

Proof. From the identity (2.5) we get by taking the integral mean that

$$\begin{aligned}
(5.6)\quad f(a) &= \frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda + \frac{1}{\beta-\alpha}\int_\gamma f'(\lambda)(a-\lambda)\,d\lambda \\
&\quad + \frac{1}{\beta-\alpha}\int_\gamma (a-\lambda)\left(\int_0^1 \left[f'((1-t)\lambda + ta) - f'(\lambda)\right] dt\right) d\lambda.
\end{aligned}$$

Since

$$\begin{aligned}
\int_\gamma f'(\lambda)(a-\lambda)\,d\lambda &= \left(\int_\gamma f'(\lambda)\,d\lambda\right) a - \int_\gamma \lambda f'(\lambda)\,d\lambda \\
&= (f(\beta) - f(\alpha))\,a - \left[\lambda f(\lambda)|_\alpha^\beta - \int_\gamma f(\lambda)\,d\lambda\right] \\
&= (f(\beta) - f(\alpha))\,a - \beta f(\beta) + \alpha f(\alpha) + \int_\gamma f(\lambda)\,d\lambda \\
&= \int_\gamma f(\lambda)\,d\lambda - f(\beta)(\beta-a) - f(\alpha)(a-\alpha),
\end{aligned}$$

hence

$$\begin{aligned}
&\frac{1}{\beta-\alpha}\int_{\gamma} f(\lambda)\,d\lambda+\frac{1}{\beta-\alpha}\int_{\gamma}(a-\lambda) f'(\lambda)\,d\lambda \\
&\quad=\frac{1}{\beta-\alpha}\int_{\gamma} f(\lambda)\,d\lambda+\frac{1}{\beta-\alpha}\int_{\gamma} f(\lambda)\,d\lambda-\frac{f(\beta)(\beta-a)+f(\alpha)(a-\alpha)}{\beta-\alpha} \\
&\quad=\frac{2}{\beta-\alpha}\int_{\gamma} f(\lambda)\,d\lambda-\frac{(\beta-a) f(\beta)+(a-\alpha) f(\alpha)}{\beta-\alpha}
\end{aligned}$$

and by (5.6) we get

$$\begin{aligned}
f(a)=&\frac{2}{\beta-\alpha}\int_{\gamma} f(\lambda)\,d\lambda-\frac{(\beta-a) f(\beta)+(a-\alpha) f(\alpha)}{\beta-\alpha} \\
&+\frac{1}{\beta-\alpha}\int_{\gamma}(a-\lambda)\left(\int_0^1\left[f'((1-t)\lambda+ta)-f'(\lambda)\right]dt\right)d\lambda,
\end{aligned}$$

or

$$\begin{aligned}
f(a)&+\frac{f(\beta)(\beta-a)+f(\alpha)(a-\alpha)}{\beta-\alpha}-\frac{2}{\beta-\alpha}\int_{\gamma} f(\lambda)\,d\lambda= \\
&+\frac{1}{\beta-\alpha}\int_{\gamma}(a-\lambda)\left(\int_0^1\left[f'((1-t)\lambda+ta)-f'(\lambda)\right]dt\right)d\lambda
\end{aligned}$$

namely, we get the following equality of interest

$$\begin{aligned}
(5.7)\quad &\frac{1}{2}\left[f(a)+\frac{f(\beta)(\beta-a)+f(\alpha)(a-\alpha)}{\beta-\alpha}\right]-\frac{1}{\beta-\alpha}\int_{\gamma} f(\lambda)\,d\lambda \\
&=\frac{1}{2(\beta-\alpha)}\int_{\gamma}(a-\lambda)\left(\int_0^1\left[f'((1-t)\lambda+ta)-f'(\lambda)\right]dt\right)d\lambda.
\end{aligned}$$

