SOME NORM INTEGRAL INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t \in [0,1]$ from $\lambda(0) = \alpha$ to $\lambda(1) = \beta$, with $\beta \neq \alpha$. If $f : G \to \mathbb{C}$ is analytic on G, then by using the analytic functional calculus we obtain among others the following result

$$\left\| f(a) - \int_0^1 f((1-t)\lambda + ta)\, dt \right\| \leq \|a - \lambda\| \int_0^1 t \left\| f'(ta + (1-t)\lambda) \right\| dt$$

$$\leq \|a-\lambda\| \begin{cases} \frac{1}{2} \sup_{t\in[0,1]} \|f'(ta + (1-t)\lambda)\|, \\ \\ \frac{1}{(q+1)^{1/q}} \left(\int_0^1 \|f'(ta + (1-t)\lambda)\|^p dt \right)^{1/p}, \\ \\ \int_0^1 \|f'(ta + (1-t)\lambda)\| dt, \end{cases}$$

for all $\lambda \in G$. Some example for the exponential function of elements in Banach algebras are also provided.

1. Introduction

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \leq \|a\| \|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*. We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\mathrm{Inv}(\mathcal{B})$. If $a, b \in \mathrm{Inv}(\mathcal{B})$ then $ab \in \mathrm{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \mathrm{Inv}(\mathcal{B})$;
(ii) $\{a \in \mathcal{B}\colon \|1 - b\| < 1\} \subset \mathrm{Inv}(\mathcal{B})$;
(iii) $\mathrm{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\mathrm{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \mathrm{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} : \ z - a \in \mathrm{Inv}(\mathcal{B})\};$$

1991 *Mathematics Subject Classification.* 47A63; 47A99.
Key words and phrases. Banach algebras, Ostrowski inequality, Norm inequalities, Analytic functional calculus.

RGMIA Res. Rep. Coll. 22 (2019), Art. 22, 17 pp. Received 20/02/19

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \text{Inv}(\mathcal{B})$, $R_a(z) := (z-a)^{-1}$. For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z-w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \leq \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_\delta f(\xi)(\xi - a)^{-1} d\xi, \tag{1.1}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \text{ins}(\delta)$, the inside of δ.

It is well known (see for instance [3, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.2}$$

holds.

Let $\mathfrak{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\mathfrak{Hol}(a)$ is an algebra where if $f, g \in \mathfrak{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f+g$ have domain $D(f) \cap D(g)$. $\mathfrak{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [1, p. 201-203]:

Theorem 1. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) *The map $f \mapsto f(a)$ of $\mathfrak{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.*
(b) *If $f(z) = \sum_{k=0}^\infty \alpha_k z^k$ has radius of convergence $r > \nu(a)$, then $f \in \mathfrak{Hol}(a)$ and $f(a) = \sum_{k=0}^\infty \alpha_k a^k$.*
(c) *If $f(z) \equiv 1$, then $f(a) = 1$.*
(d) *If $f(z) = z$ for all z, $f(a) = a$.*
(e) *If $f, f_1, ..., f_n...$ are analytic on G, $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\|f_n(a) - f(a)\| \to 0$ as $n \to \infty$.*
(f) *The Riesz Functional Calculus is unique and if a, b are commuting elements in $\mathcal{B}$ and $f \in \mathfrak{Hol}(a)$, then $f(a)b = bf(a)$.*

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t \in [0,1]$ from $\lambda(0) = \alpha$ to $\lambda(1) = \beta$, with $\beta \neq \alpha$. If $f : G \to \mathbb{C}$ is analytic on G, then by using the analytic functional calculus we obtained in the recent paper [8] the following norm integral inequality of Ostrowski type

$$\left\| f(a) - \frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda \right\| \le K\frac{1}{|\beta-\alpha|}\int_\gamma \|a-\lambda\|\,|d\lambda|, \tag{1.3}$$

provided

$$K := \sup_{(\lambda,t)\in\gamma\times[0,1]} \|f'((1-t)\lambda+ta)\| < \infty.$$

For some recent norm inequalities for functions on Banach algebras, see [1]-[2] and [5]-[12].

Motivated by the above result, in this paper we establish some other similar norm integral inequalities. Some example for the exponential function of elements in Banach algebras are also provided.

2. Some Identities

We have:

Theorem 2. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f : G\to\mathbb{C}$ is analytic on G, then for all λ, $\mu\in G$ and $\alpha\in\mathbb{C}$ we have*

$$\begin{aligned} f(a) - \alpha\int_0^1 f((1-t)\lambda+ta)\,dt - (1-\alpha)\int_0^1 f((1-t)a+t\mu)\,dt \\ = \alpha(a-\lambda)\int_0^1 tf'(ta+(1-t)\lambda)\,dt \\ + (1-\alpha)(a-\mu)\int_0^1 (1-t)f'((1-t)a+t\mu)\,dt. \end{aligned} \tag{2.1}$$

In particular, we have for $\alpha=\frac{1}{2}$ that

$$f(a) - \frac{1}{2}\left[\int_0^1 f((1-t)\lambda+ta)\,dt + \int_0^1 f((1-t)a+t\mu)\,dt\right] \tag{2.2}$$
$$= \frac{1}{2}\left[(a-\lambda)\int_0^1 tf'(ta+(1-t)\lambda)\,dt + (a-\mu)\int_0^1 (1-t)f'((1-t)a+t\mu)\,dt\right]$$

and for $\alpha=1$, the simple equality

$$f(a) - \int_0^1 f((1-t)\lambda+ta)\,dt = (a-\lambda)\int_0^1 tf'((1-t)\lambda+ta)\,dt. \tag{2.3}$$

Proof. For ξ, $\lambda\in G$ we have

$$\int_0^1 t\left(f((1-t)\lambda+t\xi)\right)'dt = (\xi-\lambda)\int_0^1 tf'((1-t)\lambda+t\xi)\,dt$$

and

$$\int_0^1 (1-t)\left(f((1-t)\xi+t\mu)\right)'dt = (\mu-\xi)\int_0^1 (1-t)f'((1-t)\xi+t\mu)\,dt.$$

Integrating by parts, we also have

$$\int_0^1 t\left(f\left((1-t)\lambda+t\xi\right)\right)' dt = tf\left((1-t)\lambda+t\xi\right)\Big|_0^1 - \int_0^1 f\left((1-t)\lambda+t\xi\right) dt$$
$$= f(\xi) - \int_0^1 f\left((1-t)\lambda+t\xi\right) dt$$

and

$$\int_0^1 (1-t)\left(f\left((1-t)\xi+t\mu\right)\right)' dt = (1-t) f\left((1-t)\xi+t\mu\right)\Big|_0^1$$
$$+\int_0^1 f\left((1-t)\xi+t\mu\right) dt = \int_0^1 f\left((1-t)\xi+t\mu\right) dt - f(\xi).$$

Then we get

$$f(\xi) = \int_0^1 f\left((1-t)\lambda+t\xi\right) dt + (\xi-\lambda)\int_0^1 tf'\left((1-t)\lambda+t\xi\right) dt \tag{2.4}$$

and

$$f(\xi) = \int_0^1 f\left((1-t)\xi+t\mu\right) dt + (\xi-\mu)\int_0^1 (1-t) f'\left((1-t)\xi+t\mu\right) dt \tag{2.5}$$

for all λ, ξ, $\mu \in G$.

