ON QUADRATIC NORM INTEGRAL INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\mu, .\lambda \in \mathbb{C}$ with $\mu \neq \lambda$. If $f : G \to \mathbb{C}$ is analytic on G, then by using the analytic functional calculus, we obtain amongst other the following quadratic inequality

$$\left\| f(a) - \frac{a-\lambda}{\mu-\lambda} \int_0^1 f((1-t)\lambda + ta)\, dt - \frac{\mu - a}{\mu - \lambda} \int_0^1 f((1-t)a + t\mu)\, dt \right\|$$
$$\leq \frac{1}{2} \frac{1}{|\mu - \lambda|} \left[\|a-\lambda\|^2 \sup_{t\in[0,1]} \left\| f'((1-t)\lambda + ta) \right\| \right.$$
$$\left. + \|a-\mu\|^2 \sup_{t\in[0,1]} \left\| f'((1-t)a + t\mu) \right\| \right].$$

Some example for the exponential function of elements in Banach algebras are also provided.

1. Introduction

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \leq \|a\| \|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*. We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\text{Inv}(\mathcal{B})$. If $a, b \in \text{Inv}(\mathcal{B})$ then $ab \in \text{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \text{Inv}(\mathcal{B})$;
(ii) $\{a \in \mathcal{B}: \|1 - b\| < 1\} \subset \text{Inv}(\mathcal{B})$;
(iii) $\text{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\text{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \text{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} : z - a \in \text{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \text{Inv}(\mathcal{B})$, $R_a(z) := (z-a)^{-1}$. For each $z, w \in \rho(a)$ we have the

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identity

$$R_a(w) - R_a(z) = (z - w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \leq \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i} \int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_\delta f(\xi) (\xi - a)^{-1} d\xi, \tag{1.1}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \text{ins}(\delta)$, the inside of δ.

It is well known (see for instance [3, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.2}$$

holds.

Let $\mathfrak{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\mathfrak{Hol}(a)$ is an algebra where if $f, g \in \mathfrak{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f + g$ have domain $D(f) \cap D(g)$. $\mathfrak{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [1, p. 201-203]:

Theorem 1. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) *The map $f \mapsto f(a)$ of $\mathfrak{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.*
(b) *If $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ has radius of convergence $r > \nu(a)$, then $f \in \mathfrak{Hol}(a)$ and $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.*
(c) *If $f(z) \equiv 1$, then $f(a) = 1$.*
(d) *If $f(z) = z$ for all z, $f(a) = a$.*
(e) *If $f, f_1, ..., f_n ...$ are analytic on G, $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\|f_n(a) - f(a)\| \to 0$ as $n \to \infty$.*
(f) *The Riesz Functional Calculus is unique and if a, b are commuting elements in $\mathcal{B}$ and $f \in \mathfrak{Hol}(a)$, then $f(a) b = b f(a)$.*

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t \in [0, 1]$ from $\lambda(0) = \alpha$ to $\lambda(1) = \beta$, with $\beta \neq \alpha$. If $f : G \to \mathbb{C}$ is analytic on G, then by using the analytic functional calculus we obtained in the recent paper [7] the following norm integral inequality of Ostrowski type

$$\left\| f(a) - \frac{1}{\beta-\alpha}\int_\gamma f(\lambda)\,d\lambda \right\| \le K \frac{1}{|\beta-\alpha|}\int_\gamma \|a-\lambda\|\,|d\lambda|, \tag{1.3}$$

provided

$$K := \sup_{(\lambda,t)\in\gamma\times[0,1]} \|f'((1-t)\lambda+ta)\| < \infty.$$

In paper [8] we establish some other similar norm integral inequalities, out of which we state here one of the simplest, namely

$$\left\| f(a) - \int_0^1 f((1-t)\lambda+ta)\,dt \right\| \le \|a-\lambda\| \int_0^1 t\,\|f'(ta+(1-t)\lambda)\|\,dt \tag{1.4}$$
$$\le \|a-\lambda\| \begin{cases} \frac{1}{2}\sup_{t\in[0,1]}\|f'(ta+(1-t)\lambda)\|, \\ \frac{1}{(q+1)^{1/q}}\left(\int_0^1 \|f'(ta+(1-t)\lambda)\|^p\,dt\right)^{1/p}, \\ \int_0^1 \|f'(ta+(1-t)\lambda)\|\,dt, \end{cases}$$

for all $\lambda \in G$. Some example for the exponential function of elements in Banach algebras were also provided, see [8].

