TWO POINTS NORM INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$, α, $\beta \in G$, $\lambda \in \mathbb{C}$ and $f : G \to \mathbb{C}$ is analytic on G. By using the analytic functional calculus we obtain among others the following result

$$\begin{aligned}
&\left\| \lambda f(\alpha) + (1-\lambda) f(\beta) - f(a) - [\lambda\alpha + (1-\lambda)\beta - a] f'(a) \right\| \\
&\leq \frac{1}{2}\left[|\lambda| \|\alpha - a\|^2 L_{\lambda,a} + |1-\lambda| \|\beta - a\|^2 L_{\beta,a} \right] \\
&\leq \frac{1}{2}\max\left\{ L_{\lambda,a}, L_{\beta,a} \right\} \left[|\lambda| \|\alpha - a\|^2 + |1-\lambda| \|\beta - a\|^2 \right]
\end{aligned}$$

provided that the derivative f' satisfies the Lipschitz conditions

$$f' \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\alpha,a}) \cap \mathfrak{Lip}_{L_{\beta,a}}(G_{\lambda,a})$$

for some $L_{\lambda,a}$, $L_{\beta,a} > 0$. In particular, we have the trapezoid type inequality

$$\begin{aligned}
&\left\| \frac{f(\alpha) + f(\beta)}{2} - f(a) - \left(\frac{\alpha+\beta}{2} - a \right) f'(a) \right\| \\
&\leq \frac{1}{4}\left[\|\alpha - a\|^2 L_{\lambda,a} + \|\beta - a\|^2 L_{\beta,a} \right] \\
&\leq \frac{1}{4}\max\left\{ L_{\lambda,a}, L_{\beta,a} \right\} \left[\|\alpha - a\|^2 + \|\beta - a\|^2 \right].
\end{aligned}$$

Some examples for the exponential function are also given.

1. Introduction

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \leq \|a\| \|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*. We assume that the Banach algebra is *unital*, this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\text{Inv}(\mathcal{B})$. If $a, b \in \text{Inv}(\mathcal{B})$ then $ab \in \text{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \text{Inv}(\mathcal{B})$;
(ii) $\{a \in \mathcal{B}\text{: } \|1 - b\| < 1\} \subset \text{Inv}(\mathcal{B})$;
(iii) $\text{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\text{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \text{Inv}(\mathcal{B})$ is continuous.

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For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} : \ z - a \in \operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \operatorname{Inv}(\mathcal{B})$, $R_a(z) := (z-a)^{-1}$. We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \le \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_\delta f(\xi)(\xi - a)^{-1} d\xi, \tag{1.1}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [3, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.2}$$

holds.

Let $\mathfrak{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\mathfrak{Hol}(a)$ is an algebra where if f, $g \in \mathfrak{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f+g$ have domain $D(f) \cap D(g)$. $\mathfrak{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [3, p. 201-203]:

Theorem 1. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) *The map $f \mapsto f(a)$ of $\mathfrak{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.*
(b) *If $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ has radius of convergence $r > \nu(a)$, then $f \in \mathfrak{Hol}(a)$ and $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.*
(c) *If $f(z) \equiv 1$, then $f(a) = 1$.*
(d) *If $f(z) = z$ for all z, $f(a) = a$.*
(e) *If f, $f_1, ..., f_n...$ are analytic on G, $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\|f_n(a) - f(a)\| \to 0$ as $n \to \infty$.*
(f) *The Riesz Functional Calculus is unique and if a, b are commuting elements in $\mathcal{B}$ and $f \in \mathfrak{Hol}(a)$, then $f(a)b = bf(a)$.*

For some recent norm inequalities for functions on Banach algebras, see [1]-[2] and [5]-[11].

In this paper, by using the analytic functional calculus, we establish some error bounds for the trapezoid and perturbed trapezoid type approximations, namely upper bounds for the quantities

$$\left\|\lambda f(\alpha)+(1-\lambda) f(\beta)-f(a)\right\|, \quad \left\|\frac{f(\alpha)+f(\beta)}{2}-f(a)\right\|,$$
$$\left\|\lambda f(\alpha)+(1-\lambda) f(\beta)-f(a)-[\lambda\alpha+(1-\lambda)\beta-a] f'(a)\right\|$$

and

$$\left\|\frac{f(\alpha)+f(\beta)}{2}-f(a)-\left(\frac{\alpha+\beta}{2}-a\right) f'(a)\right\|$$

provided that $f: G \rightarrow \mathbb{C}$ is analytic on G, a convex domain, α, $\beta \in G$ and $\lambda \in C$. Some examples for the exponential function are also given.

