Advanced Complex fractional Ostrowski inequalities

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Abstract

Here we present very general and advanced fractional complex analytic inequalities of the Ostrowski type.

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1 Introduction

Here we follow [5].

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt. \quad (1)$$

We observe that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz, \quad (2)$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt \quad (3)$$

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and the length of the curve γ is then

$$l(\gamma)=\int_{\gamma_{u,w}}|dz|:=\int_a^b |z'(t)|\,dt. \tag{4}$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma\subset G$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$. Then we have the integration by parts formula

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz=f(w)\,g(w)-f(u)\,g(u)-\int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{5}$$

We recall also the triangle inequality for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right|\le\int_\gamma |f(z)|\,|dz|\le \|f\|_{\gamma,\infty}\,l(\gamma), \tag{6}$$

where $\|f\|_{\gamma,\infty}:=\sup_{z\in\gamma}|f(z)|$.

We also define the p-norm with $p\ge 1$ by

$$\|f\|_{\gamma,p}:=\left(\int_\gamma |f(z)|^p\,|dz|\right)^{\frac{1}{p}}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1}:=\int_\gamma |f(z)|\,|dz|.$$

If $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1}\le [l(\gamma)]^{\frac{1}{q}}\,\|f\|_{\gamma,p}. \tag{7}$$

A motivation to our work follows: These are two complex Opial type inequalities.

Theorem 1 *([5]) Let f be analytic in G, a domain of complex numbers and suppose $\gamma\subset G$ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ from $z(a)=u$ to $z(b)=w$ and $z'(t)\neq 0$ for $t\in(a,b)$.*

(i) If $f(u)=0$ or $f(w)=0$, then

$$\begin{aligned}\int_\gamma |f(z)\,f'(z)|\,|dz| &\le \left(\int_\gamma l(\gamma_{u,z})\,|f'(z)|^2\,|dz|\right)^{\frac{1}{2}}\left(\int_\gamma l(\gamma_{z,w})\,|f'(z)|^2\,|dz|\right)^{\frac{1}{2}}\\ &\le \frac{1}{2}l(\gamma_{u,w})\int_\gamma |f'(z)|^2\,|dz|.\end{aligned} \tag{8}$$

(ii) If $f(u)=f(w)=0$, then

$$\int_\gamma |f(z)\,f'(z)|\,|dz|\le$$

$$\frac{1}{2}\left[\int_{\gamma}\left(l\left(\gamma_{u,w}\right)-\left|l\left(\gamma_{u,z}\right)-l\left(\gamma_{z,w}\right)\right|\right)\left|f'\left(z\right)\right|^{2}\left|dz\right|\right]^{\frac{1}{2}}\cdot \tag{9}$$

$$\left[\int_{\gamma}\left|l\left(\gamma_{u,z}\right)-l\left(\gamma_{z,w}\right)\right|\left|f'\left(z\right)\right|^{2}\left|dz\right|\right]^{\frac{1}{2}}$$

$$\leq\frac{1}{4}l\left(\gamma_{u,w}\right)\int_{\gamma}\left|f'\left(z\right)\right|^{2}\left|dz\right|.$$

In this article we utilize on $\mathbb{C}$ the results of [1] related to Ostrowski type inequalities for general Banach space valued functions. So we produce here advanced and general complex Ostrowski type inequalities.

2 Background

Here we follow [1].

We need

Definition 2 *([1]) Let $[a,b]\subset\mathbb{R}$, $(X,\|\cdot\|)$ a Banach space, $g\in C^{1}([a,b])$ and increasing, $f\in C([a,b],X)$, $\nu>0$.*

We define the left Riemann-Liouville generalized fractional Bochner integral operator

$$\left(J_{a;g}^{\nu}f\right)(x):=\frac{1}{\Gamma(\nu)}\int_{a}^{x}\left(g(x)-g(z)\right)^{\nu-1}g'(z)f(z)\,dz, \tag{10}$$

$\forall\ x\in[a,b]$, where Γ is the gamma function.

The last integral is of Bochner type ([6]). Since $f\in C([a,b],X)$, then $f\in L_{\infty}([a,b],X)$. By ([1]) we get that $\left(J_{a;g}^{\nu}f\right)\in C([a,b],X)$. Above we set $J_{a;g}^{0}f:=f$ and see that $\left(J_{a;g}^{\nu}f\right)(a)=0$.