For some recent norm inequalities for functions on Banach algebras, see [1]-[2] and [4]-[12].

2. Some Identities

We start with the following integral identity:

Theorem 2. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, then for all λ, $\mu \in G$ with $\mu \ne \lambda$ we have*

$$\begin{aligned} f(a) - \frac{a-\lambda}{\mu-\lambda}\int_0^1 f((1-t)\lambda+ta)\,dt - \frac{\mu-a}{\mu-\lambda}\int_0^1 f((1-t)a+t\mu)\,dt \\ = \frac{(a-\lambda)^2}{\mu-\lambda}\int_0^1 tf'((1-t)\lambda+ta)\,dt \\ - \frac{(a-\mu)^2}{\mu-\lambda}\int_0^1 (1-t)f'((1-t)a+t\mu)\,dt. \end{aligned} \tag{2.1}$$

Proof. We have

$$\int_0^1 t\,(f((1-t)\lambda+t\xi))'\,dt = (\xi-\lambda)\int_0^1 tf'((1-t)\lambda+t\xi)\,dt$$

and

$$\int_0^1 (1-t)\,(f((1-t)\xi+t\mu))'\,dt = (\mu-\xi)\int_0^1 (1-t)f'((1-t)\xi+t\mu)\,dt$$

for all λ, $\xi \in G$.

Integrating by parts, we also have

$$\int_0^1 t\left(f\left((1-t)\lambda+t\xi\right)\right)' dt = tf\left((1-t)\lambda+t\xi\right)\Big|_0^1 - \int_0^1 f\left((1-t)\lambda+t\xi\right) dt$$
$$= f(\xi) - \int_0^1 f\left((1-t)\lambda+t\xi\right) dt$$

and

$$\int_0^1 (1-t)\left(f\left((1-t)\xi+t\mu\right)\right)' dt = (1-t) f\left((1-t)\xi+t\mu\right)\Big|_0^1$$
$$+\int_0^1 f\left((1-t)\xi+t\mu\right) dt = \int_0^1 f\left((1-t)\xi+t\mu\right) dt - f(\xi).$$

Then we get

$$f(\xi) = \int_0^1 f\left((1-t)\lambda+t\xi\right) dt + (\xi-\lambda)\int_0^1 tf'\left((1-t)\lambda+t\xi\right) dt \tag{2.2}$$

and

$$f(\xi) = \int_0^1 f\left((1-t)\xi+t\mu\right) dt + (\xi-\mu)\int_0^1 (1-t) f'\left((1-t)\xi+t\mu\right) dt \tag{2.3}$$

for all λ, ξ, $\mu \in G$.

If we multiply (2.2) by α and (2.3) by $1-\alpha$ and we add the obtained equalities, we get

$$\begin{aligned} f(\xi) = \alpha\int_0^1 f\left((1-t)\lambda+t\xi\right) dt + (1-\alpha)\int_0^1 f\left((1-t)\xi+t\mu\right) dt \\ + \alpha(\xi-\lambda)\int_0^1 tf'\left((1-t)\lambda+t\xi\right) dt \\ + (1-\alpha)(\xi-\mu)\int_0^1 (1-t) f'\left((1-t)\xi+t\mu\right) dt \end{aligned} \tag{2.4}$$

for all λ, ξ, $\mu \in G$ and $\alpha \in \mathbb{C}$.

From the equality (2.4) we get for $\alpha = \frac{\xi-\lambda}{\mu-\lambda}$ that

$$\begin{aligned} f(\xi) = \frac{\xi-\lambda}{\mu-\lambda}\int_0^1 f\left((1-t)\lambda+t\xi\right) dt + \frac{\mu-\xi}{\mu-\lambda}\int_0^1 f\left((1-t)\xi+t\mu\right) dt \\ + \frac{(\xi-\lambda)^2}{\mu-\lambda}\int_0^1 tf'\left((1-t)\lambda+t\xi\right) dt \\ - \frac{(\xi-\mu)^2}{\mu-\lambda}\int_0^1 (1-t) f'\left((1-t)\xi+t\mu\right) dt \end{aligned} \tag{2.5}$$

for all λ, ξ, $\mu \in G$ and $\lambda \neq \mu$.