2. Some Identities

We have:

Theorem 2. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f: G \rightarrow \mathbb{C}$ is analytic on G, then for all α, $\beta \in G$ and $\lambda \in C$ we have*

$$\lambda f(\alpha)+(1-\lambda) f(\beta)-f(a)=\lambda(\alpha-a) \int_{0}^{1} f'((1-t) a+t \alpha) dt + (1-\lambda)(\beta-a) \int_{0}^{1} f'((1-t) a+t \beta) dt. \tag{2.1}$$

In particular, we have

$$\frac{f(\alpha)+f(\beta)}{2}-f(a) = \frac{1}{2}\left[(\alpha-a) \int_{0}^{1} f'((1-t) a+t \alpha) dt+(\beta-a) \int_{0}^{1} f'((1-t) a+t \beta) dt\right]. \tag{2.2}$$

Proof. Due to the convexity of D, for any ξ, $\nu \in D$ we can define the function $\varphi_{\xi,\nu}:[0,1] \rightarrow \mathbb{R}$ by $\varphi_{\xi,\nu}(t):=f((1-t)\xi+t\nu)$. The function $\varphi_{\xi,\nu}$ is differentiable on $(0,1)$ and

$$\frac{d\varphi_{\xi,\nu}(t)}{dt}=(\nu-\xi) f'((1-t)\xi+t\nu) \text{ for } t \in(0,1).$$

We have

$$f(\nu)-f(\xi)=\varphi_{\xi,\nu}(1)-\varphi_{\xi,\nu}(0)=\int_{0}^{1} \frac{d\varphi_{\xi,\nu}(t)}{dt} dt = (\nu-\xi) \int_{0}^{1} f'((1-t)\xi+t\nu) dt$$

namely

$$f(\nu)=f(\xi)+(\nu-\xi) \int_{0}^{1} f'((1-t)\xi+t\nu) dt \tag{2.3}$$

for any ξ, $\nu \in D$.

Therefore, by (2.3) we get

$$f(\alpha)=f(\xi)+(\alpha-\xi)\int_0^1 f'((1-t)\xi+t\alpha)\,dt \tag{2.4}$$

and

$$f(\beta)=f(\xi)+(\beta-\xi)\int_0^1 f'((1-t)\xi+t\beta)\,dt \tag{2.5}$$

for any $\xi\in D$.

If we multiply (2.4) and (2.5) by λ and $1-\lambda$ and add, we get the following identity that is of interest in itself

$$\begin{aligned}&\lambda f(\alpha)+(1-\lambda)f(\beta)-f(\xi)\\&=\lambda(\alpha-\xi)\int_0^1 f'((1-t)\xi+t\alpha)\,dt+(1-\lambda)(\beta-\xi)\int_0^1 f'((1-t)\xi+t\beta)\,dt\end{aligned} \tag{2.6}$$

for any $\xi\in D$ and $\lambda\in\mathbb{C}$.

From (2.6) and by analytic functional calculus (1.1) we get

$$\begin{aligned}&\frac{1}{2\pi i}\int_\delta[\lambda f(\alpha)+(1-\lambda)f(\beta)](\xi-a)^{-1}d\xi-\frac{1}{2\pi i}\int_\delta f(\xi)(\xi-a)^{-1}d\xi\\&\quad=\lambda\frac{1}{2\pi i}\int_\delta(\alpha-\xi)\left(\int_0^1 f'((1-t)\xi+t\alpha)\,dt\right)(\xi-a)^{-1}d\xi\\&\qquad+(1-\lambda)\frac{1}{2\pi i}\int_\delta(\beta-\xi)\left(\int_0^1 f'((1-t)\xi+t\beta)\,dt\right)(\xi-a)^{-1}d\xi\end{aligned}$$

namely,

$$\begin{aligned}&[\lambda f(\alpha)+(1-\lambda)f(\beta)]\frac{1}{2\pi i}\int_\delta(\xi-a)^{-1}d\xi-\frac{1}{2\pi i}\int_\delta f(\xi)(\xi-a)^{-1}d\xi\\&\quad=\lambda\int_0^1\left(\frac{1}{2\pi i}\int_\delta(\alpha-\xi)f'((1-t)\xi+t\alpha)(\xi-a)^{-1}d\xi\right)dt\\&\qquad+(1-\lambda)\int_0^1\left(\frac{1}{2\pi i}\int_\delta(\beta-\xi)f'((1-t)\xi+t\beta)(\xi-a)^{-1}d\xi\right)dt\end{aligned} \tag{2.7}$$

for all α, $\beta\in G$ and $\lambda\in\mathbb{C}$, where for the last equality we used Fubini's theorem.