Now, if we take the norm, then we get

$$\begin{aligned}
&\left\|\frac{1}{2}\left[f(a)+\frac{f(\beta)(\beta-a)+f(\alpha)(a-\alpha)}{\beta-\alpha}\right]-\frac{1}{\beta-\alpha}\int_{\gamma} f(\lambda)\,d\lambda\right\| \\
&\quad\leq\frac{1}{2|\beta-\alpha|}\int_{\gamma}\left\|(a-\lambda)\left(\int_0^1\left[f'((1-t)\lambda+ta)-f'(\lambda)\right]dt\right)\right\|d\lambda \\
&\quad\leq\frac{1}{2|\beta-\alpha|}\int_{\gamma}\|a-\lambda\|\left\|\int_0^1\left[f'((1-t)\lambda+ta)-f'(\lambda)\right]dt\right\|d\lambda \\
&\quad\leq\frac{1}{2|\beta-\alpha|}\int_{\gamma}\|a-\lambda\|\left(\int_0^1\left\|f'((1-t)\lambda+ta)-f'(\lambda)\right\|dt\right)d\lambda \\
&\quad\leq\frac{L_{\gamma,a}}{2|\beta-\alpha|}\int_{\gamma}\|a-\lambda\|^2\left(\int_0^1 tdt\right)d\lambda=\frac{L_{\gamma,a}}{4|\beta-\alpha|}\int_{\gamma}\|a-\lambda\|^2\,d\lambda,
\end{aligned}$$

which produces the desired result (5.5). □

6. Some Examples

Consider the function $f(z)=\operatorname{Log}(z)$, the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_{\ell}:=\mathbb{C}\backslash\{x+iy: x\leq 0,\ y=0\}$

and

$$f'(z) = \frac{1}{z},\ z \in \mathbb{C}_{\ell}.$$

Suppose $\gamma \subset \mathbb{C}_{\ell}$ is a *smooth path* parametrized by $z(t)$, $t \in [0,1]$ with $z(0) = \alpha$ and $z(1) = \beta$ where α, $\beta \in \mathbb{C}_{\ell}$. Then

$$\begin{aligned}
\int_{\gamma} f(z)\, dz = \int_{\gamma_{\alpha,\beta}} f(z)\, dz = \int_{\gamma_{\alpha,\beta}} \operatorname{Log}(z)\, dz = \\
&= z \operatorname{Log}(z)\big|_{\alpha}^{\beta} - \int_{\gamma_{\alpha,\beta}} (\operatorname{Log}(z))'\, z dz \\
&= \beta \operatorname{Log}(\beta) - \alpha \operatorname{Log}(\alpha) - \int_{\gamma_{\alpha,\beta}} dz \\
&= \beta \operatorname{Log}(\beta) - \alpha \operatorname{Log}(\alpha) - (\beta - \alpha).
\end{aligned}$$

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain in $\mathbb{C}_{\ell}$ with $\sigma(a) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t \in [0,1]$ from $\lambda(0) = \alpha$ to $\lambda(1) = \beta$, with $\beta \neq \alpha$. We can define $\operatorname{Log} a$ by using the functional calculus (1.2)

$$\operatorname{Log} a := \frac{1}{2\pi i} \int_{\delta} \operatorname{Log}(\xi)\, (\xi - a)^{-1}\, d\xi, \tag{6.1}$$

where $\delta \subset G \subset \mathbb{C}_{\ell}$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

By using the inequality (4.7) we get

$$\left\| \operatorname{Log} a - \frac{\beta \operatorname{Log} \beta - \alpha \operatorname{Log} \alpha}{\beta - \alpha} + 1 \right\| \leq \frac{1}{|\beta - \alpha|} M \int_{\gamma} \|a - \lambda\|\, |d\lambda|, \tag{6.2}$$

where

$$M := \sup_{(\lambda,t) \in \gamma \times [0,1]} \left\| ((1-t)\lambda + ta)^{-1} \right\|.$$

Observe that

$$((1-t)\lambda + ta)^{-1} = \lambda^{-1} \left((1-t) + \frac{t}{\lambda} a \right)^{-1} = \lambda^{-1} \left(1 - \left(1 - \frac{1}{\lambda} \right) ta \right)^{-1}.$$