Now, by using the equality (1.1), we have

$$
\begin{aligned}
f(a) &= \frac{1}{2\pi i}\int_\gamma \frac{\xi-\lambda}{\mu-\lambda}\left(\int_0^1 f((1-t)\lambda+t\xi)\,dt\right)(\xi-a)^{-1}\,d\xi \\
&+\frac{1}{2\pi i}\int_\gamma \frac{\mu-\xi}{\mu-\lambda}\left(\int_0^1 f((1-t)\xi+t\mu)\,dt\right)(\xi-a)^{-1}\,d\xi \\
&+\frac{1}{2\pi i}\int_\gamma \frac{(\xi-\lambda)^2}{\mu-\lambda}\left(\int_0^1 tf'((1-t)\lambda+t\xi)\,dt\right)(\xi-a)^{-1}\,d\xi \\
&-\frac{1}{2\pi i}\int_\gamma \frac{(\xi-\mu)^2}{\mu-\lambda}\left(\int_0^1 (1-t)f'((1-t)\xi+t\mu)\,dt\right)(\xi-a)^{-1}\,d\xi
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
&= \frac{1}{\mu-\lambda}\int_0^1\left(\frac{1}{2\pi i}\int_\gamma (\xi-\lambda)f((1-t)\lambda+t\xi)(\xi-a)^{-1}\,d\xi\right)dt \\
&+\frac{1}{\mu-\lambda}\int_0^1\left(\frac{1}{2\pi i}\int_\gamma (\mu-\xi)f((1-t)\xi+t\mu)(\xi-a)^{-1}\,d\xi\right)dt \\
&+\frac{1}{\mu-\lambda}\int_0^1 t\left(\frac{1}{2\pi i}\int_\gamma (\xi-\lambda)^2 f'((1-t)\lambda+t\xi)(\xi-a)^{-1}\,d\xi\right)dt \\
&-\frac{1}{\mu-\lambda}\int_0^1 (1-t)\left(\frac{1}{2\pi i}\int_\gamma (\xi-\mu)^2 f'((1-t)\xi+t\mu)(\xi-a)^{-1}\,d\xi\right)dt
\end{aligned}
$$

for all λ, $\mu\in G$ and $\lambda\neq\mu$, where for the last equality we used Fubini's theorem.

Since, by the analytic functional calculus for composite functions, we have

$$\frac{1}{2\pi i}\int_\gamma (\xi-\lambda)f((1-t)\lambda+t\xi)(\xi-a)^{-1}\,d\xi=(a-\lambda)f((1-t)\lambda+ta),$$

$$\frac{1}{2\pi i}\int_\gamma (\mu-\xi)f((1-t)\xi+t\mu)(\xi-a)^{-1}\,d\xi=(\mu-a)f((1-t)a+t\mu),$$

$$\frac{1}{2\pi i}\int_\gamma (\xi-\lambda)^2 f'((1-t)\lambda+t\xi)(\xi-a)^{-1}\,d\xi=(a-\lambda)^2 f'((1-t)\lambda+ta)$$

and

$$\frac{1}{2\pi i}\int_\gamma (\xi-\mu)^2 f'((1-t)\xi+t\mu)(\xi-a)^{-1}\,d\xi=(a-\mu)^2 f'((1-t)a+t\mu),$$

hence by (2.6) we get the desired result (2.1). □

We have the following perturbed identity:

Corollary 1. *With the assumptions of Theorem 2 and for any $b \in \mathcal{B}$ we have*

$$\begin{aligned}(2.7)\quad & f(a) + \left(a - \frac{\mu+\lambda}{2}\right) b \\ & - \frac{a-\lambda}{\mu-\lambda}\int_0^1 f((1-t)\lambda + ta)\,dt - \frac{\mu-a}{\mu-\lambda}\int_0^1 f((1-t)a + t\mu)\,dt \\ & = \frac{(a-\lambda)^2}{\mu-\lambda}\int_0^1 t\left[f'((1-t)\lambda + ta) - b\right]dt \\ & \quad - \frac{(a-\mu)^2}{\mu-\lambda}\int_0^1 (1-t)\left[f'((1-t)a + t\mu) - b\right]dt.\end{aligned}$$