Since the functions

$$G\ni\mapsto(\alpha-\xi)f'((1-t)\xi+t\alpha)\in\mathbb{C}$$

and

$$G\ni\mapsto(\beta-\xi)f'((1-t)\xi+t\beta)\in\mathbb{C}$$

are analytic on G for all α, $\beta\in G$ and $t\in[0,1]$, then by the analytic functional calculus we have

$$\frac{1}{2\pi i}\int_\delta(\alpha-\xi)f'((1-t)\xi+t\alpha)(\xi-a)^{-1}d\xi=(\alpha-a)f'((1-t)a+t\alpha)$$

and

$$\frac{1}{2\pi i}\int_\delta(\beta-\xi)f'((1-t)\xi+t\beta)(\xi-a)^{-1}d\xi=(\beta-a)f'((1-t)a+t\beta)$$

and by (2.7) we get the desired result (2.1). □

Corollary 1. *With the assumptions of Theorem 2 and for any b, $c \in \mathcal{B}$ we have the perturbed identity*

$$\begin{aligned}(2.8)\quad &\lambda\left[f\left(\alpha\right)-\left(\alpha-a\right)b\right]+\left(1-\lambda\right)\left[f\left(\beta\right)-\left(\beta-a\right)c\right]-f\left(a\right)\\ &=\lambda\left(\alpha-a\right)\int_0^1\left[f'\left(\left(1-t\right)a+t\alpha\right)-b\right]dt\\ &\quad+\left(1-\lambda\right)\left(\beta-a\right)\int_0^1\left[f'\left(\left(1-t\right)a+t\beta\right)-c\right]dt\end{aligned}$$

and

$$\begin{aligned}(2.9)\quad &\frac{1}{2}\left[f\left(\alpha\right)-\left(\alpha-a\right)b\right]+\frac{1}{2}\left[f\left(\beta\right)-\left(\beta-a\right)c\right]-f\left(a\right)\\ &=\frac{1}{2}\left(\alpha-a\right)\int_0^1\left[f'\left(\left(1-t\right)a+t\alpha\right)-b\right]dt\\ &\quad+\frac{1}{2}\left(\beta-a\right)\int_0^1\left[f'\left(\left(1-t\right)a+t\beta\right)-c\right]dt.\end{aligned}$$

For $c=b$, we have

$$\begin{aligned}(2.10)\quad &\lambda f\left(\alpha\right)+\left(1-\lambda\right)f\left(\beta\right)-f\left(a\right)-\left[\lambda\alpha+\left(1-\lambda\right)\beta-a\right]b\\ &=\lambda\left(\alpha-a\right)\int_0^1\left[f'\left(\left(1-t\right)a+t\alpha\right)-b\right]dt\\ &\quad+\left(1-\lambda\right)\left(\beta-a\right)\int_0^1\left[f'\left(\left(1-t\right)a+t\beta\right)-b\right]dt\end{aligned}$$

and in particular

$$\begin{aligned}(2.11)\quad &\frac{f\left(\alpha\right)+f\left(\beta\right)}{2}-f\left(a\right)-\left(\frac{\alpha+\beta}{2}-a\right)b\\ &=\frac{1}{2}\left(\alpha-a\right)\int_0^1\left[f'\left(\left(1-t\right)a+t\alpha\right)-b\right]dt\\ &\quad+\frac{1}{2}\left(\beta-a\right)\int_0^1\left[f'\left(\left(1-t\right)a+t\beta\right)-b\right]dt\end{aligned}$$

for all α, $\beta \in G$ and $\lambda \in \mathbb{C}$.

Remark 1. *If we choose in Corollary 1 $b=f'\left(\alpha\right)$ and $c=f'\left(\beta\right)$, then we get*

$$\begin{aligned}(2.12)\quad &\lambda\left[f\left(\alpha\right)-\left(\alpha-a\right)f'\left(\alpha\right)\right]+\left(1-\lambda\right)\left[f\left(\beta\right)-\left(\beta-a\right)f'\left(\beta\right)\right]-f\left(a\right)\\ &=\lambda\left(\alpha-a\right)\int_0^1\left[f'\left(\left(1-t\right)a+t\alpha\right)-f'\left(\alpha\right)\right]dt\\ &\quad+\left(1-\lambda\right)\left(\beta-a\right)\int_0^1\left[f'\left(\left(1-t\right)a+t\beta\right)-f'\left(\beta\right)\right]dt\end{aligned}$$

and, in particular,

$$\frac{1}{2}\left[f(\alpha)-(\alpha-a)f'(\alpha)\right]+\frac{1}{2}\left[f(\beta)-(\beta-a)f'(\beta)\right]-f(a) \tag{2.13}$$
$$=\frac{1}{2}(\alpha-a)\int_0^1\left[f'((1-t)a+t\alpha)-f'(\alpha)\right]dt$$
$$+\frac{1}{2}(\beta-a)\int_0^1\left[f'((1-t)a+t\beta)-f'(\beta)\right]dt.$$