If we multiply (2.4) by α and (2.5) by $1-\alpha$ and we add the obtained equalities, we get

$$\begin{aligned} f(\xi) = \alpha \int_0^1 f\left((1-t)\lambda+t\xi\right) dt + (1-\alpha)\int_0^1 f\left((1-t)\xi+t\mu\right) dt \\ + \alpha(\xi-\lambda)\int_0^1 tf'\left((1-t)\lambda+t\xi\right) dt \\ + (1-\alpha)(\xi-\mu)\int_0^1 (1-t) f'\left((1-t)\xi+t\mu\right) dt \end{aligned} \tag{2.6}$$

for all λ, ξ, $\mu \in G$ and $\alpha \in \mathbb{C}$.

Now, by using the equality (1.1), we have that

$$\begin{aligned} f(a) = \frac{\alpha}{2\pi i}\int_\gamma \left(\int_0^1 f\left((1-t)\lambda+t\xi\right) dt\right)(\xi-a)^{-1} d\xi \\ + \frac{(1-\alpha)}{2\pi i}\int_\gamma \left(\int_0^1 f\left((1-t)\xi+t\mu\right) dt\right)(\xi-a)^{-1} d\xi \\ + \frac{\alpha}{2\pi i}\int_\gamma (\xi-\lambda)\left(\int_0^1 tf'\left((1-t)\lambda+t\xi\right) dt\right)(\xi-a)^{-1} d\xi \\ + \frac{(1-\alpha)}{2\pi i}\int_\gamma (\xi-\mu)\left(\int_0^1 (1-t) f'\left((1-t)\xi+t\mu\right) dt\right)(\xi-a)^{-1} d\xi \end{aligned} \tag{2.7}$$

$$= \alpha \int_0^1 \left(\frac{1}{2\pi i} \int_\gamma f\left((1-t)\lambda + t\xi\right)(\xi - a)^{-1} d\xi \right) dt$$
$$+ (1-\alpha) \int_0^1 \left(\frac{1}{2\pi i} \int_\gamma f\left((1-t)\xi + t\mu\right)(\xi - a)^{-1} d\xi \right) dt$$
$$+ \alpha \int_0^1 t \left(\frac{1}{2\pi i} \int_\gamma (\xi - \lambda) f'\left((1-t)\lambda + t\xi\right)(\xi - a)^{-1} d\xi \right) dt$$
$$+ (1-\alpha) \int_0^1 (1-t) \left(\frac{1}{2\pi i} \int_\gamma (\xi - \mu) f'\left((1-t)\xi + t\mu\right)(\xi - a)^{-1} d\xi \right) dt,$$

where for the last equality we used Fubini's theorem.

Since, by the analytic functional calculus for composite functions, we have

$$\frac{1}{2\pi i} \int_\gamma f\left((1-t)\lambda + t\xi\right)(\xi - a)^{-1} d\xi = f\left((1-t)\lambda + ta\right),$$

$$\frac{1}{2\pi i} \int_\gamma f\left((1-t)\xi + t\mu\right)(\xi - a)^{-1} d\xi = f\left((1-t)a + t\mu\right),$$

$$\frac{1}{2\pi i} \int_\gamma (\xi - \lambda) f'\left((1-t)\lambda + t\xi\right)(\xi - a)^{-1} d\xi = (a - \lambda) f'\left(ta + (1-t)\lambda\right)$$

and

$$\frac{1}{2\pi i} \int_\gamma (\xi - \mu) f'\left((1-t)\xi + t\mu\right)(\xi - a)^{-1} d\xi = (a - \mu) f'\left((1-t)a + t\mu\right)$$

for any λ, $\mu \in G$ and $t \in [0,1]$, then by (2.7) we get the desired result (2.7). □

Remark 1. *We observe that by changing the variable* $s = 1 - t$, $t \in [0,1]$ *we have for* $\mu \in G$ *that*

$$\int_0^1 f\left((1-t)a + t\mu\right) dt = \int_0^1 f\left(sa + (1-s)\mu\right) ds$$

and

$$\int_0^1 (1-t) f'\left((1-t)a + t\mu\right) dt = \int_0^1 s f'\left(sa + (1-s)\mu\right) ds.$$

By taking $\mu = \lambda \in G$ *in (2.1) we get*

$$f(a) - \alpha \int_0^1 f\left((1-t)\lambda + ta\right) dt - (1-\alpha) \int_0^1 f\left(sa + (1-s)\lambda\right) ds$$
$$= \alpha (a - \lambda) \int_0^1 t f'\left(ta + (1-t)\lambda\right) dt$$
$$+ (1-\alpha)(a - \lambda) \int_0^1 s f'\left(sa + (1-s)\lambda\right) ds,$$

which is equivalent to (2.3).