We mention

Theorem 3 *([1]) Let all as in Definition 2. Let $m,n>0$ and $f\in C([a,b],X)$. Then*

$$J_{a;g}^{m}J_{a;g}^{n}f=J_{a;g}^{m+n}f=J_{a;g}^{n}J_{a;g}^{m}f. \tag{11}$$

We need

Definition 4 *([1]) Let $[a,b]\subset\mathbb{R}$, $(X,\|\cdot\|)$ a Banach space, $g\in C^{1}([a,b])$ and increasing, $f\in C([a,b],X)$, $\nu>0$.*

We define the right Riemann-Liouville generalized fractional Bochner integral operator

$$\left(J_{b-;g}^{\nu}f\right)(x):=\frac{1}{\Gamma(\nu)}\int_{x}^{b}\left(g(z)-g(x)\right)^{\nu-1}g'(z)f(z)\,dz, \tag{12}$$

$\forall\ x\in[a,b]$, where Γ is the gamma function.

The last integral is of Bochner type. Since $f \in C([a,b], X)$, then $f \in L_\infty([a,b], X)$. By ([1]) we get that $\left(J_{b-;g}^{\nu} f\right) \in C([a,b], X)$. Above we set $J_{b-;g}^{0} f := f$ and see that $\left(J_{b-;g}^{\nu} f\right)(b) = 0$.

We mention

Theorem 5 *([1]) Let all as in Definition 4. Let $\alpha, \beta > 0$ and $f \in C([a,b], X)$. Then*

$$\left(J_{b-;g}^{\alpha} J_{b-;g}^{\beta} f\right)(x) = \left(J_{b-;g}^{\alpha+\beta} f\right)(x) = \left(J_{b-;g}^{\beta} J_{b-;g}^{\alpha} f\right)(x), \tag{13}$$

$\forall\, x \in [a,b]$.

We need

Definition 6 *([1]) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$.*

We define the left generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{a+;g}^{\alpha} f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{14}$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type.

Derivatives for vector valued functions are defined according to [8], p. 83, similar to numerical ones.

If $\alpha \notin \mathbb{N}$, by [1], we have that $\left(D_{a+;g}^{\alpha} f\right) \in C([a,b], X)$.

We see that

$$\left(J_{a;g}^{n-\alpha}\left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(D_{a+;g}^{\alpha} f\right)(x), \quad \forall\, x \in [a,b]. \tag{15}$$

We set

$$D_{a+;g}^{n} f(x) := \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C([a,b], X), \quad n \in \mathbb{N}, \tag{16}$$

$$D_{a+;g}^{0} f(x) = f(x), \quad \forall\, x \in [a,b].$$

When $g = id$, then

$$D_{a+;g}^{\alpha} f = D_{a+;id}^{\alpha} f = D_{*a}^{\alpha} f, \tag{17}$$

the usual left X-valued Caputo fractional derivative, see [2].

We need

Definition 7 *([1]) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$.*

We define the right generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{b-;g}^{\alpha}f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t)-g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{18}$$

$\forall\ x \in [a,b]$. The last integral is of Bochner type.

If $\alpha \notin \mathbb{N}$, by [1], we have that $\left(D_{b-;g}^{\alpha}f\right) \in C([a,b],X)$.

We see that

$$J_{b-;g}^{n-\alpha}\left((-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x) = \left(D_{b-;g}^{\alpha}f\right)(x), \quad a \leq x \leq b. \tag{19}$$

We set

$$D_{b-;g}^{n}f(x) := (-1)^n \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C([a,b],X), \quad n \in \mathbb{N}, \tag{20}$$

$$D_{b-;g}^{0}f(x) := f(x), \quad \forall\ x \in [a,b].$$

When $g = id$, then

$$D_{b-;g}^{\alpha}f(x) = D_{b-;id}^{\alpha}f(x) = D_{b-}^{\alpha}f, \tag{21}$$

the usual right X-valued Caputo fractional derivative, see [3], [4].

We mention the following general left fractional Taylor's formula:

Theorem 8 *([1]) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a),g(b)])$, $a \leq x \leq b$. Then*

$$\begin{gathered} f(x) = f(a) + \sum_{i=1}^{n-1} \frac{(g(x)-g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) + \\ \frac{1}{\Gamma(\alpha)} \int_a^x (g(x)-g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^{\alpha}f\right)(t)\, dt = \\ f(a) + \sum_{i=1}^{n-1} \frac{(g(x)-g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) + \\ \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x)-z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha}f\right) \circ g^{-1}\right)(z)\, dz. \end{gathered} \tag{22}$$

We also mention the following general right fractional Taylor's formula:

Theorem 9 *([1]) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a),g(b)])$, $a \leq x \leq b$. Then*

$$f(x) = f(b) + \sum_{i=1}^{n-1} \frac{(g(x)-g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) +$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1} g'(t)\left(D_{b-;g}^{\alpha}f\right)(t)\,dt =$$

$$f(b)+\sum_{i=1}^{n-1}\frac{(g(x)-g(b))^i}{i!}\left(f\circ g^{-1}\right)^{(i)}(g(b))+ \tag{23}$$

$$\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)} (z-g(x))^{\alpha-1}\left(\left(D_{b-;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz.$$

From Theorem 8 when $0<\alpha\leq 1$, we get that

$$\left(I_{a+;g}^{\alpha}D_{a+;g}^{\alpha}f\right)(x)=f(x)-f(a)=$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1} g'(t)\left(D_{a+;g}^{\alpha}f\right)(t)\,dt = \tag{24}$$

$$\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)} (g(x)-z)^{\alpha-1}\left(\left(D_{a+;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz,$$

and by Theorem 9 when $0<\alpha\leq 1$ we get

$$\left(I_{b-;g}^{\alpha}D_{b-;g}^{\alpha}f\right)(x)=f(x)-f(b)=$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1} g'(t)\left(D_{b-;g}^{\alpha}f\right)(t)\,dt = \tag{25}$$

$$\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)} (z-g(x))^{\alpha-1}\left(\left(D_{b-;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz,$$

all $a\leq x\leq b$.

Above we considered $f\in C^1([a,b],X)$, $g\in C^1([a,b])$, strictly increasing, such that $g^{-1}\in C^1([g(a),g(b)])$.

Denote by

$$D_{a+;g}^{n\alpha}:=D_{a+;g}^{\alpha}D_{a+;g}^{\alpha}...D_{a+;g}^{\alpha}\quad (n \text{ times}),\ n\in\mathbb{N}. \tag{26}$$

Also denote by

$$I_{a+;g}^{n\alpha}:=I_{a+;g}^{\alpha}I_{a+;g}^{\alpha}...I_{a+;g}^{\alpha}\quad (n \text{ times}), \tag{27}$$

and remind

$$\left(I_{a+;g}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1} g'(t)\,f(t)\,dt,\quad x\geq a. \tag{28}$$

By convention $I_{a+;g}^{0}=D_{a+;g}^{0}=I$ (identity operator).

We mention the following g-left generalized modified X-valued Taylor's formula.

Theorem 10 *([1]) Let* $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, *strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$. *Let* $F_k := D_{a+;g}^{k\alpha} f$, $k = 1, ..., n$, *that fulfill* $F_k \in C^1([a,b], X)$, *and* $F_{n+1} \in C([a,b], X)$.

Then

$$f(x) = \sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{a+;g}^{i\alpha} f\right)(a) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D_{a+;g}^{(n+1)\alpha} f\right)(t)\, dt, \tag{29}$$

$\forall\, x \in [a,b]$.

Denote by

$$D_{b-;g}^{n\alpha} := D_{b-;g}^{\alpha} D_{b-;g}^{\alpha} ... D_{b-;g}^{\alpha} \quad (n \text{ times}),\ n \in \mathbb{N}. \tag{30}$$

Also denote by

$$I_{b-;g}^{n\alpha} := I_{b-;g}^{\alpha} I_{b-;g}^{\alpha} ... I_{b-;g}^{\alpha} \quad (n \text{ times}), \tag{31}$$

and remind

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha - 1} g'(t) f(t)\, dt, \quad x \leq b. \tag{32}$$

We also mention the following g-right generalized modified X-valued Taylor's formula.

Theorem 11 *([1]) Let* $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, *strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$. *Suppose that* $F_k := D_{b-;g}^{k\alpha} f$, $k = 1, ..., n$, *fulfill* $F_k \in C^1([a,b], X)$, *and* $F_{n+1} \in C([a,b], X)$, *where* $0 < \alpha \leq 1$, $n \in \mathbb{N}$.

Then

$$f(x) = \sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{b-;g}^{i\alpha} f\right)(b) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left(D_{b-;g}^{(n+1)\alpha} f\right)(t)\, dt, \tag{33}$$

$\forall\, x \in [a,b]$.

Next we refer to a related generalized fractional Ostrowski type inequality:

Theorem 12 *([1]) Let* $g \in C^1([a,b])$ *and strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$, *and* $0 < \alpha < 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, *where* $(X, \|\cdot\|)$ *is a Banach space. Let* $x_0 \in [a,b]$ *be fixed. Assume that* $F_k^{x_0} := D_{x_0-;g}^{k\alpha} f$, *for* $k = 1, ..., n$, *fulfill* $F_k^{x_0} \in C^1([a,b], X)$ *and* $F_{n+1}^{x_0} \in C([a, x_0], X)$ *and* $\left(D_{x_0-;g}^{i\alpha} f\right)(x_0) = 0$, $i = 1, ..., n$.

Similarly, we assume that $G_k^{x_0} := D_{x_0+;g}^{k\alpha} f$, *for* $k = 1, ..., n$, *fulfill* $G_k^{x_0} \in C^1([x_0, b], X)$ *and* $G_{n+1}^{x_0} \in C([x_0, b], X)$ *and* $\left(D_{x_0+;g}^{i\alpha} f\right)(x_0) = 0$, $i = 1, ..., n$.

Then

$$\left\| \frac{1}{b-a} \int_a^b f(x)\,dx - f(x_0) \right\| \leq \frac{1}{(b-a)\,\Gamma((n+1)\alpha+1)} \cdot$$

$$\left\{ (g(b) - g(x_0))^{(n+1)\alpha} (b - x_0) \left\| D_{x_0+;g}^{(n+1)\alpha} f \right\|_{\infty,[x_0,b]} + \right.$$

$$\left. (g(x_0) - g(a))^{(n+1)\alpha} (x_0 - a) \left\| D_{x_0-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,x_0]} \right\}. \tag{34}$$

We mention

Remark 13 *Some examples for g follow:*

$$\begin{array}{l} g(x) = x, \quad x \in [a,b], \\ g(x) = e^x, \quad x \in [a,b] \subset \mathbb{R}, \end{array} \tag{35}$$

also

$$\begin{array}{l} g(x) = \sin x, \\ g(x) = \tan x, \ \text{when } x \in [a,b] := \left[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon\right], \ \varepsilon > 0 \text{ small}, \end{array} \tag{36}$$

and

$$g(x) = \cos x, \ \text{when } x \in [a,b] := [\pi + \varepsilon, 2\pi - \varepsilon], \ \varepsilon > 0 \text{ small}. \tag{37}$$

Above all g's are strictly increasing, $g \in C^1([a,b])$, and $g^{-1} \in C^n([g(a), g(b)])$, for any $n \in \mathbb{N}$.

Applications od Theorem 12 follow:

We give the following exponential Ostrowski type fractional inequality:

Theorem 14 *([1]) Let $0 < \alpha < 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, where $(X, \|\cdot\|)$ is a Banach space, $x_0 \in [a,b]$. Assume that $F_k^{x_0} := D_{x_0-;e^t}^{k\alpha} f$, for $k = 1, ..., n$, fulfill $F_k^{x_0} \in C^1([a, x_0], X)$ and $F_{n+1}^{x_0} \in C([a, x_0], X)$ and $\left(D_{x_0-;e^t}^{i\alpha} f\right)(x_0) = 0$, $i = 1, ..., n$.*

Similarly, we assume that $G_k^{x_0} := D_{x_0+;e^t}^{k\alpha} f$, for $k = 1, ..., n$, fulfill $G_k^{x_0} \in C^1([x_0, b], X)$ and $G_{n+1}^{x_0} \in C([x_0, b], X)$ and $\left(D_{x_0+;e^t}^{i\alpha} f\right)(x_0) = 0$, $i = 1, ..., n$.

Then

$$\left\| \frac{1}{b-a} \int_a^b f(x)\,dx - f(x_0) \right\| \leq \frac{1}{(b-a)\,\Gamma((n+1)\alpha+1)} \cdot \tag{38}$$

$$\left\{ \left(e^b - e^{x_0}\right)^{(n+1)\alpha} (b - x_0) \left\| D_{x_0+;e^t}^{(n+1)\alpha} f \right\|_{\infty,[x_0,b]} + \right.$$

$$\left. (e^{x_0} - e^a)^{(n+1)\alpha} (x_0 - a) \left\| D_{x_0-;e^t}^{(n+1)\alpha} f \right\|_{\infty,[a,x_0]} \right\}.$$

We finish this section with the following trigonometric Ostrowski type fractional inequality:

Theorem 15 *([1]) Let* $0 < \alpha < 1$*,* $n \in \mathbb{N}$*,* $f \in C^1([\pi+\varepsilon, 2\pi-\varepsilon], X)$*,* $\varepsilon > 0$ *small, where* $(X, \|\cdot\|)$ *is a Banach space,* $x_0 \in [\pi+\varepsilon, 2\pi-\varepsilon]$*. Assume that* $F_k^{x_0} := D_{x_0-;\cos}^{k\alpha} f$*, for* $k = 1, ..., n$*, fulfill* $F_k^{x_0} \in C^1([\pi+\varepsilon, x_0], X)$ *and* $F_{n+1}^{x_0} \in C([\pi+\varepsilon, x_0], X)$ *and* $\left(D_{x_0-;\cos}^{i\alpha} f\right)(x_0) = 0$*,* $i = 1, ..., n$*.*

Similarly, we assume that $G_k^{x_0} := D_{x_0+;\cos}^{k\alpha} f$*, for* $k = 1, ..., n$*, fulfill* $G_k^{x_0} \in C^1([x_0, 2\pi-\varepsilon], X)$ *and* $G_{n+1}^{x_0} \in C([x_0, 2\pi-\varepsilon], X)$ *and* $\left(D_{x_0+;\cos}^{i\alpha} f\right)(x_0) = 0$*,* $i = 1, ..., n$*.*

Then

$$\left\| \frac{1}{\pi - 2\varepsilon} \int_{\pi+\varepsilon}^{2\pi-\varepsilon} f(x)\, dx - f(x_0) \right\| \leq \frac{1}{(\pi - 2\varepsilon)\Gamma((n+1)\alpha + 1)} \cdot$$

$$\left\{ (\cos(2\pi - \varepsilon) - \cos x_0)^{(n+1)\alpha} (2\pi - \varepsilon - x_0) \left\| D_{x_0+;\cos}^{(n+1)\alpha} f \right\|_{\infty,[x_0,2\pi-\varepsilon]} + \right.$$

$$\left. (\cos x_0 - \cos(\pi + \varepsilon))^{(n+1)\alpha} (x_0 - \pi - \varepsilon) \left\| D_{x_0-;\cos}^{(n+1)\alpha} f \right\|_{\infty,[\pi+\varepsilon,x_0]} \right\}. \tag{39}$$

Important results of this background: Theorems 12, 14, 15 next are applied for $X = \mathbb{C}$, the Banach space of complex numbers with $\|\cdot\| = |\cdot|$, the absolute value.

3 Main Results

We start with some history of the topic of Ostrowski type inequalities:

In 1938, A. Ostrowski [7], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\, dt$ and the value $f(x)$, $x \in [a, b]$.

Theorem 16 *(Ostrowski, 1938 [7]) Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b) *such that* $f' : (a, b) \to \mathbb{R}$ *is bounded on* (a, b)*, i.e.,* $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$*. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b - a), \tag{40}$$

for all $x \in [a, b]$ *and the constant* $\frac{1}{4}$ *is the ebst possible.*

We present the following advanced generalized fractional $\mathbb{C}$-Ostrowski type inequalities:

Theorem 17 *Let $g \in C^1([a,b])$ and strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$, and $0 < \alpha < 1$, $n \in \mathbb{N}$, $h \in C^1([a,b], \mathbb{C})$. Let $x_0 \in [a,b]$ be fixed. Assume that $F_k^{x_0} := D_{x_0-;g}^{k\alpha} h$, for $k = 1, ..., n$, fulfill $F_k^{x_0} \in C^1([a,b], \mathbb{C})$ and $F_{n+1}^{x_0} \in C([a,x_0], \mathbb{C})$ and $\left(D_{x_0-;g}^{i\alpha} h\right)(x_0) = 0$, $i = 1, ..., n$.*

Similarly, we assume that $G_k^{x_0} := D_{x_0+;g}^{k\alpha} h$, for $k = 1, ..., n$, fulfill $G_k^{x_0} \in C^1([x_0,b], \mathbb{C})$ and $G_{n+1}^{x_0} \in C([x_0,b], \mathbb{C})$ and $\left(D_{x_0+;g}^{i\alpha} h\right)(x_0) = 0$, $i = 1, ..., n$.

Then

$$\left|\frac{1}{b-a}\int_a^b h(x)\,dx - h(x_0)\right| \leq \frac{1}{(b-a)\,\Gamma((n+1)\alpha+1)}\cdot \tag{41}$$

$$\Big\{(g(b)-g(x_0))^{(n+1)\alpha}(b-x_0)\left\|D_{x_0+;g}^{(n+1)\alpha} h\right\|_{\infty,[x_0,b]} +$$

$$(g(x_0)-g(a))^{(n+1)\alpha}(x_0-a)\left\|D_{x_0-;g}^{(n+1)\alpha} h\right\|_{\infty,[a,x_0]}\Big\}.$$

Proof. By Theorem 12. ■

Theorem 18 *Let $0 < \alpha < 1$, $n \in \mathbb{N}$, $h \in C^1([a,b], \mathbb{C})$, $x_0 \in [a,b]$. Assume that $F_k^{x_0} := D_{x_0-;e^t}^{k\alpha} h$, for $k = 1, ..., n$, fulfill $F_k^{x_0} \in C^1([a,x_0], \mathbb{C})$ and $F_{n+1}^{x_0} \in C([a,x_0], \mathbb{C})$ and $\left(D_{x_0-;e^t}^{i\alpha} h\right)(x_0) = 0$, $i = 1, ..., n$.*

Similarly, we assume that $G_k^{x_0} := D_{x_0+;e^t}^{k\alpha} h$, for $k = 1, ..., n$, fulfill $G_k^{x_0} \in C^1([x_0,b], \mathbb{C})$ and $G_{n+1}^{x_0} \in C([x_0,b], \mathbb{C})$ and $\left(D_{x_0+;e^t}^{i\alpha} h\right)(x_0) = 0$, $i = 1, ..., n$.

Then

$$\left|\frac{1}{b-a}\int_a^b h(x)\,dx - h(x_0)\right| \leq \frac{1}{(b-a)\,\Gamma((n+1)\alpha+1)}\cdot \tag{42}$$

$$\Big\{\left(e^b-e^{x_0}\right)^{(n+1)\alpha}(b-x_0)\left\|D_{x_0+;e^t}^{(n+1)\alpha} h\right\|_{\infty,[x_0,b]} +$$

$$\left(e^{x_0}-e^a\right)^{(n+1)\alpha}(x_0-a)\left\|D_{x_0-;e^t}^{(n+1)\alpha} h\right\|_{\infty,[a,x_0]}\Big\}.$$

Proof. By Theorem 14. ■

Theorem 19 *Let $0 < \alpha < 1$, $n \in \mathbb{N}$, $h \in C^1([\pi+\varepsilon, 2\pi-\varepsilon], \mathbb{C})$, $\varepsilon > 0$ small, $x_0 \in [\pi+\varepsilon, 2\pi-\varepsilon]$. Assume that $F_k^{x_0} := D_{x_0-;\cos}^{k\alpha} h$, for $k = 1, ..., n$, fulfill $F_k^{x_0} \in C^1([\pi+\varepsilon, x_0], \mathbb{C})$ and $F_{n+1}^{x_0} \in C([\pi+\varepsilon, x_0], \mathbb{C})$ and $\left(D_{x_0-;\cos}^{i\alpha} h\right)(x_0) = 0$, $i = 1, ..., n$.*

Similarly, we assume that $G_k^{x_0} := D_{x_0+;\cos}^{k\alpha} h$, for $k = 1, ..., n$, fulfill $G_k^{x_0} \in C^1([x_0, 2\pi-\varepsilon], \mathbb{C})$ and $G_{n+1}^{x_0} \in C([x_0, 2\pi-\varepsilon], \mathbb{C})$ and $\left(D_{x_0+;\cos}^{i\alpha} h\right)(x_0) = 0$, $i = 1, ..., n$.

Then

$$\left|\frac{1}{\pi-2\varepsilon}\int_{\pi+\varepsilon}^{2\pi-\varepsilon}h(x)\,dx-h(x_0)\right|\leq\frac{1}{(\pi-2\varepsilon)\,\Gamma((n+1)\alpha+1)}\cdot\tag{43}$$

$$\left\{(\cos(2\pi-\varepsilon)-\cos x_0)^{(n+1)\alpha}(2\pi-\varepsilon-x_0)\left\|D_{x_0+;\cos}^{(n+1)\alpha}h\right\|_{\infty,[x_0,2\pi-\varepsilon]}+\right.$$

$$\left.(\cos x_0-\cos(\pi+\varepsilon))^{(n+1)\alpha}(x_0-\pi-\varepsilon)\left\|D_{x_0-;\cos}^{(n+1)\alpha}h\right\|_{\infty,[\pi+\varepsilon,x_0]}\right\}.$$

Proof. By Theorem 15. ■

From now on $f(z)$, $z(t)$, $t\in(a,b)$, γ will be as in section 1. Introduction. Put $z(a)=u$, $z(b)=w$ and $z(c)=v$, where $u,w,v\in\mathbb{C}$, with $c\in[a,b]$.

We will use here $h(t):=f(z(t))\,z'(t)$, $t\in[a,b]$.

In that case we will have

$$\left|\frac{1}{b-a}\int_a^b h(t)\,dt-h(c)\right|=\left|\frac{1}{b-a}\int_a^b f(z(t))\,z'(t)\,dt-f(z(c))\,z'(c)\right|\overset{(1)}{=}$$

$$\left|\frac{1}{b-a}\int_{\gamma_{u,w}}f(z)\,dz-f(v)\,z'(c)\right|\overset{(1)}{=}\left|\frac{1}{b-a}\int_{\gamma}f(z)\,dz-f(v)\,z'(c)\right|,\tag{44}$$

where $\gamma_{u,w}=\gamma$.

We have the following advanced generalized fractional complete $\mathbb{C}$-Ostrowski type inequalities:

Theorem 20 *Let $g\in C^1([a,b])$ and strictly increasing, such that $g^{-1}\in C^1([g(a),g(b)])$, and $0<\alpha<1$, $n\in\mathbb{N}$, $f(z(\cdot))\,z'(\cdot)\in C^1([a,b],\mathbb{C})$. Let $c\in[a,b]$ be fixed. Assume that $F_k^c:=D_{c-;g}^{k\alpha}(f(z(\cdot))\,z'(\cdot))$, for $k=1,...,n$, fulfill $F_k^c\in C^1([a,b],\mathbb{C})$ and $F_{n+1}^c\in C([a,c],\mathbb{C})$ and $\left(D_{c-;g}^{i\alpha}(f(z(\cdot))\,z'(\cdot))\right)(c)=0$, $i=1,...,n$.*

Similarly, we assume that $G_k^c:=D_{c+;g}^{k\alpha}(f(z(\cdot))\,z'(\cdot))$, for $k=1,...,n$, fulfill $G_k^c\in C^1([c,b],\mathbb{C})$ and $G_{n+1}^c\in C([c,b],\mathbb{C})$ and $\left(D_{c+;g}^{i\alpha}(f(z(\cdot))\,z'(\cdot))\right)(c)=0$, $i=1,...,n$.

Then

$$\left|\frac{1}{b-a}\int_{\gamma_{u,w}}f(z)\,dz-f(v)\,z'(c)\right|\leq\frac{1}{(b-a)\,\Gamma((n+1)\alpha+1)}\cdot\tag{45}$$

$$\left\{(g(b)-g(c))^{(n+1)\alpha}(b-c)\left\|D_{c+;g}^{(n+1)\alpha}(f(z(\cdot))\,z'(\cdot))\right\|_{\infty,[c,b]}+\right.$$

$$\left.(g(c)-g(a))^{(n+1)\alpha}(c-a)\left\|D_{c-;g}^{(n+1)\alpha}(f(z(\cdot))\,z'(\cdot))\right\|_{\infty,[a,c]}\right\}.$$

Proof. By Theorem 17. ■

We continue with

Theorem 21 *Let* $0 < \alpha < 1$, $n \in \mathbb{N}$, $f(z(\cdot)) z'(\cdot) \in C^1([a,b], \mathbb{C})$, $c \in [a,b]$. *Assume that* $F_k^c := D_{c-;e^t}^{k\alpha}(f(z(\cdot)) z'(\cdot))$, *for* $k = 1, ..., n$, *fulfill* $F_k^c \in C^1([a,c], \mathbb{C})$ *and* $F_{n+1}^c \in C([a,c], \mathbb{C})$ *and* $\left(D_{c-;e^t}^{i\alpha}(f(z(\cdot)) z'(\cdot))\right)(c) = 0$, $i = 1, ..., n$.

Similarly, we assume that $G_k^c := D_{c+;e^t}^{k\alpha}(f(z(\cdot)) z'(\cdot))$, *for* $k = 1, ..., n$, *fulfill* $G_k^c \in C^1([c,b], \mathbb{C})$ *and* $G_{n+1}^c \in C([c,b], \mathbb{C})$ *and* $\left(D_{c+;e^t}^{i\alpha}(f(z(\cdot)) z'(\cdot))\right)(c) = 0$, $i = 1, ..., n$.

Then

$$\left| \frac{1}{b-a} \int_{\gamma_{u,w}} f(z) dz - f(v) z'(c) \right| \leq \frac{1}{(b-a) \Gamma((n+1)\alpha + 1)} \cdot \tag{46}$$

$$\left\{ \left(e^b - e^c\right)^{(n+1)\alpha} (b-c) \left\| D_{c+;e^t}^{(n+1)\alpha}(f(z(\cdot)) z'(\cdot)) \right\|_{\infty,[c,b]} + \right.$$

$$\left. (e^c - e^a)^{(n+1)\alpha} (c-a) \left\| D_{c-;e^t}^{(n+1)\alpha}(f(z(\cdot)) z'(\cdot)) \right\|_{\infty,[a,c]} \right\}.$$

Proof. By Theorem 18. ■

Finally and additionally, we choose that $a = \pi + \varepsilon$, $b = 2\pi - \varepsilon$, where $\varepsilon > 0$ is small, and $c \in [\pi + \varepsilon, 2\pi - \varepsilon]$. So here it is $z(\pi + \varepsilon) = u$, $z(2\pi - \varepsilon) = w$ and $z(c) = v$, where $u, w, u \in \mathbb{C}$.

We present

Theorem 22 *Let* $0 < \alpha < 1$, $n \in \mathbb{N}$, $f(z(\cdot)) z'(\cdot) \in C^1([\pi + \varepsilon, 2\pi - \varepsilon], \mathbb{C})$, $\varepsilon > 0$ *small*, $c \in [\pi + \varepsilon, 2\pi - \varepsilon]$. *Assume that* $F_k^c := D_{c-;\cos}^{k\alpha}(f(z(\cdot)) z'(\cdot))$, *for* $k = 1, ..., n$, *fulfill* $F_k^c \in C^1([\pi + \varepsilon, c], \mathbb{C})$ *and* $F_{n+1}^c \in C([\pi + \varepsilon, c], \mathbb{C})$ *and* $\left(D_{c-;\cos}^{i\alpha}(f(z(\cdot)) z'(\cdot))\right)(c) = 0$, $i = 1, ..., n$.

Similarly, we assume that $G_k^c := D_{c+;\cos}^{k\alpha}(f(z(\cdot)) z'(\cdot))$, *for* $k = 1, ..., n$, *fulfill* $G_k^c \in C^1([c, 2\pi - \varepsilon], \mathbb{C})$ *and* $G_{n+1}^c \in C([c, 2\pi - \varepsilon], \mathbb{C})$ *and* $\left(D_{c+;\cos}^{i\alpha}(f(z(\cdot)) z'(\cdot))\right)(c) = 0$, $i = 1, ..., n$.

Then

$$\left| \frac{1}{\pi - 2\varepsilon} \int_{\gamma_{u,w}} f(z) dz - f(v) z'(c) \right| \leq \frac{1}{(\pi - 2\varepsilon) \Gamma((n+1)\alpha + 1)} \cdot \tag{47}$$

$$\left\{ (\cos(2\pi - \varepsilon) - \cos c)^{(n+1)\alpha} (2\pi - \varepsilon - c) \left\| D_{c+;\cos}^{(n+1)\alpha}(f(z(\cdot)) z'(\cdot)) \right\|_{\infty,[c,2\pi-\varepsilon]} + \right.$$

$$\left. (\cos c - \cos(\pi + \varepsilon))^{(n+1)\alpha} (c - \pi - \varepsilon) \left\| D_{c-;\cos}^{(n+1)\alpha}(f(z(\cdot)) z'(\cdot)) \right\|_{\infty,[\pi+\varepsilon,c]} \right\}.$$

Proof. By Theorem 19. ■

References

[1] G.A. Anastassiou, *Principles of General Fractional Analysis for Banach space valued functions*, Bulletin of Allahabad Math. Soc., 32 (1) (2017), 71-145.

[2] G.A. Anastassiou, *A strong Fractional Calculus Theory for Banach space valued functions*, Nonlinear Functional Analysis and Applications, 22 (3) (2017), 495-524.

[3] G.A. Anastassiou, *Strong Right Fractional Calculus for Banach space valued functions*, Revista Proyecciones, 36(1) (2017), 149-186.

[4] G.A. Anastassiou, *Strong mixed and generalized fractional calculus for Banach space valued functions*, Mat. Vesnik, 69 (3) (2017), 176-191.

[5] S.S. Dragomir, *An extension of Opial's inequality to the complex integral*, RGMIA Res. Rep. Coll. 22 (2019), Art 9, 9 pp., rgmia.org/v22.php.

[6] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[7] A.M. Ostrowski, *On an integral inequality*, Aequat. Math. 4 (1970), 358-373.

[8] G.E. Shilov, *Elementary Functional Analysis*, Dover Publications, Inc., New York, 1996.