Also, if we choose in Corollary 1 $b=c=f'(a)$, *then we get*

$$\lambda f(\alpha)+(1-\lambda)f(\beta)-f(a)-\left[\lambda\alpha+(1-\lambda)\beta-a\right]f'(a) \tag{2.14}$$
$$=\lambda(\alpha-a)\int_0^1\left[f'((1-t)a+t\alpha)-f'(a)\right]dt$$
$$+(1-\lambda)(\beta-a)\int_0^1\left[f'((1-t)a+t\beta)-f'(a)\right]dt$$

and, in particular,

$$\frac{f(\alpha)+f(\beta)}{2}-f(a)-\left(\frac{\alpha+\beta}{2}-a\right)f'(a) \tag{2.15}$$
$$=\frac{1}{2}(\alpha-a)\int_0^1\left[f'((1-t)a+t\alpha)-f'(a)\right]dt$$
$$+\frac{1}{2}(\beta-a)\int_0^1\left[f'((1-t)a+t\beta)-f'(a)\right]dt.$$

3. Norm Inequalities

We have:

Theorem 3. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic on G, then for all α, $\beta\in G$ and $\lambda\in C$ we have*

$$\|\lambda f(\alpha)+(1-\lambda)f(\beta)-f(a)\| \tag{3.1}$$
$$\le|\lambda|\,\|\alpha-a\|\sup_{t\in[0,1]}\|f'((1-t)a+t\alpha)\|+|1-\lambda|\,\|\beta-a\|\sup_{t\in[0,1]}\|f'((1-t)a+t\beta)\|$$
$$\le\max\{|\lambda|,|1-\lambda|\}$$
$$\times\left[\|\alpha-a\|\sup_{t\in[0,1]}\|f'((1-t)a+t\alpha)\|+\|\beta-a\|\sup_{t\in[0,1]}\|f'((1-t)a+t\beta)\|\right].$$

In particular, we have

$$\left\|\frac{f(\alpha)+f(\beta)}{2}-f(a)\right\| \tag{3.2}$$
$$\le\frac{1}{2}\left[\|\alpha-a\|\sup_{t\in[0,1]}\|f'((1-t)a+t\alpha)\|+\|\beta-a\|\sup_{t\in[0,1]}\|f'((1-t)a+t\beta)\|\right].$$

Moreover, if

$$\|f'\|_{a,G,\infty} := \sup_{t\in[0,1],\alpha\in G} \|f'((1-t)a+t\alpha)\| < \infty,$$

then

$$\begin{aligned}(3.3)\quad & \|\lambda f(\alpha)+(1-\lambda)f(\beta)-f(a)\| \\ &\le [|\lambda|\|\alpha-a\|+|1-\lambda|\|\beta-a\|]\|f'\|_{a,G,\infty} \\ &\le \max\{|\lambda|,|1-\lambda|\}[\|\alpha-a\|+\|\beta-a\|]\|f'\|_{a,G,\infty}\end{aligned}$$

and, in particular,

$$\left\|\frac{f(\alpha)+f(\beta)}{2}-f(a)\right\| \le \frac{1}{2}[\|\alpha-a\|+\|\beta-a\|]\|f'\|_{a,G,\infty}. \tag{3.4}$$

Proof. By taking the norm in the identity (2.1) we get

$$\begin{aligned}&\|\lambda f(\alpha)+(1-\lambda)f(\beta)-f(a)\| \\ &\le |\lambda|\left\|(\alpha-a)\int_0^1 f'((1-t)a+t\alpha)\,dt\right\|+|1-\lambda|\left\|(\beta-a)\int_0^1 f'((1-t)a+t\beta)\,dt\right\| \\ &\le |\lambda|\|\alpha-a\|\left\|\int_0^1 f'((1-t)a+t\alpha)\,dt\right\|+|1-\lambda|\|\beta-a\|\left\|\int_0^1 f'((1-t)a+t\beta)\,dt\right\| \\ &\le |\lambda|\|\alpha-a\|\int_0^1 \|f'((1-t)a+t\alpha)\|\,dt+|1-\lambda|\|\beta-a\|\int_0^1 \|f'((1-t)a+t\beta)\|\,dt \\ &\le |\lambda|\|\alpha-a\|\sup_{t\in[0,1]}\|f'((1-t)a+t\alpha)\|+|1-\lambda|\|\beta-a\|\sup_{t\in[0,1]}\|f'((1-t)a+t\beta)\|,\end{aligned}$$

which proves (3.1).