Corollary 1. *With the assumptions of Theorem 2 and if* $b \in \mathcal{B}$, *then we have the equality*

$$f(a) - \frac{1}{2}(a-\lambda)b - \int_0^1 f((1-t)\lambda + ta)\,dt = (a-\lambda)\int_0^1 t\left[f'((1-t)\lambda + ta) - b\right]dt. \tag{2.8}$$

In particular, we have

$$f(a) - \frac{1}{2}(a-\lambda)f'(a) - \int_0^1 f((1-t)\lambda + ta)\,dt = (a-\lambda)\int_0^1 t\left[f'((1-t)\lambda + ta) - f'(a)\right]dt, \tag{2.9}$$

$$f(a) - \frac{1}{2}f'(\lambda)(a-\lambda) - \int_0^1 f((1-t)\lambda + ta)\,dt = (a-\lambda)\int_0^1 t\left[f'((1-t)\lambda + ta) - f'(\lambda)\right]dt, \tag{2.10}$$

$$f(a) - (a-\lambda)\frac{f'(a)+f'(\lambda)}{4} - \int_0^1 f((1-t)\lambda + ta)\,dt = (a-\lambda)\int_0^1 t\left[f'((1-t)\lambda + ta) - \frac{f'(a)+f'(\lambda)}{2}\right]dt, \tag{2.11}$$

and

$$f(a) - \frac{1}{2}(a-\lambda)f'\left(\frac{a+\lambda}{2}\right) - \int_0^1 f((1-t)\lambda + ta)\,dt = (a-\lambda)\int_0^1 t\left[f'((1-t)\lambda + ta) - f'\left(\frac{a+\lambda}{2}\right)\right]dt. \tag{2.12}$$

Corollary 2. *With the assumptions of Theorem 2 and if* $\gamma \subset G$ *is a piecewise smooth path parametrized by* $\lambda(t)$, $t \in [0,1]$ *from* $\lambda(0) = \alpha$ *to* $\lambda(1) = \beta$, *with* $\beta \neq \alpha$, *then*

$$f(a) - \frac{1}{2}\left(a - \frac{\alpha+\beta}{2}\right)b - \frac{1}{\beta-\alpha}\int_\gamma\left(\int_0^1 f((1-t)\lambda + ta)\,dt\right)d\lambda = \frac{1}{\beta-\alpha}\int_\gamma (a-\lambda)\left(\int_0^1 t\left[f'((1-t)\lambda + ta) - b\right]dt\right)d\lambda. \tag{2.13}$$

In particular,

$$f(a) - \frac{1}{2}\left(a - \frac{\alpha+\beta}{2}\right)f'(a) - \frac{1}{\beta-\alpha}\int_\gamma\left(\int_0^1 f((1-t)\lambda + ta)\,dt\right)d\lambda = \frac{1}{\beta-\alpha}\int_\gamma (a-\lambda)\left(\int_0^1 t\left[f'((1-t)\lambda + ta) - f'(a)\right]dt\right)d\lambda. \tag{2.14}$$

The proof follows by the identity (2.8) by taking the integral mean $\frac{1}{\beta-\alpha}\int_\gamma$.

Corollary 3. *With the assumptions of Corollary 2 we have*

$$f(a)+\frac{f(\beta)(\beta-a)+f(\alpha)(a-\alpha)}{2}-\frac{1}{2}\frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda \tag{2.15}$$
$$-\frac{1}{\beta-\alpha}\int_\gamma\left(\int_0^1 f((1-t)\lambda+ta)\,dt\right)d\lambda$$
$$=\frac{1}{\beta-\alpha}\int_\gamma(a-\lambda)\left(\int_0^1 t\left[f'((1-t)\lambda+ta)-f'(\lambda)\right]dt\right)d\lambda.$$

Proof. If we take the integral mean $\frac{1}{\beta-\alpha}\int_\gamma$ in (2.10), then we get

$$f(a)-\frac{1}{2}\frac{1}{\beta-\alpha}\int_\gamma f'(\lambda)(a-\lambda)\,d\lambda \tag{2.16}$$
$$-\frac{1}{\beta-\alpha}\int_\gamma\left(\int_0^1 f((1-t)\lambda+ta)\,dt\right)d\lambda$$
$$=\frac{1}{\beta-\alpha}\int_\gamma(a-\lambda)\left(\int_0^1 t\left[f'((1-t)\lambda+ta)-f'(\lambda)\right]dt\right)d\lambda.$$

Since

$$\int_\gamma f'(\lambda)(a-\lambda)\,d\lambda=\left(\int_\gamma f'(\lambda)\,d\lambda\right)a-\int_\gamma \lambda f'(\lambda)\,d\lambda$$
$$=(f(\beta)-f(\alpha))\,a-\left[\lambda f(\lambda)|_\alpha^\beta-\int_\gamma f(\lambda)\,d\lambda\right]$$
$$=(f(\beta)-f(\alpha))\,a-\beta f(\beta)+\alpha f(\alpha)+\int_\gamma f(\lambda)\,d\lambda$$
$$=\int_\gamma f(\lambda)\,d\lambda-f(\beta)(\beta-a)-f(\alpha)(a-\alpha),$$

hence by (2.16) we get

$$f(a)-\frac{1}{2}\frac{\int_\gamma f(\lambda)\,d\lambda-f(\beta)(\beta-a)-f(\alpha)(a-\alpha)}{\beta-\alpha}$$
$$-\frac{1}{\beta-\alpha}\int_\gamma\left(\int_0^1 f((1-t)\lambda+ta)\,dt\right)d\lambda$$
$$=\frac{1}{\beta-\alpha}\int_\gamma(a-\lambda)\left(\int_0^1 t\left[f'((1-t)\lambda+ta)-f'(\lambda)\right]dt\right)d\lambda,$$

which is equivalent to (2.15). □

3. Norm Integral Inequalities

We have:

Theorem 3. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$ and $\gamma\subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t\in[0,1]$ from $\lambda(0)=\alpha$ to $\lambda(1)=\beta$, with $\beta\neq\alpha$. If $f:G\to\mathbb{C}$ is analytic on G,* *then for all* λ, $\mu \in G$ *and* $\alpha \in \mathbb{C}$ *we have*

$$\left\| f(a) - \alpha \int_0^1 f((1-t)\lambda + ta)\,dt - (1-\alpha)\int_0^1 f((1-t)a + t\mu)\,dt \right\| \tag{3.1}$$

$$\leq |\alpha| \|a-\lambda\| \int_0^1 t \|f'(ta + (1-t)\lambda)\|\,dt$$

$$+ |1-\alpha| \|a-\mu\| \int_0^1 (1-t)\|f'((1-t)a + t\mu)\|\,dt$$

$$\leq |\alpha| \|a-\lambda\| \begin{cases} \frac{1}{2}\sup_{t\in[0,1]} \|f'(ta+(1-t)\lambda)\| \\ \frac{1}{(q+1)^{1/q}} \left(\int_0^1 \|f'(ta+(1-t)\lambda)\|^p\,dt\right)^{1/p} \\ \int_0^1 \|f'(ta+(1-t)\lambda)\|\,dt \end{cases}$$

$$+ |1-\alpha| \|a-\mu\| \begin{cases} \frac{1}{2}\sup_{t\in[0,1]} \|f'((1-t)a+t\mu)\| \\ \frac{1}{(q+1)^{1/q}} \left(\int_0^1 \|f'((1-t)a+t\mu)\|^p\,dt\right)^{1/p} \\ \int_0^1 \|f'((1-t)a+t\mu)\|\,dt. \end{cases}$$