In particular, we have

$$\begin{aligned}(2.8)\quad & f(a) + \left(a - \frac{\mu+\lambda}{2}\right) f'(a) \\ & - \frac{a-\lambda}{\mu-\lambda}\int_0^1 f((1-t)\lambda + ta)\,dt - \frac{\mu-a}{\mu-\lambda}\int_0^1 f((1-t)a + t\mu)\,dt \\ & = \frac{(a-\lambda)^2}{\mu-\lambda}\int_0^1 t\left[f'((1-t)\lambda + ta) - f'(a)\right]dt \\ & \quad - \frac{(a-\mu)^2}{\mu-\lambda}\int_0^1 (1-t)\left[f'((1-t)a + t\mu) - f'(a)\right]dt.\end{aligned}$$

Proof. Observe that

$$\begin{aligned}& \frac{(a-\lambda)^2}{\mu-\lambda}\int_0^1 t\left[f'((1-t)\lambda + ta) - b\right]dt \\ & = \frac{(a-\lambda)^2}{\mu-\lambda}\int_0^1 tf'((1-t)\lambda + ta) - \frac{(a-\lambda)^2}{2(\mu-\lambda)}b\end{aligned}$$

and

$$\begin{aligned}& \frac{(a-\mu)^2}{\mu-\lambda}\int_0^1 (1-t)\left[f'((1-t)a + t\mu) - b\right]dt \\ & = \frac{(a-\mu)^2}{\mu-\lambda}\int_0^1 (1-t)f'((1-t)a + t\mu)\,dt - \frac{(a-\mu)^2}{2(\mu-\lambda)}b.\end{aligned}$$

Therefore

$$\begin{aligned}& \frac{(a-\lambda)^2}{\mu-\lambda}\int_0^1 t\left[f'((1-t)\lambda + ta) - b\right]dt \\ & \quad - \frac{(a-\mu)^2}{\mu-\lambda}\int_0^1 (1-t)\left[f'((1-t)a + t\mu) - b\right]dt \\ & = \frac{(a-\lambda)^2}{\mu-\lambda}\int_0^1 tf'((1-t)\lambda + ta) - \frac{(a-\lambda)^2}{2(\mu-\lambda)}b \\ & \quad - \frac{(a-\mu)^2}{\mu-\lambda}\int_0^1 (1-t)f'((1-t)a + t\mu)\,dt + \frac{(a-\mu)^2}{2(\mu-\lambda)}b\end{aligned}$$

$$= \frac{(a-\lambda)^2}{\mu-\lambda}\int_0^1 tf'((1-t)\lambda+ta) - \frac{(a-\mu)^2}{\mu-\lambda}\int_0^1 (1-t) f'((1-t)a+t\mu)\,dt + \frac{1}{2(\mu-\lambda)}\left[(a-\mu)^2-(a-\lambda)^2\right]b$$
$$= \frac{(a-\lambda)^2}{\mu-\lambda}\int_0^1 tf'((1-t)\lambda+ta) - \frac{(a-\mu)^2}{\mu-\lambda}\int_0^1 (1-t) f'((1-t)a+t\mu)\,dt + \left(\frac{\mu+\lambda}{2}-a\right)b,$$

which proves the desired identity (2.7). □

3. Norm Inequalities

We have the norm inequalities:

Theorem 3. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic on G, then for all λ, $\mu\in G$ with $\mu\neq\lambda$ we have the norm inequality*

$$\left\|f(a) - \frac{a-\lambda}{\mu-\lambda}\int_0^1 f((1-t)\lambda+ta)\,dt - \frac{\mu-a}{\mu-\lambda}\int_0^1 f((1-t)a+t\mu)\,dt\right\| \tag{3.1}$$
$$\leq \frac{1}{|\mu-\lambda|}\|a-\lambda\|^2\int_0^1 t\,\|f'((1-t)\lambda+ta)\|\,dt + \frac{1}{|\mu-\lambda|}\|a-\mu\|^2\int_0^1 (1-t)\,\|f'((1-t)a+t\mu)\|\,dt$$
$$\leq \frac{1}{|\mu-\lambda|}\|a-\lambda\|^2\begin{cases}\frac{1}{2}\sup_{t\in[0,1]}\|f'((1-t)\lambda+ta)\| \\ \frac{1}{(q+1)^{1/q}}\left(\int_0^1\|f'((1-t)\lambda+ta)\|^p\,dt\right)^{1/p} \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \int_0^1\|f'((1-t)\lambda+ta)\|\,dt\end{cases}$$
$$+\frac{1}{|\mu-\lambda|}\|a-\mu\|^2\begin{cases}\frac{1}{2}\sup_{t\in[0,1]}\|f'((1-t)a+t\mu)\| \\ \frac{1}{(q+1)^{1/q}}\left(\int_0^1\|f'((1-t)a+t\mu)\|^p\,dt\right)^{1/p} \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \int_0^1\|f'((1-t)a+t\mu)\|\,dt.\end{cases}$$