The rest follows from (3.1). □

Let $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$ and $\lambda\in G$. We define $G_{\lambda,a} := \{(1-t)\lambda+ta \mid \text{with } t\in[0,1]\}$. We observe that $G_{\lambda,a}$ is a convex subset in $\mathcal{B}$ for every $\lambda\in G$.

For two distinct elements u, v in the Banach algebra B we say that the function $g: G_{\lambda,a}\to\mathcal{B}$ belongs to the class $\Delta_{u,v}(G_{\lambda,a})$ if it satisfies the boundedness condition

$$\left\|g((1-t)\lambda+ta)-\frac{u+v}{2}\right\| \le \frac{1}{2}\|v-u\| \tag{3.5}$$

for all $t\in[0,1]$. We write $g\in\Delta_{u,v}(G_{\lambda,a})$. This definition is an extension to Banach algebras valued functions of the scalar case, see [4].

We say that the function $g: G_{\lambda,a}\to B$ is Lipschitzian on $G_{\lambda,a}$ with the constant $L_{\lambda,a}>0$, if for all x, $y\in G_{\lambda,a}$ we have

$$\|g(x)-g(y)\| \le L_{\lambda,a}\|x-y\|.$$

This is equivalent to

$$\|g((1-t)\lambda+ta)-g((1-s)\lambda+sa)\| \le L_{\lambda,a}|t-s|\|a-\lambda\| \tag{3.6}$$

for all t, $s\in[0,1]$. We write this by $g\in\mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$.

Let $h : G \to \mathbb{C}$ be an analytic function on G. For $t \in [0,1]$ and $\lambda \in G$, the auxiliary function $h_{t,\lambda}$ defined on G by $h_{t,\lambda}(\xi) := h((1-t)\lambda + t\xi)$ is also analytic and using the analytic functional calculus (1.1) for the element $a \in \mathcal{B}$, we can define

$$\widetilde{h}((1-t)\lambda + ta) := h_{t,\lambda}(a) = \frac{1}{2\pi i}\int_\gamma h_{t,\lambda}(\xi)(\xi - a)^{-1} d\xi \tag{3.7}$$
$$= \frac{1}{2\pi i}\int_\gamma h((1-t)\lambda + t\xi)(\xi - a)^{-1} d\xi.$$

We say that the scalar function $h \in \Delta_{u,v}(G_{\lambda,a})$ if its extension $\widetilde{h} : G_{\lambda,a} \to B$ satisfies the boundedness condition (3.5). Also, we say that the scalar function $h \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$ if its extension $\widetilde{h} : G_{\lambda,a} \to B$ satisfies the Lipschitz condition (3.6).

Theorem 4. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and α, $\beta \in G$, $\lambda \in \mathbb{C}$. If $f : G \to \mathbb{C}$ is analytic on G and there exists u, $v \in \mathcal{B}$ such that $f' \in \Delta_{u,v}(G_{\alpha,a}) \cap \Delta_{u,v}(G_{\beta,a})$, then we have*

$$\left\|\lambda f(\alpha) + (1-\lambda) f(\beta) - f(a) - [\lambda\alpha + (1-\lambda)\beta - a]\frac{u+v}{2}\right\| \tag{3.8}$$
$$\leq \frac{1}{2}\|v-u\|\left[|\lambda|\|\alpha - a\| + |1-\lambda|\|\beta - a\|\right]$$

and, in particular

$$\left\|\frac{f(\alpha) + f(\beta)}{2} - f(a) - \left(\frac{\alpha+\beta}{2} - a\right)\frac{u+v}{2}\right\| \tag{3.9}$$
$$\leq \frac{1}{4}\|v-u\|\left[\|\alpha - a\| + \|\beta - a\|\right].$$