If

$$\left\| \left(1 - \frac{1}{\lambda} \right) ta \right\| < 1$$

namely

$$\left| 1 - \frac{1}{\lambda} \right| t \|a\| < 1,$$

then

$$\left(1 - \left(1 - \frac{1}{\lambda} \right) ta \right)^{-1} = \sum_{n=0}^{\infty} \left(1 - \frac{1}{\lambda} \right)^{n} t^n a^n.$$

If we take the norm and use the triangle inequality, then we get

$$\begin{aligned}
&\left\|\left((1-t)\lambda+ta\right)^{-1}\right\| \\
&=|\lambda|^{-1}\left\|\left(1-\left(1-\frac{1}{\lambda}\right)ta\right)^{-1}\right\| \le |\lambda|^{-1}\sum_{n=0}^{\infty}\left|1-\frac{1}{\lambda}\right|^{n}t^{n}\|a\|^{n} \\
&=|\lambda|^{-1}\frac{1}{1-\left|1-\frac{1}{\lambda}\right|t\|a\|}=|\lambda|^{-1}\frac{1}{1-\left|\frac{\lambda-1}{\lambda}\right|t\|a\|} \\
&=\frac{1}{|\lambda|-|\lambda-1|t\|a\|}.
\end{aligned}$$

So, if

$$\left|1-\frac{1}{\lambda}\right|t\|a\|<1$$

for all $(\lambda,t)\in\gamma\times[0,1]$ and if we put

$$K:=\sup_{(\lambda,t)\in\gamma\times[0,1]}\frac{1}{|\lambda|-|\lambda-1|t\|a\|}$$

and assume that $K<\infty$, then by (6.2) we get

$$\left\|\operatorname{Log}a-\frac{\beta\operatorname{Log}\beta-\alpha\operatorname{Log}\alpha}{\beta-\alpha}+1\right\|\le\frac{1}{|\beta-\alpha|}K\int_{\gamma}\|a-\lambda\|\,|d\lambda|. \tag{6.3}$$

Suppose $\gamma\subset\mathbb{C}$ is a *smooth path* parametrized by $\lambda(t)$, $t\in[0,1]$ with $\lambda(0)=\alpha$ and $\lambda(1)=\beta$, α, $\beta\in\mathbb{C}$ with $\beta\neq\alpha$. Then by (4.7) we get

$$\left\|\exp a-\frac{\exp\beta-\exp\alpha}{\beta-\alpha}\right\|\le\frac{1}{|\beta-\alpha|}T\int_{\gamma}\|a-\lambda\|\,|d\lambda|, \tag{6.4}$$

where

$$T:=\sup_{(\lambda,t)\in\gamma\times[0,1]}\|\exp((1-t)\lambda+ta)\|.$$

Observe that

$$\exp((1-t)\lambda+ta)=\exp[(1-t)\lambda]\exp(ta),$$

which gives

$$\begin{aligned}
&\|\exp((1-t)\lambda+ta)\| \\
&=|\exp[(1-t)\lambda]|\,\|\exp(ta)\|=\exp[(1-t)\operatorname{Re}\lambda]\,\|\exp(ta)\| \\
&\le\exp[(1-t)\operatorname{Re}\lambda]\exp(t\|a\|)=\exp[(1-t)\operatorname{Re}\lambda+t\|a\|].
\end{aligned}$$

So, if we put

$$S:=\sup_{(\lambda,t)\in\gamma\times[0,1]}\exp[(1-t)\operatorname{Re}\lambda+t\|a\|],$$

then we get by (6.4) that

$$\left\|\exp a-\frac{\exp\beta-\exp\alpha}{\beta-\alpha}\right\|\le\frac{1}{|\beta-\alpha|}S\int_{\gamma}\|a-\lambda\|\,|d\lambda|. \tag{6.5}$$

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