Proof. By taking the norm in the identity (2.1) we get

$$\left\|f(a) - \frac{a-\lambda}{\mu-\lambda}\int_0^1 f((1-t)\lambda+ta)\,dt - \frac{\mu-a}{\mu-\lambda}\int_0^1 f((1-t)a+t\mu)\,dt\right\| \tag{3.2}$$
$$\leq \left\|\frac{(a-\lambda)^2}{\mu-\lambda}\int_0^1 tf'((1-t)\lambda+ta)\,dt\right\|$$

$$+\left\|\frac{(a-\mu)^2}{\mu-\lambda}\int_0^1(1-t)\,f'((1-t)\,a+t\mu)\,dt\right\|$$
$$\leq\frac{1}{|\mu-\lambda|}\left\|(a-\lambda)^2\right\|\left\|\int_0^1 tf'((1-t)\,\lambda+ta)\,dt\right\|$$
$$+\frac{1}{|\mu-\lambda|}\left\|(a-\mu)^2\right\|\left\|\int_0^1(1-t)\,f'((1-t)\,a+t\mu)\,dt\right\|$$
$$\leq\frac{1}{|\mu-\lambda|}\|a-\lambda\|^2\int_0^1 t\,\|f'((1-t)\,\lambda+ta)\|\,dt$$
$$+\frac{1}{|\mu-\lambda|}\|a-\mu\|^2\int_0^1(1-t)\,\|f'((1-t)\,a+t\mu)\|\,dt=:A.$$

Using Hölder's integral inequality, we have

$$\int_0^1 t\,\|f'((1-t)\,\lambda+ta)\|\,dt$$
$$\leq\begin{cases}\sup_{t\in[0,1]}\|f'((1-t)\,\lambda+ta)\|\int_0^1 tdt\\ \\ \left(\int_0^1\|f'((1-t)\,\lambda+ta)\|^p\,dt\right)^{1/p}\left(\int_0^1 t^q dt\right)^{1/q}\\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \max_{t\in[0,1]}\{t\}\int_0^1\|f'((1-t)\,\lambda+ta)\|\,dt\end{cases}$$
$$=\begin{cases}\frac{1}{2}\sup_{t\in[0,1]}\|f'((1-t)\,\lambda+ta)\|\\ \\ \frac{1}{(q+1)^{1/q}}\left(\int_0^1\|f'((1-t)\,\lambda+ta)\|^p\,dt\right)^{1/p}\\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \int_0^1\|f'((1-t)\,\lambda+ta)\|\,dt\end{cases}$$

and

$$\int_0^1(1-t)\,\|f'((1-t)\,a+t\mu)\|\,dt$$
$$\leq\begin{cases}\frac{1}{2}\sup_{t\in[0,1]}\|f'((1-t)\,a+t\mu)\|\\ \\ \frac{1}{(q+1)^{1/q}}\left(\int_0^1\|f'((1-t)\,a+t\mu)\|^p\,dt\right)^{1/p}\\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \int_0^1\|f'((1-t)\,a+t\mu)\|\,dt,\end{cases}$$

which implies that

$$A \leq \frac{1}{|\mu-\lambda|}\|a-\lambda\|^{2}\begin{cases}\frac{1}{2}\sup_{t\in[0,1]}\|f'((1-t)\lambda+ta)\| \\ \\ \frac{1}{(q+1)^{1/q}}\left(\int_0^1\|f'((1-t)\lambda+ta)\|^{p}dt\right)^{1/p} \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_0^1\|f'((1-t)\lambda+ta)\|dt\end{cases}$$
$$+\frac{1}{|\mu-\lambda|}\|a-\mu\|^{2}\begin{cases}\frac{1}{2}\sup_{t\in[0,1]}\|f'((1-t)a+t\mu)\| \\ \\ \frac{1}{(q+1)^{1/q}}\left(\int_0^1\|f'((1-t)a+t\mu)\|^{p}dt\right)^{1/p} \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_0^1\|f'((1-t)a+t\mu)\|dt.\end{cases} \tag{3.3}$$