In particular, we have

$$\left\| f(a) - \frac{1}{2}\int_0^1 f((1-t)\lambda + ta)\,dt - \frac{1}{2}\int_0^1 f((1-t)a + t\mu)\,dt \right\| \tag{3.2}$$

$$\leq \frac{1}{2}\|a-\lambda\| \int_0^1 t\|f'(ta+(1-t)\lambda)\|\,dt$$

$$+ \frac{1}{2}\|a-\mu\| \int_0^1 (1-t)\|f'((1-t)a+t\mu)\|\,dt$$

$$\leq \frac{1}{2}\|a-\lambda\| \begin{cases} \frac{1}{2}\sup_{t\in[0,1]} \|f'(ta+(1-t)\lambda)\| \\ \frac{1}{(q+1)^{1/q}} \left(\int_0^1 \|f'(ta+(1-t)\lambda)\|^p\,dt\right)^{1/p} \\ \int_0^1 \|f'(ta+(1-t)\lambda)\|\,dt \end{cases}$$

$$+ \frac{1}{2}\|a-\mu\| \begin{cases} \frac{1}{2}\sup_{t\in[0,1]} \|f'((1-t)a+t\mu)\| \\ \frac{1}{(q+1)^{1/q}} \left(\int_0^1 \|f'((1-t)a+t\mu)\|^p\,dt\right)^{1/p} \\ \int_0^1 \|f'((1-t)a+t\mu)\|\,dt \end{cases}$$

and

$$\left\| f(a) - \int_0^1 f((1-t)\lambda + ta)\,dt \right\| \le \|a-\lambda\| \int_0^1 t \|f'(ta+(1-t)\lambda)\|\,dt \tag{3.3}$$
$$\le \|a-\lambda\| \begin{cases} \frac{1}{2}\sup_{t\in[0,1]} \|f'(ta+(1-t)\lambda)\| \\ \frac{1}{(q+1)^{1/q}} \left(\int_0^1 \|f'(ta+(1-t)\lambda)\|^p\,dt\right)^{1/p} \\ \int_0^1 \|f'(ta+(1-t)\lambda)\|\,dt. \end{cases}$$

Proof. From the identity (2.1) we get

$$\left\| f(a) - \alpha\int_0^1 f((1-t)\lambda + ta)\,dt - (1-\alpha)\int_0^1 f((1-t)a + t\mu)\,dt \right\| \tag{3.4}$$
$$\le |\alpha| \left\| (a-\lambda)\int_0^1 tf'(ta+(1-t)\lambda)\,dt \right\|$$
$$+ |1-\alpha| \left\| (a-\mu)\int_0^1 (1-t) f'((1-t)a+t\mu)\,dt \right\|$$
$$\le |\alpha|\,\|a-\lambda\| \left\| \int_0^1 tf'(ta+(1-t)\lambda)\,dt \right\|$$
$$+ |1-\alpha|\,\|a-\mu\| \left\| \int_0^1 (1-t) f'((1-t)a+t\mu)\,dt \right\|$$
$$\le |\alpha|\,\|a-\lambda\| \int_0^1 t\|f'(ta+(1-t)\lambda)\|\,dt$$
$$+ |1-\alpha|\,\|a-\mu\| \int_0^1 (1-t)\|f'((1-t)a+t\mu)\|\,dt$$

that proves the first inequality in (3.4).

By Hölder's inequality we get

$$\int_0^1 t\|f'(ta+(1-t)\lambda)\|\,dt$$
$$\le \begin{cases} \sup_{t\in[0,1]} \|f'(ta+(1-t)\lambda)\| \int_0^1 tdt \\ \left(\int_0^1 \|f'(ta+(1-t)\lambda)\|^p\,dt\right)^{1/p} \left(\int_0^1 t^q dt\right)^{1/q} \\ \int_0^1 \|f'(ta+(1-t)\lambda)\|\,dt \sup_{t\in[0,1]}\{t\} \end{cases}$$
$$= \begin{cases} \frac{1}{2}\sup_{t\in[0,1]} \|f'(ta+(1-t)\lambda)\| \\ \frac{1}{(q+1)^{1/q}} \left(\int_0^1 \|f'(ta+(1-t)\lambda)\|^p\,dt\right)^{1/p} \\ \int_0^1 \|f'(ta+(1-t)\lambda)\|\,dt \end{cases}$$

and

$$\int_0^1 (1-t)\|f'((1-t)a+t\mu)\| dt$$
$$\leq \begin{cases} \frac{1}{2}\sup_{t\in[0,1]}\|f'((1-t)a+t\mu)\| \\ \\ \frac{1}{(q+1)^{1/q}}\left(\int_0^1 \|f'((1-t)a+t\mu)\|^p dt\right)^{1/p} \\ \\ \int_0^1 \|f'((1-t)a+t\mu)\| dt, \end{cases}$$

which proves the last part of (3.1). □

Let $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$ and $\lambda\in G$. We define $G_{\lambda,a} := \{(1-t)\lambda+ta \mid \text{with } t\in[0,1]\}$. We observe that $G_{\lambda,a}$ is a convex subset in $\mathcal{B}$ for every $\lambda\in G$.

For two distinct elements u, v in the Banach algebra B we say that the function $g: G_{\lambda,a}\to\mathcal{B}$ belongs to the class $\Delta_{u,v}(G_{\lambda,a})$ if it satisfies the boundedness condition

$$\left\|g((1-t)\lambda+ta)-\frac{u+v}{2}\right\|\leq\frac{1}{2}\|v-u\| \tag{3.5}$$

for all $t\in[0,1]$. We write $g\in\Delta_{u,v}(G_{\lambda,a})$. This definition is an extension to Banach algebras valued functions of the scalar case, see [4].

We say that the function $g: G_{\lambda,a}\to B$ is Lipschitzian on $G_{\lambda,a}$ with the constant $L_{\lambda,a}>0$, if for all x, $y\in G_{\lambda,a}$ we have

$$\|g(x)-g(y)\|\leq L_{\lambda,a}\|x-y\|.$$

This is equivalent to

$$\|g((1-t)\lambda+ta)-g((1-s)\lambda+sa)\|\leq L_{\lambda,a}|t-s|\|a-\lambda\| \tag{3.6}$$

for all t, $s\in[0,1]$. We write this by $g\in\mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$.