Proof. From the identity (2.10) and since $f' \in \Delta_{u,v}(G_{\alpha,a}) \cap \Delta_{u,v}(G_{\beta,a})$, hence

$$\left\|\lambda f(\alpha) + (1-\lambda) f(\beta) - f(a) - [\lambda\alpha + (1-\lambda)\beta - a]\frac{u+v}{2}\right\|$$
$$\leq |\lambda|\|\alpha - a\|\int_0^1\left\|f'((1-t)a + t\alpha) - \frac{u+v}{2}\right\|dt$$
$$+ |1-\lambda|\|\beta - a\|\int_0^1\left\|f'((1-t)a + t\beta) - \frac{u+v}{2}\right\|dt$$
$$\leq \frac{1}{2}\|v-u\|\left[|\lambda|\|\alpha - a\| + |1-\lambda|\|\beta - a\|\right],$$

which proves (3.8). □

We also have:

Theorem 5. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and α, $\beta \in G$, $\lambda \in \mathbb{C}$. If $f : G \to \mathbb{C}$ is analytic on G and*

$f' \in \mathfrak{Lip}_{L_{\alpha,a}}(G_{\alpha,a}) \cap \mathfrak{Lip}_{L_{\beta,a}}(G_{\lambda,a})$ *for some* $L_{\alpha,a},\ L_{\beta,a} > 0$, *then*

$$\begin{aligned} (3.10)\quad & \|\lambda f(\alpha) + (1-\lambda) f(\beta) - f(a) - [\lambda\alpha + (1-\lambda)\beta - a] f'(a)\| \\ &\leq \frac{1}{2}\left[|\lambda| \|\alpha - a\|^2 L_{\lambda,a} + |1-\lambda| \|\beta - a\|^2 L_{\beta,a}\right] \\ &\leq \frac{1}{2}\max\{L_{\lambda,a}, L_{\beta,a}\}\left[|\lambda| \|\alpha - a\|^2 + |1-\lambda| \|\beta - a\|^2\right] \\ &\leq \frac{1}{2}\max\{L_{\lambda,a}, L_{\beta,a}\}\max\{|\lambda|, |1-\lambda|\}\left[\|\alpha - a\|^2 + \|\beta - a\|^2\right] \end{aligned}$$

In particular, we have

$$\begin{aligned} (3.11)\quad & \left\|\frac{f(\alpha)+f(\beta)}{2} - f(a) - \left(\frac{\alpha+\beta}{2} - a\right) f'(a)\right\| \\ &\leq \frac{1}{4}\left[\|\alpha - a\|^2 L_{\lambda,a} + \|\beta - a\|^2 L_{\beta,a}\right] \\ &\leq \frac{1}{4}\max\{L_{\lambda,a}, L_{\beta,a}\}\left[\|\alpha - a\|^2 + \|\beta - a\|^2\right]. \end{aligned}$$

Proof. Using the identity (2.14) we get

$$\begin{aligned} & \|\lambda f(\alpha) + (1-\lambda) f(\beta) - f(a) - [\lambda\alpha + (1-\lambda)\beta - a] f'(a)\| \\ &\leq |\lambda| \|\alpha - a\| \int_0^1 \|f'((1-t)a + t\alpha) - f'(a)\| dt \\ &+ |1-\lambda| \|\beta - a\| \int_0^1 \|f'((1-t)a + t\beta) - f'(a)\| dt \\ &\leq |\lambda| \|\alpha - a\| L_{\lambda,a} \int_0^1 \|(1-t)a + t\alpha - a\| dt \\ &+ |1-\lambda| \|\beta - a\| L_{\beta,a} \int_0^1 \|(1-t)a + t\beta - a\| dt \\ &= |\lambda| \|\alpha - a\|^2 L_{\lambda,a} \int_0^1 t dt + |1-\lambda| \|\beta - a\|^2 L_{\beta,a} \int_0^1 t dt \\ &= \frac{1}{2}\left[|\lambda| \|\alpha - a\|^2 L_{\lambda,a} + |1-\lambda| \|\beta - a\|^2 L_{\beta,a}\right], \end{aligned}$$

which proves (3.10). □

4. Examples for Exponential Function

Using the inequality (3.1) for the exponential function we get

$$\begin{aligned} (4.1)\quad & \|\lambda \exp\alpha + (1-\lambda)\exp\beta - \exp a\| \\ &\leq |\lambda| \|\alpha - a\| \sup_{t\in[0,1]} \|\exp((1-t)a + t\alpha)\| \\ &+ |1-\lambda| \|\beta - a\| \sup_{t\in[0,1]} \|\exp((1-t)a + t\beta)\| \\ &\leq \max\{|\lambda|, |1-\lambda|\} \\ &\times \left[\|\alpha - a\| \sup_{t\in[0,1]} \|\exp((1-t)a + t\alpha)\| + \|\beta - a\| \sup_{t\in[0,1]} \|\exp((1-t)a + t\beta)\|\right] \end{aligned}$$

for all α, $\beta \in G$, $\lambda \in \mathbb{C}$ and $a \in \mathcal{B}$.