On utilising (3.2) and (3.3) we get the desired result (3.1). □

Theorem 4. *With the assumptions of Theorem 3 and if there exists $L_{\lambda,a}$, $L_{\mu,a}>0$ such that*

$$\|f'((1-t)\lambda+ta)-f'(a)\|\leq(1-t)L_{\lambda,a}\|a-\lambda\| \tag{3.4}$$

and

$$\|f'((1-t)a+t\mu)-f'(a)\|\leq tL_{\mu,a}\|a-\mu\| \tag{3.5}$$

for all $t\in[0,1]$, then

$$\left\|f(a)+\left(a-\frac{\mu+\lambda}{2}\right)f'(a)\right.$$
$$\left.-\frac{a-\lambda}{\mu-\lambda}\int_0^1 f((1-t)\lambda+ta)\,dt-\frac{\mu-a}{\mu-\lambda}\int_0^1 f((1-t)a+t\mu)\,dt\right\|$$
$$\leq\frac{1}{6}\frac{1}{|\mu-\lambda|}\left[L_{\lambda,a}\|a-\lambda\|^{3}+L_{\mu,a}\|a-\mu\|^{3}\right]. \tag{3.6}$$

Proof. Using the identity (2.8) we have

$$\left\|f(a)+\left(a-\frac{\mu+\lambda}{2}\right)f'(a)\right.$$
$$\left.-\frac{a-\lambda}{\mu-\lambda}\int_0^1 f((1-t)\lambda+ta)\,dt-\frac{\mu-a}{\mu-\lambda}\int_0^1 f((1-t)a+t\mu)\,dt\right\| \tag{3.7}$$

$$
\begin{aligned}
&\leq \left\| \frac{(a-\lambda)^2}{\mu-\lambda} \int_0^1 t\left[f'\left((1-t)\lambda+ta\right)-f'(a)\right]dt \right\| \\
&\qquad + \left\| \frac{(a-\mu)^2}{\mu-\lambda} \int_0^1 (1-t)\left[f'\left((1-t)a+t\mu\right)-f'(a)\right]dt \right\| \\
&\leq \frac{1}{|\mu-\lambda|} \left\|(a-\lambda)^2\right\| \left\| \int_0^1 t\left[f'\left((1-t)\lambda+ta\right)-f'(a)\right]dt \right\| \\
&\quad + \frac{1}{|\mu-\lambda|} \left\|(a-\mu)^2\right\| \left\| \int_0^1 (1-t)\left[f'\left((1-t)a+t\mu\right)-f'(a)\right]dt \right\| \\
&\leq \frac{1}{|\mu-\lambda|} \|a-\lambda\|^2 \int_0^1 t\left\|\left[f'\left((1-t)\lambda+ta\right)-f'(a)\right]\right\| dt \\
&\quad + \frac{1}{|\mu-\lambda|} \|a-\mu\|^2 \int_0^1 (1-t)\left\|\left[f'\left((1-t)a+t\mu\right)-f'(a)\right]\right\| dt =: B.
\end{aligned}
$$

Since by (3.4) and (3.5) we have

$$
\begin{aligned}
\int_0^1 t\left\|\left[f'\left((1-t)\lambda+ta\right)-f'(a)\right]\right\| dt &\leq L_{\lambda,a}\|a-\lambda\| \int_0^1 t(1-t)\,dt \\
&= \frac{1}{6} L_{\lambda,a}\|a-\lambda\|
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 (1-t)\left\|\left[f'\left((1-t)a+t\mu\right)-f'(a)\right]\right\| dt &\leq L_{\mu,a}\|a-\mu\| \int_0^1 (1-t)\,tdt \\
&= \frac{1}{6} L_{\mu,a}\|a-\mu\|,
\end{aligned}
$$