Assume that $h: G\to\mathbb{C}$ is an analytic function on G. For $t\in[0,1]$ and $\lambda\in G$, the auxiliary function $h_{t,\lambda}$ defined on G by $h_{t,\lambda}(\xi):=h((1-t)\lambda+t\xi)$ is also analytic and using the analytic functional calculus (1.1) for the element $a\in\mathcal{B}$, we can define

$$\begin{aligned}\widetilde{h}((1-t)\lambda+ta) &:= h_{t,\lambda}(a)=\frac{1}{2\pi i}\int_\gamma h_{t,\lambda}(\xi)(\xi-a)^{-1}d\xi \\ &= \frac{1}{2\pi i}\int_\gamma h((1-t)\lambda+t\xi)(\xi-a)^{-1}d\xi.\end{aligned} \tag{3.7}$$

We say that the scalar function $h\in\Delta_{u,v}(G_{\lambda,a})$ if its extension $\widetilde{h}: G_{\lambda,a}\to B$ satisfies the boundedness condition (3.5). Also, we say that the scalar function $h\in\mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$ if its extension $\widetilde{h}: G_{\lambda,a}\to B$ satisfies the Lipschitz condition (3.6).

Theorem 4. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$. Assume also that $f: G\to\mathbb{C}$ is analytic on G and $\lambda\in G$. If there exists u, $v\in\mathcal{B}$ with $u\neq v$ such that $f'\in\Delta_{u,v}(G_{\lambda,a})$, then*

$$\left\|f(a)-\frac{1}{2}(a-\lambda)\frac{u+v}{2}-\int_0^1 f((1-t)\lambda+ta)\,dt\right\|\leq\frac{1}{4}\|a-\lambda\|\|v-u\|. \tag{3.8}$$

Proof. Since $f' \in \Delta_{u,v}(G_{\lambda,a})$, then from (2.8)

$$\begin{aligned}
&\left\| f(a) - \frac{1}{2}(a-\lambda)\frac{u+v}{2} - \int_0^1 f((1-t)\lambda + ta)\,dt \right\| \\
&= \left\| (a-\lambda)\int_0^1 t\left[f'((1-t)\lambda + ta) - \frac{u+v}{2} \right] dt \right\| \\
&\le \|a-\lambda\| \left\| \int_0^1 t\left[f'((1-t)\lambda + ta) - \frac{u+v}{2} \right] dt \right\| \\
&\le \|a-\lambda\| \int_0^1 t\left\| f'((1-t)\lambda + ta) - \frac{u+v}{2} \right\| dt \\
&\le \frac{1}{2}\|a-\lambda\|\,\|v-u\| \int_0^1 t\,dt = \frac{1}{4}\|a-\lambda\|\,\|v-u\|,
\end{aligned}$$

which gives (3.8). □

We also have:

Theorem 5. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. Assume also that $f : G \to \mathbb{C}$ is analytic on G and $\lambda \in G$. If $f' \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$, then*

$$\left\| f(a) - \frac{1}{2}(a-\lambda) f'(a) - \int_0^1 f((1-t)\lambda + ta)\,dt \right\| \le \frac{1}{6}\|a-\lambda\|^2 L_{\lambda,a}, \tag{3.9}$$

$$\left\| f(a) - \frac{1}{2} f'(\lambda)(a-\lambda) - \int_0^1 f((1-t)\lambda + ta)\,dt \right\| \le \frac{1}{3}\|a-\lambda\|^2 L_{\lambda,a} \tag{3.10}$$

and

$$\begin{aligned}
&\left\| f(a) - \frac{1}{2}(a-\lambda) f'\left(\frac{a+\lambda}{2}\right) - \int_0^1 f((1-t)\lambda + ta)\,dt \right\| \\
&\qquad \le \frac{1}{8}\|a-\lambda\|^2 L_{\lambda,a}.
\end{aligned} \tag{3.11}$$

Proof. From the (2.9) and since $f' \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$, hence we have

$$\begin{aligned}
&\left\| f(a) - \frac{1}{2}(a-\lambda) f'(a) - \int_0^1 f((1-t)\lambda + ta)\,dt \right\| \\
&\le \|a-\lambda\| \left\| \int_0^1 t\left[f'((1-t)\lambda + ta) - f'(a) \right] dt \right\| \\
&\le \|a-\lambda\| \int_0^1 t\left\| f'((1-t)\lambda + ta) - f'(a) \right\| dt \\
&\le \|a-\lambda\| L_{\lambda,a} \|a-\lambda\| \int_0^1 (1-t)\,t\,dt = \frac{1}{6}\|a-\lambda\|^2 L_{\lambda,a},
\end{aligned} \tag{3.12}$$

which proves (3.9).

From the identity (2.10) and since $f' \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$, hence we have

$$\begin{aligned}(3.13)\quad & \left\| f(a) - \frac{1}{2} f'(\lambda)(a-\lambda) - \int_0^1 f((1-t)\lambda + ta)\,dt \right\| \\ & \leq \|a-\lambda\| \left\| \int_0^1 t\left[f'((1-t)\lambda + ta) - f'(\lambda)\right] dt \right\| \\ & \leq \|a-\lambda\| \int_0^1 t \left\| f'((1-t)\lambda + ta) - f'(\lambda) \right\| dt \\ & \leq \|a-\lambda\| L_{\lambda,a} \|a-\lambda\| \int_0^1 t^2 dt = \frac{1}{3} \|a-\lambda\|^2 L_{\lambda,a},\end{aligned}$$

which proves (3.10).

From (2.12) we have

$$\begin{aligned}(3.14)\quad & \left\| f(a) - \frac{1}{2}(a-\lambda) f'\left(\frac{a+\lambda}{2}\right) - \int_0^1 f((1-t)\lambda + ta)\,dt \right\| \\ & \leq \|a-\lambda\| \left\| \int_0^1 t\left[f'((1-t)\lambda + ta) - f'\left(\frac{a+\lambda}{2}\right)\right] dt \right\| \\ & \leq \|a-\lambda\| \int_0^1 t \left\| f'((1-t)\lambda + ta) - f'\left(\frac{a+\lambda}{2}\right) \right\| dt \\ & \leq \|a-\lambda\| L_{\lambda,a} \|a-\lambda\| \int_0^1 t\left|t-\frac{1}{2}\right| dt = \frac{1}{8} \|a-\lambda\|^2 L_{\lambda,a},\end{aligned}$$

which proves (3.11). □

4. Norm Double Integral Inequalities

Let $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t \in [0,1]$ from $\lambda(0) = \alpha$ to $\lambda(1) = \beta$, with $\beta \neq \alpha$. We define the following subset of the Banach algebra

$$G_{\gamma,a} := \cup_{\lambda\in\gamma} G_{\lambda,a} = \cup_{\lambda\in\gamma} \{(1-t)\lambda + ta \mid \text{with } t \in [0,1]\}.$$

For two distinct elements u, v in the Banach algebra B we say that the function $g : G_{\gamma,a} \to \mathcal{B}$ belongs to the class $\Delta_{u,v}(G_{\gamma,a})$ if it satisfies the boundedness condition

$$(4.1)\qquad \left\| g((1-t)\lambda + ta) - \frac{u+v}{2} \right\| \leq \frac{1}{2}\|v-u\|$$

for all $t \in [0,1]$ and $\lambda \in \gamma$. We write $g \in \Delta_{u,v}(G_{\gamma,a})$.