Observe that

$$\begin{aligned}\|\exp(ta+(1-t)\mu)\| &= \|\exp[(1-t)\mu]\exp(ta)\| = |\exp[(1-t)\mu]|\,\|\exp(ta)\| \\ &= \exp[(1-t)\operatorname{Re}\mu]\,\|\exp(ta)\| \le \exp[(1-t)\operatorname{Re}\mu]\exp(t\|a\|) \\ &= \exp[(1-t)\operatorname{Re}\mu + t\|a\|] \le \exp\{\operatorname{Re}\mu, \|a\|\}\end{aligned}$$

for all $t \in [0,1]$, $\mu \in \mathbb{C}$ and $a \in \mathcal{B}$.

Therefore by the first inequality in (4.1) we get

$$\begin{aligned}&\|\lambda\exp\alpha + (1-\lambda)\exp\beta - \exp a\| \\ &\quad\le |\lambda|\,\|\alpha - a\|\exp\{\operatorname{Re}\alpha, \|a\|\} + |1-\lambda|\,\|\beta - a\|\exp\{\operatorname{Re}\beta, \|a\|\} \\ &\quad\le \exp\{\operatorname{Re}\alpha, \operatorname{Re}\beta, \|a\|\}\left[|\lambda|\,\|\alpha - a\| + |1-\lambda|\,\|\beta - a\|\right]\end{aligned} \tag{4.2}$$

and, in particular,

$$\begin{aligned}&\left\|\frac{\exp\alpha + \exp\beta}{2} - \exp a\right\| \\ &\quad\le \frac{1}{2}\left[\|\alpha - a\|\exp\{\operatorname{Re}\alpha, \|a\|\} + \|\beta - a\|\exp\{\operatorname{Re}\beta, \|a\|\}\right] \\ &\quad\le \frac{1}{2}\exp\{\operatorname{Re}\alpha, \operatorname{Re}\beta, \|a\|\}\left[\|\alpha - a\| + \|\beta - a\|\right].\end{aligned} \tag{4.3}$$

Using the identity (2.13) we get

$$\begin{aligned}&\frac{1}{2}\exp(\alpha)(1-\alpha+a) + \frac{1}{2}\exp(\beta)(1-\beta+a) - \exp a \\ &\quad= \frac{1}{2}(\alpha - a)\exp(\alpha)\int_0^1\left[\exp((1-t)(a-\alpha)) - 1\right]dt \\ &\qquad+ \frac{1}{2}(\beta - a)\exp(\beta)\int_0^1\left[\exp((1-t)(a-\beta)) - 1\right]dt,\end{aligned} \tag{4.4}$$

for all α, $\beta \in \mathbb{C}$, $\lambda \in \mathbb{C}$ and $a \in \mathcal{B}$.

If we take the norm in (4.4), then we get

$$\begin{aligned}&\left\|\frac{1}{2}\left[\exp(\alpha)(1-\alpha+a) + \exp(\beta)(1-\beta+a)\right] - \exp a\right\| \\ &\quad\le \frac{1}{2}\|\alpha - a\|\,|\exp(\alpha)|\int_0^1\|\exp((1-t)(a-\alpha)) - 1\|\,dt \\ &\qquad+ \frac{1}{2}\|\beta - a\|\,|\exp(\beta)|\int_0^1\|\exp((1-t)(a-\beta)) - 1\|\,dt,\end{aligned} \tag{4.5}$$

for all α, $\beta \in \mathbb{C}$, $\lambda \in \mathbb{C}$ and $a \in \mathcal{B}$.

In the recent paper [6] we obtained the following norm inequality for the exponential function

$$\|\exp y - \exp x\| \le \|y - x\|\int_0^1\exp(\|(1-s)x + sy\|)\,ds. \tag{4.6}$$

Therefore

$$\begin{aligned}
(4.7)\quad & \|\exp((1-t)(a-\alpha))-1\| \\
&\leq \|((1-t)(a-\alpha))\| \int_0^1 \exp(\|s((1-t)(a-\alpha))\|)\,ds \\
&= (1-t)\|a-\alpha\| \int_0^1 \exp(s(1-t)\|a-\alpha\|)\,ds
\end{aligned}$$

for $t\in[0,1]$, $\alpha\in\mathbb{C}$ and $a\in\mathcal{B}$.

If $t\neq 1$ and $a\neq\alpha$, then

$$\int_0^1 \exp(s(1-t)\|a-\alpha\|)\,ds = \frac{\exp((1-t)\|a-\alpha\|)-1}{(1-t)\|a-\alpha\|}$$

and by (4.7) we get

$$\begin{aligned}
\|\exp((1-t)(a-\alpha))-1\| &\leq \|((1-t)(a-\alpha))\| \int_0^1 \exp(\|s((1-t)(a-\alpha))\|)\,ds \\
&= \exp((1-t)\|a-\alpha\|)-1,
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_0^1 \|\exp((1-t)(a-\alpha))-1\|\,dt &\leq \int_0^1 [\exp((1-t)\|a-\alpha\|)-1]\,dt \\
&= \frac{\exp(\|a-\alpha\|)-\|a-\alpha\|-1}{\|a-\alpha\|}
\end{aligned}$$

and, similarly,

$$\int_0^1 \|\exp((1-t)(a-\beta))-1\|\,dt \leq \frac{\exp(\|a-\beta\|)-\|a-\beta\|-1}{\|a-\beta\|}.$$

Since $|\exp(\alpha)|=\exp(\operatorname{Re}\alpha)$ and $|\exp(\beta)|=\exp(\operatorname{Re}\beta)$, hence by (4.5) we get

$$\begin{aligned}
(4.8)\quad & \left\|\frac{1}{2}[\exp(\alpha)(1-\alpha+a)+\exp(\beta)(1-\beta+a)]-\exp a\right\| \\
&\leq \frac{1}{2}\exp(\operatorname{Re}\alpha)[\exp(\|a-\alpha\|)-\|a-\alpha\|-1] \\
&\quad + \frac{1}{2}\exp(\operatorname{Re}\beta)[\exp(\|a-\beta\|)-\|a-\beta\|-1],
\end{aligned}$$

for all α, $\beta\in\mathbb{C}$ and $a\in\mathcal{B}$.

If we use the identity (2.15) for the exponential function, we get

$$\begin{aligned}
(4.9)\quad & \frac{\exp(\alpha)+\exp(\beta)}{2}-\left(1+\frac{\alpha+\beta}{2}-a\right)\exp(a) \\
&= \frac{1}{2}(\alpha-a)\exp(a)\int_0^1[\exp(t(\alpha-a))-1]\,dt \\
&\quad + \frac{1}{2}(\beta-a)\exp(a)\int_0^1[\exp(t(\beta-a))-1]\,dt
\end{aligned}$$

for all α, $\beta\in\mathbb{C}$ and $a\in\mathcal{B}$.

By taking the norm in (4.9) we get

$$\left\| \frac{\exp(\alpha)+\exp(\beta)}{2} - \left(1+\frac{\alpha+\beta}{2}-a\right)\exp(a) \right\| \le \frac{1}{2}\|\alpha-a\|\,\|\exp(a)\| \int_0^1 \|\exp(t(\alpha-a))-1\|\,dt + \frac{1}{2}\|\beta-a\|\,\|\exp(a)\| \int_0^1 \|\exp(t(\beta-a))-1\|\,dt. \tag{4.10}$$

From the first inequality in (4.6) we get for $t \neq 0$

$$\|\exp(t(\alpha-a))-1\| \le t\|\alpha-a\| \int_0^1 \exp(st\|\alpha-a\|)\,ds = \exp(t\|\alpha-a\|)-1,$$

which gives

$$\int_0^1 \|\exp(t(\alpha-a))-1\|\,dt \le \int_0^1 [\exp(t\|\alpha-a\|)-1]\,dt = \frac{\exp(\|\alpha-a\|)-\|\alpha-a\|-1}{\|\alpha-a\|}$$

and, similarly

$$\int_0^1 \|\exp(t(\beta-a))-1\|\,dt \le \frac{\exp(\|\beta-a\|)-\|\beta-a\|-1}{\|\beta-a\|}.$$

Therefore, by (4.10), we have

$$\left\| \frac{\exp(\alpha)+\exp(\beta)}{2} - \left(1+\frac{\alpha+\beta}{2}-a\right)\exp(a) \right\| \le \frac{1}{2}\|\exp(a)\| [\exp(\|\alpha-a\|)+\exp(\|\beta-a\|)-\|\alpha-a\|-\|\beta-a\|-2] \tag{4.11}$$

for all α, $\beta \in \mathbb{C}$ and $a \in \mathcal{B}$.

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