hence

$$
B \leq \frac{1}{6}\frac{1}{|\mu-\lambda|}\left[L_{\lambda,a}\|a-\lambda\|^3 + L_{\mu,a}\|a-\mu\|^3\right]
$$

and by (3.7) we get the desired result (3.6). □

4. Examples for Exponential Function

Let $\mathcal{B}$ be a unital Banach algebra, $b \in \mathcal{B}$ and the exponential of b defined by

$$
\exp b := \sum_{n=0}^{\infty} \frac{1}{n!} b^n.
$$

We observe that, if b is invertible, then

$$
\begin{aligned}
\int_0^1 \exp(tb)\,dt &= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{1}{n!} t^n b^n\right) dt = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int_0^1 t^n dt\right) b^n \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} b^n = b^{-1}(\exp b - 1).
\end{aligned}
$$

Therefore for any $\lambda \in \mathbb{C}$ and for $a \in \mathcal{B}$ such that $a - \lambda$ is invertible, then we have

$$\begin{aligned}\int_0^1 \exp\left((1-t)\lambda + ta\right) dt &= \exp\lambda \int_0^1 \exp\left[t(a-\lambda)\right] dt \\ &= (\exp\lambda)(a-\lambda)^{-1}(\exp(a-\lambda)-1) \\ &= (a-\lambda)^{-1}(\exp a - \exp\lambda).\end{aligned}$$

and if $a - \mu$ is invertible, then

$$\int_0^1 f\left((1-t)a + t\mu\right) dt = \int_0^1 f\left(ta + (1-t)\mu\right) dt = (a-\mu)^{-1}(\exp a - \exp\mu).$$

Since

$$\begin{aligned}\|\exp(ta + (1-t)\lambda)\| &= \|\exp[(1-t)\lambda]\exp(ta)\| = |\exp[(1-t)\lambda]|\,\|\exp(ta)\| \\ &= \exp[(1-t)\operatorname{Re}\lambda]\,\|\exp(ta)\| \le \exp[(1-t)\operatorname{Re}\lambda]\exp(t\|a\|) \\ &= \exp[(1-t)\operatorname{Re}\lambda + t\|a\|]\end{aligned}$$

hence, for $\|a\| \neq \operatorname{Re}\lambda$

$$\begin{aligned}\int_0^1 t\,\|\exp(ta + (1-t)\lambda)\|\,dt &\le \int_0^1 t\exp[(1-t)\operatorname{Re}\lambda + t\|a\|]\,dt \\ &= \int_0^1 t\exp[t(\|a\| - \operatorname{Re}\lambda) + \operatorname{Re}\lambda]\,dt \\ &= \frac{1}{\|a\| - \operatorname{Re}\lambda}\int_0^1 td\left(\exp[t(\|a\| - \operatorname{Re}\lambda) + \operatorname{Re}\lambda]\right) \\ &= \frac{1}{\|a\| - \operatorname{Re}\lambda}\left[\exp\|a\| - \frac{\exp\|a\| - \operatorname{Re}\lambda}{\|a\| - \operatorname{Re}\lambda}\right] \\ &= \|a-\lambda\|\left[\frac{(\|a\| - \operatorname{Re}\lambda - 1)\exp\|a\| + \operatorname{Re}\lambda}{(\|a\| - \operatorname{Re}\lambda)^2}\right].\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^1 (1-t)\,\|\exp((1-t)a + t\mu)\|\,dt &\le \int_0^1 (1-t)\exp[(1-t)\|a\| + t\operatorname{Re}\lambda]\,dt \\ &\le \|a-\mu\|\left[\frac{(\|a\| - \operatorname{Re}\mu - 1)\exp\|a\| + \operatorname{Re}\mu}{(\|a\| - \operatorname{Re}\mu)^2}\right]\end{aligned}$$

provided $\|a\| \neq \operatorname{Re}\mu$.

Then from (3.1) we get

$$\begin{aligned}&\left\|\exp a - \frac{\exp\mu - \exp\lambda}{\mu - \lambda}\right\| \\ &\le \frac{1}{|\mu-\lambda|}\|a-\lambda\|^3\left[\frac{(\|a\| - \operatorname{Re}\lambda - 1)\exp\|a\| + \operatorname{Re}\lambda}{(\|a\| - \operatorname{Re}\lambda)^2}\right] \\ &\quad + \frac{1}{|\mu-\lambda|}\|a-\mu\|^3\left[\frac{(\|a\| - \operatorname{Re}\mu - 1)\exp\|a\| + \operatorname{Re}\mu}{(\|a\| - \operatorname{Re}\mu)^2}\right]\end{aligned} \tag{4.1}$$

for $a - \lambda$, $a - \mu$ invertible, $\|a\| \neq \operatorname{Re}\lambda$, $\|a\| \neq \operatorname{Re}\mu$ and $\mu \neq \lambda$.

We observe also that

$$(1-t)\operatorname{Re}\lambda + t\|a\| \leq \max\{\operatorname{Re}\lambda, \|a\|\}$$

giving that

$$\exp[(1-t)\operatorname{Re}\lambda + t\|a\|] \leq \exp[\max\{\operatorname{Re}\lambda, \|a\|\}]$$

implying

$$\int_0^1 t\exp[(1-t)\operatorname{Re}\lambda + t\|a\|]\,dt \leq \frac{1}{2}\exp[\max\{\operatorname{Re}\lambda, \|a\|\}]$$

and, similarly

$$\int_0^1 (1-t)\exp[(1-t)\|a\| + t\operatorname{Re}\mu]\,dt \leq \frac{1}{2}\exp[\max\{\operatorname{Re}\mu, \|a\|\}].$$

Therefore by (3.1) we get the simpler inequality

$$\begin{aligned}(4.2)\quad &\left\|\exp a - \frac{\exp\mu - \exp\lambda}{\mu-\lambda}\right\| \\ &\leq \frac{1}{2}\frac{1}{|\mu-\lambda|}\Big\{\|a-\lambda\|^2\exp[\max\{\operatorname{Re}\lambda, \|a\|\}] \\ &\qquad + \|a-\mu\|^2\exp[\max\{\operatorname{Re}\mu, \|a\|\}]\Big\} \\ &\leq \frac{1}{2}\frac{1}{|\mu-\lambda|}\exp[\max\{\operatorname{Re}\lambda, \operatorname{Re}\mu, \|a\|\}]\Big\{\|a-\lambda\|^2 + \|a-\mu\|^2\Big\}\end{aligned}$$

for $a-\lambda$, $a-\mu$ invertible and $\mu \neq \lambda$.

References

[1] M. V. Boldea, Inequalities of Čebyšev type for Lipschitzian functions in Banach algebras. *An. Univ. Vest Timiş. Ser. Mat.-Inform.* **54** (2016), no. 2, 59–74.

[2] M. V. Boldea, S. S. Dragomir and M. Megan, New bounds for Čebyšev functional for power series in Banach algebras via a Grüss-Lupaş type inequality. *PanAmer. Math. J.* **26** (2016), no. 3, 71–88.

[3] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.

[4] S. S. Dragomir, Inequalities for power series in Banach algebras. *SUT J. Math.* **50** (2014), no. 1, 25–45

[5] S. S. Dragomir, Inequalities of Lipschitz type for power series in Banach algebras. *Ann. Math. Sil.* **No**. 29 (2015), 61–83.

[6] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 1, 283 pp.

[7] S. S. Dragomir, Norm inequalities of Ostrowski type for analytic functions in Banach algebras, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art.

[8] S. S. Dragomir, Some norm integral inequalities for analytic functions in Banach algebras, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art.

[9] S. S. Dragomir, M. V. Boldea and M. Megan, New norm inequalities of Čebyšev type for power series in Banach algebras. *Sarajevo J. Math.* **11** (24) (2015), no. 2, 253–266.

[10] S. S. Dragomir, M. V. Boldea, C. Buşe and M. Megan, Norm inequalities of Čebyšev type for power series in Banach algebras. *J. Inequal. Appl.* **2014**, 2014:294, 19 pp.

[11] S. S. Dragomir, M. V. Boldea and M. Megan, Further bounds for Čebyšev functional for power series in Banach algebras via Grüss-Lupaş type inequalities for *p*-norms. *Mem. Grad. Sch. Sci. Eng. Shimane Univ. Ser. B Math.* **49** (2016), 15–34.

[12] S. S. Dragomir, M. V. Boldea and M. Megan, Inequalities for Chebyshev functional in Banach algebras. *Cubo* **19** (2017), no. 1, 53–77.

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