We say that the function $g : G_{\gamma,a} \to B$ is Lipschitzian on $G_{\gamma,a}$ with the constant $L_{\gamma,a} > 0$, if for all x, $y \in G_{\gamma,a}$ we have

$$\|g(x) - g(y)\| \leq L_{\gamma,a}\|x-y\|.$$

This is equivalent to

$$(4.2)\qquad \|g((1-t)\lambda + ta) - g((1-s)\lambda + sa)\| \leq L_{\gamma,a}|t-s|\,\|a-\lambda\|$$

for all t, $s \in [0,1]$ and $\lambda \in \gamma$. We write this by $g \in \mathfrak{Lip}_{L_{\gamma,a}}(G_{\gamma,a})$.

We say that the scalar function $h \in \Delta_{u,v}(G_{\gamma,a})$ if its extension $\widetilde{h} : G_{\gamma,a} \to B$ defined by (3.7) satisfies the boundedness condition (4.1). Also, we say that the scalar function $h \in \mathfrak{Lip}_{L_{\gamma,a}}(G_{\gamma,a})$ if its extension $\widetilde{h} : G_{\gamma,a} \to B$ satisfies the Lipschitz condition (4.2).

Theorem 6. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t \in [0,1]$ from $\lambda(0) = \alpha$ to $\lambda(1) = \beta$, with $\beta \neq \alpha$. If $f : G \to \mathbb{C}$ is analytic on G and there exists $u, v \in \mathcal{B}$ with $u \neq v$ such that $f' \in \Delta_{u,v}(G_{\gamma,a})$, then*

$$\left\| f(a) - \frac{1}{2}\left(a - \frac{\alpha+\beta}{2}\right)\frac{u+v}{2} - \frac{1}{\beta-\alpha}\int_\gamma \left(\int_0^1 f((1-t)\lambda + ta)\,dt\right) d\lambda \right\| \leq \frac{1}{4}\frac{\|v-u\|}{|\beta-\alpha|}\int_\gamma \|a-\lambda\|\,|d\lambda|. \tag{4.3}$$

Proof. Using the identity (2.13) and the fact that $f' \in \Delta_{u,v}(G_{\gamma,a})$, we have

$$\begin{aligned}
&\left\| f(a) - \frac{1}{2}\left(a - \frac{\alpha+\beta}{2}\right) b - \frac{1}{\beta-\alpha}\int_\gamma \left(\int_0^1 f((1-t)\lambda + ta)\,dt\right) d\lambda \right\| \\
&\quad = \frac{1}{|\beta-\alpha|}\left\| \int_\gamma (a-\lambda)\left(\int_0^1 t\left[f'((1-t)\lambda+ta) - b\right] dt\right) d\lambda \right\| \\
&\quad \leq \frac{1}{|\beta-\alpha|}\int_\gamma \left\| (a-\lambda)\left(\int_0^1 t\left[f'((1-t)\lambda+ta) - b\right] dt\right) \right\| |d\lambda| \\
&\quad \leq \frac{1}{|\beta-\alpha|}\int_\gamma \|a-\lambda\| \left\| \int_0^1 t\left[f'((1-t)\lambda+ta) - b\right] dt \right\| |d\lambda| \\
&\quad \leq \frac{1}{|\beta-\alpha|}\int_\gamma \|a-\lambda\| \left(\int_0^1 t\left\|f'((1-t)\lambda+ta) - b\right\| dt\right) |d\lambda| \\
&\quad \leq \frac{1}{2}\frac{\|v-u\|}{|\beta-\alpha|}\int_\gamma \|a-\lambda\|\,|d\lambda| \left(\int_0^1 t dt\right) = \frac{1}{4}\frac{\|v-u\|}{|\beta-\alpha|}\int_\gamma \|a-\lambda\|\,|d\lambda|,
\end{aligned}$$

which proves the desired inequality (4.3). □

We also have:

Theorem 7. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t \in [0,1]$ from $\lambda(0) = \alpha$ to $\lambda(1) = \beta$, with $\beta \neq \alpha$. If $f : G \to \mathbb{C}$ is analytic on G and there exists $L_{\gamma,a} > 0$ so that $f' \in \mathfrak{Lip}_{L_{\gamma,a}}(G_{\gamma,a})$, then*

$$\left\| f(a) - \frac{1}{2}\left(a - \frac{\alpha+\beta}{2}\right) f'(a) - \frac{1}{\beta-\alpha}\int_\gamma \left(\int_0^1 f((1-t)\lambda + ta)\,dt\right) d\lambda \right\| \leq \frac{L_{\gamma,a}}{6|\beta-\alpha|}\int_\gamma \|a-\lambda\|^2\,|d\lambda| \tag{4.4}$$

and

$$\left\| f(a) + \frac{f(\beta)(\beta - a) + f(\alpha)(a - \alpha)}{2} - \frac{1}{2}\frac{1}{\beta - \alpha}\int_\gamma f(\lambda)\,d\lambda - \frac{1}{\beta - \alpha}\int_\gamma \left(\int_0^1 f((1-t)\lambda + ta)\,dt\right)d\lambda \right\| \leq \frac{L_{\gamma,a}}{3|\beta - \alpha|}\int_\gamma \|a - \lambda\|^2 |d\lambda|. \tag{4.5}$$

Proof. Using the identity (2.14) and the fact that $f' \in \mathfrak{Lip}_{L_{\gamma,a}}(G_{\gamma,a})$, we get

$$\begin{aligned}
&\left\| f(a) - \frac{1}{2}\left(a - \frac{\alpha+\beta}{2}\right)f'(a) - \frac{1}{\beta - \alpha}\int_\gamma \left(\int_0^1 f((1-t)\lambda + ta)\,dt\right)d\lambda \right\| \\
&= \frac{1}{|\beta - \alpha|}\left\| \int_\gamma (a - \lambda)\left(\int_0^1 t\left[f'((1-t)\lambda + ta) - f'(a)\right]dt\right)d\lambda \right\| \\
&\leq \frac{1}{|\beta - \alpha|}\int_\gamma \left\| (a - \lambda)\int_0^1 t\left[f'((1-t)\lambda + ta) - f'(a)\right]dt \right\| |d\lambda| \\
&\leq \frac{1}{|\beta - \alpha|}\int_\gamma \|a - \lambda\| \left\| \int_0^1 t\left[f'((1-t)\lambda + ta) - f'(a)\right]dt \right\| |d\lambda| \\
&\leq \frac{1}{|\beta - \alpha|}\int_\gamma \|a - \lambda\| \left(\int_0^1 t\left\|f'((1-t)\lambda + ta) - f'(a)\right\| dt\right)|d\lambda| \\
&\leq \frac{L_{\gamma,a}}{|\beta - \alpha|}\int_\gamma \|a - \lambda\|^2 |d\lambda| \int_0^1 t(1-t)\,dt = \frac{L_{\gamma,a}}{6|\beta - \alpha|}\int_\gamma \|a - \lambda\|^2 |d\lambda|,
\end{aligned}$$

which proves the desired inequality (4.4).

The proof of (4.5) follows in a similar way from the identity (2.15). □

5. Examples for Exponential Function

Let $\mathcal{B}$ be a unital Banach algebra, $b \in \mathcal{B}$ and the exponential of b defined by

$$\exp b := \sum_{n=0}^{\infty} \frac{1}{n!}b^n.$$

We observe that, if b is invertible, then

$$\begin{aligned}
\int_0^1 \exp(tb)\,dt &= \int_0^1 \left(\sum_{n=0}^{\infty}\frac{1}{n!}t^n b^n\right)dt = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_0^1 t^n dt\right)b^n \\
&= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}b^n = b^{-1}(\exp b - 1).
\end{aligned}$$

Therefore for any $\lambda \in \mathbb{C}$ and for $a \in \mathcal{B}$ such that $a - \lambda$ is invertible, then we have

$$\begin{aligned}
\int_0^1 \exp((1-t)\lambda + ta)\,dt &= \exp\lambda \int_0^1 \exp[t(a - \lambda)]\,dt = (\exp\lambda)(a - \lambda)^{-1}(\exp(a - \lambda) - 1) \\
&= (a - \lambda)^{-1}(\exp a - \exp\lambda).
\end{aligned}$$

Now, by using the norm inequality (3.3), we get

$$\left\|\exp a-(a-\lambda)^{-1}(\exp a-\exp\lambda)\right\| \le \|a-\lambda\|\int_0^1 t\,\|\exp(ta+(1-t)\lambda)\|\,dt \tag{5.1}$$
$$\le \|a-\lambda\|\begin{cases}\frac{1}{2}\sup_{t\in[0,1]}\|\exp(ta+(1-t)\lambda)\| \\ \\ \frac{1}{(q+1)^{1/q}}\left(\int_0^1\|\exp(ta+(1-t)\lambda)\|^p\,dt\right)^{1/p} \\ \\ \int_0^1\|\exp(ta+(1-t)\lambda)\|\,dt.\end{cases}$$

Since

$$\begin{aligned}\|\exp(ta+(1-t)\lambda)\| &= \|\exp[(1-t)\lambda]\exp(ta)\| = |\exp[(1-t)\lambda]|\,\|\exp(ta)\| \\ &= \exp[(1-t)\operatorname{Re}\lambda]\,\|\exp(ta)\| \le \exp[(1-t)\operatorname{Re}\lambda]\exp(t\|a\|) \\ &= \exp[(1-t)\operatorname{Re}\lambda+t\|a\|]\end{aligned}$$

hence, for $\|a\|\ne\operatorname{Re}\lambda$

$$\begin{aligned}\int_0^1 t\,\|\exp(ta+(1-t)\lambda)\|\,dt &\le \int_0^1 t\exp[(1-t)\operatorname{Re}\lambda+t\|a\|]\,dt \\ &= \int_0^1 t\exp[t(\|a\|-\operatorname{Re}\lambda)+\operatorname{Re}\lambda]\,dt \\ &= \frac{1}{\|a\|-\operatorname{Re}\lambda}\int_0^1 t\,d(\exp[t(\|a\|-\operatorname{Re}\lambda)+\operatorname{Re}\lambda]) \\ &= \frac{1}{\|a\|-\operatorname{Re}\lambda}\left[\exp\|a\|-\frac{\exp\|a\|-\operatorname{Re}\lambda}{\|a\|-\operatorname{Re}\lambda}\right].\end{aligned}$$

Therefore by the first inequality in (5.1) we get

$$\left\|\exp a-(a-\lambda)^{-1}(\exp a-\exp\lambda)\right\| \le \|a-\lambda\|\left[\frac{(\|a\|-\operatorname{Re}\lambda-1)\exp\|a\|+\operatorname{Re}\lambda}{(\|a\|-\operatorname{Re}\lambda)^2}\right] \tag{5.2}$$

for all $\lambda\in\mathbb{C}$ and $a\in\mathcal{B}$ and provided that $a-\lambda$ is invertible and $\|a\|\ne\operatorname{Re}\lambda$.

From (2.14) we get for the exponential function

$$\begin{aligned}&\exp a-\frac{1}{2}\left(a-\frac{\alpha+\beta}{2}\right)\exp a-\frac{1}{\beta-\alpha}\int_\gamma(a-\lambda)^{-1}(\exp a-\exp\lambda)\,d\lambda \\ &\quad=\frac{1}{\beta-\alpha}\int_\gamma(a-\lambda)\left(\int_0^1 t\,[\exp((1-t)\lambda+ta)-\exp(a)]\,dt\right)d\lambda.\end{aligned} \tag{5.3}$$

By taking the norm, we get

$$\begin{aligned}&\left\|\exp a-\frac{1}{2}\left(a-\frac{\alpha+\beta}{2}\right)\exp a-\frac{1}{\beta-\alpha}\int_\gamma(a-\lambda)^{-1}(\exp a-\exp\lambda)\,d\lambda\right\| \\ &\quad\le\frac{1}{|\beta-\alpha|}\int_\gamma\|a-\lambda\|\left(\int_0^1 t\,\|\exp((1-t)\lambda+ta)-\exp(a)\|\,dt\right)|d\lambda|.\end{aligned} \tag{5.4}$$

In the recent paper [6] we obtained the following norm inequality for the exponential function

$$\|\exp y-\exp x\| \leq \|y-x\| \int_0^1 \exp(\|(1-s)x+sy\|)\,ds \leq \|y-x\| \begin{cases} \frac{\exp(\|y\|)-\exp(\|x\|)}{\|y\|-\|x\|} & \text{if } \|y\| \neq \|x\| \\ \exp(\|x\|) & \text{if } \|y\| \neq \|x\| . \end{cases} \tag{5.5}$$

Now observe that

$$\begin{aligned} \|\exp((1-t)\lambda+ta)-\exp(a)\| &= \|\exp(a)[\exp[(1-t)(\lambda-a)]-1]\| \\ &\leq \|\exp(a)\| \|\exp[(1-t)(\lambda-a)]-1\| \end{aligned} \tag{5.6}$$

and by (5.5) we also have

$$\begin{aligned} &\|\exp[(1-t)(\lambda-a)]-1\| \\ &= \|\exp[(1-t)(\lambda-a)]-\exp 0\| \\ &\leq \|(1-t)(\lambda-a)\| \int_0^1 \exp(\|(1-s)(1-t)(\lambda-a)\|)\,ds \\ &= (1-t)\|\lambda-a\| \int_0^1 \exp((1-s)(1-t)\|\lambda-a\|)\,ds \\ &= (1-t)\|\lambda-a\| \left[\frac{\exp((1-t)\|\lambda-a\|)}{(1-t)\|\lambda-a\|}\right] = \exp((1-t)\|\lambda-a\|). \end{aligned}$$

Therefore by (5.6) we get

$$\begin{aligned} &\frac{1}{|\beta-\alpha|}\int_\gamma \|a-\lambda\| \left(\int_0^1 t\|\exp((1-t)\lambda+ta)-\exp(a)\|\,dt\right)|d\lambda| \\ &\leq \frac{\|\exp(a)\|}{|\beta-\alpha|}\int_\gamma \|a-\lambda\| \left(\int_0^1 t\exp((1-t)\|\lambda-a\|)\,dt\right)|d\lambda| . \end{aligned} \tag{5.7}$$

Now, since

$$\begin{aligned} &\int_0^1 t\exp((1-t)\|\lambda-a\|)\,dt \\ &= -\frac{1}{\|\lambda-a\|}\int_0^1 t d(\exp((1-t)\|\lambda-a\|)) \\ &= -\frac{1}{\|\lambda-a\|}\left[1-\int_0^1 \exp((1-t)\|\lambda-a\|)\,dt\right] \\ &= \frac{1}{\|\lambda-a\|}\left[\int_0^1 \exp((1-t)\|\lambda-a\|)\,dt-1\right] \\ &= \frac{1}{\|\lambda-a\|}\left[\frac{-1+\exp\|\lambda-a\|}{\|\lambda-a\|}-1\right] \\ &= \frac{\exp\|\lambda-a\|-\|\lambda-a\|-1}{\|\lambda-a\|^2}. \end{aligned}$$

Therefore by (5.7) we get

$$\frac{1}{|\beta-\alpha|}\int_{\gamma}\|a-\lambda\|\left(\int_{0}^{1}t\,\|\exp((1-t)\lambda+ta)-\exp(a)\|\,dt\right)|d\lambda| \leq \frac{\|\exp(a)\|}{|\beta-\alpha|}\int_{\gamma}\frac{\exp\|\lambda-a\|-\|\lambda-a\|-1}{\|\lambda-a\|}|d\lambda|.$$

Therefore, by (5.4) we get

$$\left\|\exp a-\frac{1}{2}\left(a-\frac{\alpha+\beta}{2}\right)\exp a-\frac{1}{\beta-\alpha}\int_{\gamma}(a-\lambda)^{-1}(\exp a-\exp\lambda)\,d\lambda\right\| \leq \frac{\|\exp(a)\|}{|\beta-\alpha|}\int_{\gamma}\frac{\exp\|\lambda-a\|-\|\lambda-a\|-1}{\|\lambda-a\|}|d\lambda|, \tag{5.8}$$

provided $a-\lambda$ is invertible for $\lambda\in\gamma$.

References

[1] M. V. Boldea, Inequalities of Čebyšev type for Lipschitzian functions in Banach algebras. *An. Univ. Vest Timiş. Ser. Mat.-Inform.* **54** (2016), no. 2, 59–74.

[2] M. V. Boldea, S. S. Dragomir and M. Megan, New bounds for Čebyšev functional for power series in Banach algebras via a Grüss-Lupaş type inequality. *PanAmer. Math. J.* **26** (2016), no. 3, 71–88.

[3] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.

[4] S. S. Dragomir, A counterpart of Schwarz's inequality in inner product spaces, *East Asian Math. J.*, **20** (1) (2004), 1-10. Preprint, https://arxiv.org/abs/math/0305373.

[5] S. S. Dragomir, Inequalities for power series in Banach algebras. *SUT J. Math.* **50** (2014), no. 1, 25–45

[6] S. S. Dragomir, Inequalities of Lipschitz type for power series in Banach algebras. *Ann. Math. Sil.* **No**. 29 (2015), 61–83.

[7] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 1, 283 pp.

[8] S. S. Dragomir, Norm inequalities of Ostrowski type for analytic functions in Banach algebras, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art.

[9] S. S. Dragomir, M. V. Boldea and M. Megan, New norm inequalities of Čebyšev type for power series in Banach algebras. *Sarajevo J. Math.* **11** (24) (2015), no. 2, 253–266.

[10] S. S. Dragomir, M. V. Boldea, C. Buşe and M. Megan, Norm inequalities of Čebyšev type for power series in Banach algebras. *J. Inequal. Appl.* **2014**, 2014:294, 19 pp.

[11] S. S. Dragomir, M. V. Boldea and M. Megan, Further bounds for Čebyšev functional for power series in Banach algebras via Grüss-Lupaş type inequalities for p-norms. *Mem. Grad. Sch. Sci. Eng. Shimane Univ. Ser. B Math.* **49** (2016), 15–34.

[12] S. S. Dragomir, M. V. Boldea and M. Megan, Inequalities for Chebyshev functional in Banach algebras. *Cubo* **19** (2017), no. 1, 53–77.

[13] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helv.,* **10** (1938), 226-227.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa