NORM INEQUALITIES FOR THE ERROR IN APPROXIMATING ANALYTIC FUNCTIONS IN BANACH ALGEBRAS BY COMPLEX CHORDS

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ABSTRACT. Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$, α, $\beta \in G$, $\alpha \neq \beta$ and $f : G \to \mathbb{C}$ is analytic on G. By using the analytic functional calculus in $\mathcal{B}$ we can define the errors in approximating analytic functions in Banach algebras by complex chords as follows

$$\Phi_f(a; \alpha, \beta) := \frac{f(\alpha)(\beta - a) + f(\beta)(a - \alpha)}{\beta - \alpha} - f(a)$$

and

$$\widetilde{\Phi}_f(a; \alpha, \beta) := \frac{f(\alpha)(a - \alpha) + f(\beta)(\beta - a)}{\beta - \alpha} - f(a).$$

In this paper we provide some norm inequalities involving the functions $\Phi_f(a; \alpha, \beta)$ and $\widetilde{\Phi}_f(a; \alpha, \beta)$ defined above.

1. Introduction

Consider a function $f : [a, b] \to \mathbb{R}$ and assume that it is bounded on $[a, b]$. The chord that connects its end points $A = (a, f(a))$ and $B = (b, f(b))$ has the equation

$$d_f : [a, b] \to \mathbb{R}, \qquad d_f(x) = \frac{1}{b - a}[f(a)(b - x) + f(b)(x - a)].$$

In [7], we introduced the error in approximating the value of the function $f(x)$ by $d_f(x)$ with $x \in [a, b]$ by $\Phi_f(x)$, i.e., $\Phi_f(x)$ is defined by:

$$\Phi_f(x) := \frac{b - x}{b - a} \cdot f(a) + \frac{x - a}{b - a} \cdot f(b) - f(x). \tag{1.1}$$

The following simple result, which provides a sharp upper bound for the case of bounded functions, has been stated in [6] as an intermediate result needed to obtain a Grüss type inequality:

If $f : [a, b] \to \mathbb{R}$ is a bounded function with $-\infty < m \leq f(x) \leq M < \infty$ for any $x \in [a, b]$, then

$$|\Phi_f(x)| \leq M - m. \tag{1.2}$$

The multiplicative constant 1 *in front of $M - m$ cannot be replaced by a smaller quantity.*

The case of convex functions has been considered in [6] in order to prove another Grüss type inequality:

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If $f:[a,b]\to\mathbb{R}$ *is a convex function on* $[a,b]$*, then*

$$0\leq\Phi_f(x)\leq\frac{(b-x)(x-a)}{b-a}\left[f'_-(b)-f'_+(a)\right]\leq\frac{1}{4}(b-a)\left[f'_-(b)-f'_+(a)\right]\tag{1.3}$$

for any $x\in[a,b]$*.*

If the lateral derivatives $f'_-(b)$ *and* $f'_+(a)$ *are finite, then the second inequality and the constant* $\frac{1}{4}$ *are sharp.*

The following estimation result holds [7]:

Theorem 1. *If* $f:[a,b]\to\mathbb{R}$ *is of bounded variation, then*

$$|\Phi_f(x)|\leq\left(\frac{b-x}{b-a}\right)\cdot\bigvee_a^x(f)+\left(\frac{x-a}{b-a}\right)\cdot\bigvee_x^b(f)\tag{1.4}$$
$$\leq\begin{cases}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f);\\[2ex] \left[\left(\frac{b-x}{b-a}\right)^p+\left(\frac{x-a}{b-a}\right)^p\right]^{\frac{1}{p}}\left[\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right]^{\frac{1}{q}}\\ \qquad\qquad\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|.\end{cases}$$

The first inequality in (1.4) is sharp. The constant $\frac{1}{2}$ *is best possible in the first and third branches.*

Corollary 1. *If* $f:[a,b]\to\mathbb{R}$ *is* L_1*-Lipschitzian on* $[a,x]$ *and* L_2*-Lipschitzian on* $[x,b]$*,* $L_1,L_2>0$*, then*

$$|\Phi_f(x)|\leq\frac{(b-x)(x-a)}{b-a}(L_1+L_2)\leq\frac{1}{4}(b-a)(L_1+L_2)\tag{1.5}$$

for any $x\in[a,b]$*.*

In particular, if f *is* L*-Lipschitzian on* $[a,b]$*, then*

$$|\Phi_f(x)|\leq\frac{2(b-x)(x-a)}{b-a}L\leq\frac{1}{2}(b-a)L.\tag{1.6}$$

The constants $\frac{1}{4},2$ *and* $\frac{1}{2}$ *are best possible.*

When more information on the derivative of the function is available, then we can state the following results as well [7]:

Theorem 2. *Assume that* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous on* $[a,b]$*. If* f' *is of bounded variation on* $[a,b]$*, then*

$$|\Phi_f(x)|\leq\frac{(x-a)(b-x)}{b-a}\cdot\bigvee_a^b(f')\leq\frac{1}{4}(b-a)\bigvee_a^b(f'),\tag{1.7}$$

where $\bigvee_a^b(f')$ *denotes the total variation of* f' *on* $[a,b]$*.*

The inequalities are sharp and the constant $\frac{1}{4}$ *is best possible.*

In order to extend some of these results for functions defined on Banach algebras, we need the following preparation.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|:\mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further: $\|ab\|\leq\|a\|\|b\|$ for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\mathrm{Inv}\,(\mathcal{B})$. If $a, b \in \mathrm{Inv}\,(\mathcal{B})$ then $ab \in \mathrm{Inv}\,(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \mathrm{Inv}\,(\mathcal{B})$;
(ii) $\{a \in \mathcal{B}: \ \|1 - b\| < 1\} \subset \mathrm{Inv}\,(\mathcal{B})$;
(iii) $\mathrm{Inv}\,(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\mathrm{Inv}\,(\mathcal{B}) \ni a \longmapsto a^{-1} \in \mathrm{Inv}\,(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho\,(a) := \{z \in \mathbb{C} : \ z - a \in \mathrm{Inv}\,(\mathcal{B})\}\,;$$

the *spectrum* of a is $\sigma\,(a)$, the complement of $\rho\,(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho\,(a) \to \mathrm{Inv}\,(\mathcal{B})$, $R_a\,(z) := (z - a)^{-1}$. For each z, $w \in \rho\,(a)$ we have the identity

$$R_a\,(w) - R_a\,(z) = (z - w)\, R_a\,(z)\, R_a\,(w)\,.$$

We also have that

$$\sigma\,(a) \subset \{z \in \mathbb{C} : \ |z| \le \|a\|\}\,.$$

The *spectral radius* of a is defined as

$$\nu\,(a) = \sup\,\{|z| : z \in \sigma\,(a)\}\,.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho\,(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho\,(a)$;
(iii) The spectrum $\sigma\,(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu\,(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n\,(\xi - a)^{-1}\, d\xi$;
(v) We have $\nu\,(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma\,(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f\,(a)$ in $\mathcal{B}$ by

$$f\,(a) := \frac{1}{2\pi i}\int_\delta f\,(\xi)\,(\xi - a)^{-1}\, d\xi, \tag{1.8}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma\,(a) \subset \mathrm{ins}\,(\delta)$, the inside of δ.

It is well known (see for instance [3, pp. 201-204]) that $f\,(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma\,(f\,(a)) = f\,(\sigma\,(a)) \tag{1.9}$$

holds.

Let $\mathfrak{Hol}\,(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma\,(a)$. Note that $\mathfrak{Hol}\,(a)$ is an algebra where if f, $g \in \mathfrak{Hol}\,(a)$ and f and g have domains $D\,(f)$ and $D\,(g)$, then fg and $f + g$ have domain $D\,(f) \cap D\,(g)$. $\mathfrak{Hol}\,(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [3, p. 201-203]:

Theorem 3. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) *The map $f \mapsto f(a)$ of $\mathfrak{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.*
(b) *If $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ has radius of convergence $r > \nu(a)$, then $f \in \mathfrak{Hol}(a)$ and $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.*
(c) *If $f(z) \equiv 1$, then $f(a) = 1$.*
(d) *If $f(z) = z$ for all z, $f(a) = a$.*
(e) *If $f, f_1, ..., f_n...$ are analytic on G, $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\|f_n(a) - f(a)\| \to 0$ as $n \to \infty$.*
(f) *The Riesz Functional Calculus is unique and if a, b are commuting elements in $\mathcal{B}$ and $f \in \mathfrak{Hol}(a)$, then $f(a)b = bf(a)$.*

For some recent norm inequalities for functions on Banach algebras, see [1]-[2] and [10]-[16].

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G and α, $\beta \in G$ with $\alpha \neq \beta$,then we can define the errors in approximating analytic functions in Banach algebras by complex chords as follows

$$\Phi_f(a; \alpha, \beta) := \frac{f(\alpha)(\beta - a) + f(\beta)(a - \alpha)}{\beta - \alpha} - f(a)$$

and

$$\widetilde{\Phi}_f(a; \alpha, \beta) := \frac{f(\alpha)(a - \alpha) + f(\beta)(\beta - a)}{\beta - \alpha} - f(a).$$

Motivated by the above results, in this paper we provide some norm inequalities involving the functions $\Phi_f(a; \alpha, \beta)$ and $\widetilde{\Phi}_f(a; \alpha, \beta)$ defined above.

2. Some Identities

We have the following simple identities:

Theorem 4. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, then for all α, $\beta \in G$ with $\alpha \neq \beta$, we have*

$$\widetilde{\Phi}_f(a; \alpha, \beta) = \frac{1}{\beta - \alpha}(\beta - a)^2 \int_0^1 f'((1-t)a + t\beta)\,dt - \frac{1}{\beta - \alpha}(a - \alpha)^2 \int_0^1 f'((1-t)a + t\alpha)\,dt \tag{2.1}$$

and

$$\Phi_f(a; \alpha, \beta) = \frac{(\beta - a)(a - \alpha)}{\beta - \alpha} \int_0^1 \left[f'((1-t)a + t\beta) - f'((1-t)a + t\alpha)\right] dt. \tag{2.2}$$

Proof. Due to the convexity of D, for any ξ, $\nu \in D$ we can define the function $\varphi_{\xi,\nu} : [0,1] \to \mathbb{R}$ by $\varphi_{\xi,\nu}(t) := f((1-t)\xi + t\nu)$. The function $\varphi_{\xi,\nu}$ is differentiable on $(0,1)$ and

$$\frac{d\varphi_{\xi,\nu}(t)}{dt} = (\nu - \xi) f'((1-t)\xi + t\nu) \text{ for } t \in (0,1).$$

We have

$$f(\nu) - f(\xi) = \varphi_{\xi,\nu}(1) - \varphi_{\xi,\nu}(0) = \int_0^1 \frac{d\varphi_{\xi,\nu}(t)}{dt} dt$$
$$= (\nu - \xi) \int_0^1 f'((1-t)\xi + t\nu)\, dt$$

namely

$$f(\nu) = f(\xi) + (\nu - \xi) \int_0^1 f'((1-t)\xi + t\nu)\, dt \tag{2.3}$$

for any ξ, $\nu \in D$.

Therefore, by (2.3) we get

$$f(\alpha) = f(\xi) + (\alpha - \xi) \int_0^1 f'((1-t)\xi + t\alpha)\, dt \tag{2.4}$$

and

$$f(\beta) = f(\xi) + (\beta - \xi) \int_0^1 f'((1-t)\xi + t\beta)\, dt \tag{2.5}$$

for any $\xi \in D$.

If we multiply (2.4) and (2.5) by λ and $1-\lambda$ and add, we get the following identity that is of interest in itself

$$\begin{aligned} &\lambda f(\alpha) + (1-\lambda) f(\beta) - f(\xi) \\ &= \lambda(\alpha - \xi) \int_0^1 f'((1-t)\xi + t\alpha)\, dt + (1-\lambda)(\beta - \xi) \int_0^1 f'((1-t)\xi + t\beta)\, dt \end{aligned} \tag{2.6}$$

for any $\xi \in D$.

If we take $\lambda = \frac{\xi - \alpha}{\beta - \alpha}$, then $1 - \lambda = \frac{\beta - \xi}{\beta - \alpha}$ and by (2.6) we get

$$\begin{aligned} &\frac{\xi - \alpha}{\beta - \alpha} f(\alpha) + \frac{\beta - \xi}{\beta - \alpha} f(\beta) - f(\xi) \\ &= \frac{(\beta - \xi)^2}{\beta - \alpha} \int_0^1 f'((1-t)\xi + t\beta)\, dt - \frac{(\xi - \alpha)^2}{\beta - \alpha} \int_0^1 f'((1-t)\xi + t\alpha)\, dt, \end{aligned} \tag{2.7}$$

for any $\xi \in D$.

Also, if we take $\lambda = \frac{\beta - \xi}{\beta - \alpha}$, then $1 - \lambda = \frac{\xi - \alpha}{\beta - \alpha}$ and by (2.6) we get

$$\begin{aligned} &\frac{\beta - \xi}{\beta - \alpha} f(\alpha) + \frac{\xi - \alpha}{\beta - \alpha} f(\beta) - f(\xi) \\ &= \frac{(\beta - \xi)(\xi - \alpha)}{\beta - \alpha} \left[\int_0^1 f'((1-t)\xi + t\beta)\, dt - \int_0^1 f'((1-t)\xi + t\alpha)\, dt \right], \end{aligned} \tag{2.8}$$

for any $\xi \in D$.

From the identity (2.7) we get

$$\begin{aligned}(2.9)\quad & \frac{f(\alpha)}{\beta-\alpha}\frac{1}{2\pi i}\int_{\delta}(\xi-\alpha)(\xi-a)^{-1}d\xi+\frac{f(\beta)}{\beta-\alpha}\frac{1}{2\pi i}\int_{\delta}(\beta-\xi)(\xi-a)^{-1}d\xi \\ & \qquad -\frac{1}{2\pi i}\int_{\delta}f(\xi)(\xi-a)^{-1}d\xi \\ &=\frac{1}{\beta-\alpha}\frac{1}{2\pi i}\int_{\delta}(\beta-\xi)^{2}\left(\int_{0}^{1}f'((1-t)\xi+t\beta)\,dt\right)(\xi-a)^{-1}d\xi \\ &-\frac{1}{\beta-\alpha}\frac{1}{2\pi i}\int_{\delta}(\xi-\alpha)^{2}\left(\int_{0}^{1}f'((1-t)\xi+t\alpha)\,dt\right)(\xi-a)^{-1}d\xi \\ &=\frac{1}{\beta-\alpha}\int_{0}^{1}\left(\frac{1}{2\pi i}\int_{\delta}(\beta-\xi)^{2}f'((1-t)\xi+t\beta)(\xi-a)^{-1}d\xi\right)dt \\ &\quad-\frac{1}{\beta-\alpha}\int_{0}^{1}\left(\frac{1}{2\pi i}\int_{\delta}(\xi-\alpha)^{2}f'((1-t)\xi+t\alpha)(\xi-a)^{-1}d\xi\right)dt,\end{aligned}$$

where for the last equality we used Fubini's theorem.

Since the functions

$$G\ni\mapsto(\beta-\xi)^{2}f'((1-t)\xi+t\beta)\in\mathbb{C}$$

and

$$G\ni\mapsto(\beta-\xi)^{2}f'((1-t)\xi+t\beta)\in\mathbb{C}$$

are analytic on G for all α, $\beta\in G$ and $t\in[0,1]$, then by the analytic functional calculus we have

$$\frac{1}{2\pi i}\int_{\delta}(\beta-\xi)^{2}f'((1-t)\xi+t\beta)(\xi-a)^{-1}d\xi=(\beta-a)^{2}f'((1-t)a+t\beta)$$

and

$$\frac{1}{2\pi i}\int_{\delta}(\xi-\alpha)^{2}f'((1-t)\xi+t\alpha)(\xi-a)^{-1}d\xi=(a-\alpha)^{2}f'((1-t)a+t\alpha).$$

Also, we have

$$\frac{1}{2\pi i}\int_{\delta}(\xi-\alpha)(\xi-a)^{-1}d\xi=a-\alpha,\ \frac{1}{2\pi i}\int_{\delta}(\beta-\xi)(\xi-a)^{-1}d\xi=\beta-a$$

and by (2.9) we get (2.1).

The identity (2.2) follows in a similar way from (2.8) and we omit the details. □

We have the following perturbed versions of the identities above:

Corollary 2. *With the assumptions of Theorem 4 and if b, $c\in\mathcal{B}$, then*

$$\begin{aligned}(2.10)\quad & \widetilde{\Phi}_{f}(a;\alpha,\beta)-\frac{1}{\beta-\alpha}(\beta-a)^{2}b+\frac{1}{\beta-\alpha}(a-\alpha)^{2}c \\ &=\frac{1}{\beta-\alpha}(\beta-a)^{2}\int_{0}^{1}\left[f'((1-t)a+t\beta)-b\right]dt \\ &\qquad-\frac{1}{\beta-\alpha}(a-\alpha)^{2}\int_{0}^{1}\left[f'((1-t)a+t\alpha)-c\right]dt\end{aligned}$$

and

$$\Phi_f(a;\alpha,\beta)-\frac{(\beta-a)(a-\alpha)}{\beta-\alpha}(b-c)=\frac{(\beta-a)(a-\alpha)}{\beta-\alpha}\times\left[\int_0^1\left[f'((1-t)a+t\beta)-b\right]dt-\int_0^1\left[f'((1-t)a+t\alpha)-c\right]dt\right]. \tag{2.11}$$

In particular, for $c=b$, we have

$$\widetilde{\Phi}_f(a;\alpha,\beta)+2(\beta-\alpha)\left(a-\frac{\beta+\alpha}{2}\right)b=\frac{1}{\beta-\alpha}(\beta-a)^2\int_0^1\left[f'((1-t)a+t\beta)-b\right]dt-\frac{1}{\beta-\alpha}(a-\alpha)^2\int_0^1\left[f'((1-t)a+t\alpha)-b\right]dt \tag{2.12}$$

and

$$\Phi_f(a;\alpha,\beta)=\frac{(\beta-a)(a-\alpha)}{\beta-\alpha}\times\left[\int_0^1\left[f'((1-t)a+t\beta)-b\right]dt-\int_0^1\left[f'((1-t)a+t\alpha)-b\right]dt\right]. \tag{2.13}$$

Remark 1. *If we take $b=f'(\beta)$ and $c=f'(\alpha)$ in (2.10) and (2.11), then we get*

$$\widetilde{\Phi}_f(a;\alpha,\beta)-\frac{f'(\beta)}{\beta-\alpha}(\beta-a)^2+\frac{f'(\alpha)}{\beta-\alpha}(a-\alpha)^2=\frac{1}{\beta-\alpha}(\beta-a)^2\int_0^1\left[f'((1-t)a+t\beta)-f'(\beta)\right]dt-\frac{1}{\beta-\alpha}(a-\alpha)^2\int_0^1\left[f'((1-t)a+t\alpha)-f'(\alpha)\right]dt \tag{2.14}$$

and

$$\Phi_f(a;\alpha,\beta)-\frac{f'(\beta)-f'(\alpha)}{\beta-\alpha}(\beta-a)(a-\alpha)=\frac{(\beta-a)(a-\alpha)}{\beta-\alpha}\times\left[\int_0^1\left[f'((1-t)a+t\beta)-f'(\beta)\right]dt-\int_0^1\left[f'((1-t)a+t\alpha)-f'(\alpha)\right]dt\right]. \tag{2.15}$$

If we take $b=f'(a)$ in (2.12) and (2.13), then we get

$$\widetilde{\Phi}_f(a;\alpha,\beta)+2(\beta-\alpha)\left(a-\frac{\beta+\alpha}{2}\right)f'(a)=\frac{1}{\beta-\alpha}(\beta-a)^2\int_0^1\left[f'((1-t)a+t\beta)-f'(a)\right]dt-\frac{1}{\beta-\alpha}(a-\alpha)^2\int_0^1\left[f'((1-t)a+t\alpha)-f'(a)\right]dt \tag{2.16}$$

and

$$\Phi_f(a;\alpha,\beta)=\frac{(\beta-a)(a-\alpha)}{\beta-\alpha} \times\left[\int_0^1\left[f'((1-t)a+t\beta)-f'(a)\right]dt-\int_0^1\left[f'((1-t)a+t\alpha)-f'(a)\right]dt\right]. \tag{2.17}$$

3. Norm Inequalities

We have:

Theorem 5. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic on G, then for all α, $\beta\in G$ with $\alpha\neq\beta$, we have*

$$\begin{aligned}\left\|\widetilde{\Phi}_f(a;\alpha,\beta)\right\| &\le \frac{1}{|\beta-\alpha|}\|\beta-a\|^2\int_0^1\|f'((1-t)a+t\beta)\|\,dt \\ &\quad+\frac{1}{|\beta-\alpha|}\|a-\alpha\|^2\int_0^1\|f'((1-t)a+t\alpha)\|\,dt \\ &\le \frac{1}{|\beta-\alpha|}\|\beta-a\|^2\sup_{t\in[0,1]}\|f'((1-t)a+t\beta)\| \\ &\quad+\frac{1}{|\beta-\alpha|}\|a-\alpha\|^2\sup_{t\in[0,1]}\|f'((1-t)a+t\alpha)\| \\ &\le \frac{1}{|\beta-\alpha|}\max\left\{\sup_{t\in[0,1]}\|f'((1-t)a+t\beta)\|,\sup_{t\in[0,1]}\|f'((1-t)a+t\alpha)\|\right\} \\ &\qquad\times\left[\|\beta-a\|^2+\|a-\alpha\|^2\right]\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&\|\Phi_f(a;\alpha,\beta)\| \\ &\le \frac{\|(\beta-a)(a-\alpha)\|}{|\beta-\alpha|}\int_0^1\|f'((1-t)a+t\beta)-f'((1-t)a+t\alpha)\|\,dt \\ &\le \frac{\|\beta-a\|\,\|a-\alpha\|}{|\beta-\alpha|}\int_0^1\left[\|f'((1-t)a+t\beta)\|+\|f'((1-t)a+t\alpha)\|\right]dt \\ &\le \frac{\|\beta-a\|\,\|a-\alpha\|}{|\beta-\alpha|}\sup_{t\in[0,1]}\left[\|f'((1-t)a+t\beta)\|+\|f'((1-t)a+t\alpha)\|\right].\end{aligned} \tag{3.2}$$

Proof. By taking the norm in the identity (2.1), we get

$$\begin{aligned}\left\|\widetilde{\Phi}_f(a;\alpha,\beta)\right\| &\leq \frac{1}{|\beta-\alpha|}\left\|(\beta-a)^2\int_0^1 f'((1-t)a+t\beta)\,dt\right\| \\ &\quad+\frac{1}{|\beta-\alpha|}\left\|(a-\alpha)^2\int_0^1 f'((1-t)a+t\alpha)\,dt\right\| \\ &\leq \frac{1}{|\beta-\alpha|}\left\|(\beta-a)^2\right\|\left\|\int_0^1 f'((1-t)a+t\beta)\,dt\right\| \\ &\quad+\frac{1}{|\beta-\alpha|}\left\|(a-\alpha)^2\right\|\left\|\int_0^1 f'((1-t)a+t\alpha)\,dt\right\| \\ &\leq \frac{1}{|\beta-\alpha|}\|\beta-a\|^2\int_0^1\|f'((1-t)a+t\beta)\|\,dt \\ &\qquad+\frac{1}{|\beta-\alpha|}\|a-\alpha\|^2\int_0^1\|f'((1-t)a+t\alpha)\|\,dt,\end{aligned}$$

which proves the desired inequality (3.1).

The inequality (3.2) follows by (2.2). □

Corollary 3. *With the assumptions of Theorem 5 and if*

$$\|f'\|_{a,G} := \sup_{(t,\alpha)\in[0,1]\times G}\|f'((1-t)a+t\alpha)\| < \infty,$$

then

$$\left\|\widetilde{\Phi}_f(a;\alpha,\beta)\right\| \leq \frac{1}{|\beta-\alpha|}\|f'\|_{a,G}\left[\|\beta-a\|^2+\|a-\alpha\|^2\right]. \tag{3.3}$$

Corollary 4. *With the assumptions of Theorem 5 and if*

$$\|f'((1-t)a+t\beta)-f'((1-t)a+t\alpha)\| \leq tL_{\alpha,\beta}|\beta-\alpha| \text{ for } t\in[0,1],$$

then

$$\|\Phi_f(a;\alpha,\beta)\| \leq \frac{1}{2}\|(\beta-a)(a-\alpha)\|L_{\alpha,\beta} \leq \frac{1}{2}\|\beta-a\|\|a-\alpha\|L_{\alpha,\beta}. \tag{3.4}$$

Let $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$ and $\lambda\in G$. We define $G_{\lambda,a} := \{(1-t)\lambda+ta \mid \text{with } t\in[0,1]\}$. We observe that $G_{\lambda,a}$ is a convex subset in $\mathcal{B}$ for every $\lambda\in G$.

We say that the function $g: G_{\lambda,a}\to B$ is Lipschitzian on $G_{\lambda,a}$ with the constant $L_{\lambda,a}>0$, if for all x, $y\in G_{\lambda,a}$ we have

$$\|g(x)-g(y)\| \leq L_{\lambda,a}\|x-y\|.$$

This is equivalent to

$$\|g((1-t)\lambda+ta)-g((1-s)\lambda+sa)\| \leq L_{\lambda,a}|t-s|\|a-\lambda\| \tag{3.5}$$

for all t, $s\in[0,1]$. We write this by $g\in\mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$.

Let $h: G\to\mathbb{C}$ be an analytic function on G. For $t\in[0,1]$ and $\lambda\in G$, the auxiliary function $h_{t,\lambda}$ defined on G by $h_{t,\lambda}(\xi) := h((1-t)\lambda+t\xi)$ is also analytic and using the analytic functional calculus (1.8) for the element $a \in \mathcal{B}$, we can define

$$\widetilde{h}\left((1-t)\lambda+ta\right) := h_{t,\lambda}(a) = \frac{1}{2\pi i}\int_{\gamma} h_{t,\lambda}(\xi)(\xi-a)^{-1}d\xi \tag{3.6}$$
$$= \frac{1}{2\pi i}\int_{\gamma} h\left((1-t)\lambda+t\xi\right)(\xi-a)^{-1}d\xi.$$

We say that the scalar function $h \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$ if its extension $\widetilde{h} : G_{\lambda,a} \to B$ satisfies the Lipschitz condition (3.5).

Theorem 6. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, α, $\beta \in G$ with $\alpha \neq \beta$ and $f' \in \mathfrak{Lip}_{L_{\alpha,a}}(G_{\lambda,a}) \cap \mathfrak{Lip}_{L_{\beta,a}}(G_{\lambda,a})$ for some $L_{\alpha,a}$, $L_{\beta,a} > 0$, then we have*

$$\left\|\widetilde{\Phi}_f(a;\alpha,\beta) - \frac{f'(\beta)}{\beta-\alpha}(\beta-a)^2 + \frac{f'(\alpha)}{\beta-\alpha}(a-\alpha)^2\right\| \tag{3.7}$$
$$\leq \frac{1}{2|\beta-\alpha|}\left[\|\beta-a\|^3 L_{\beta,a} + \|a-\alpha\|^3 L_{\alpha,a}\right]$$
$$\leq \frac{1}{2|\beta-\alpha|}\max\{L_{\beta,a}, L_{\alpha,a}\}\left[\|\beta-a\|^3 + \|a-\alpha\|^3\right]$$

and

$$\left\|\Phi_f(a;\alpha,\beta) - \frac{f'(\beta)-f'(\alpha)}{\beta-\alpha}(\beta-a)(a-\alpha)\right\| \tag{3.8}$$
$$\leq \frac{\|(\beta-a)(a-\alpha)\|}{2|\beta-\alpha|}\left[\|\beta-a\| L_{\beta,a} + \|a-\alpha\| L_{\alpha,a}\right]$$
$$\leq \frac{\|(\beta-a)(a-\alpha)\|}{2|\beta-\alpha|}\max\{L_{\beta,a}, L_{\alpha,a}\}\left[\|\beta-a\| + \|a-\alpha\|\right].$$

Proof. From the identity (2.14) and by the fact that $f' \in \mathfrak{Lip}_{L_{\alpha,a}}(G_{\lambda,a}) \cap \mathfrak{Lip}_{L_{\beta,a}}(G_{\lambda,a})$, we have

$$\left\|\widetilde{\Phi}_f(a;\alpha,\beta) - \frac{1}{\beta-\alpha}f'(\beta)(\beta-a)^2 + \frac{1}{\beta-\alpha}f'(\alpha)(a-\alpha)^2\right\|$$
$$\leq \frac{1}{|\beta-\alpha|}\left\|(\beta-a)^2\int_0^1\left[f'((1-t)a+t\beta) - f'(\beta)\right]dt\right\|$$
$$+ \frac{1}{|\beta-\alpha|}\left\|(a-\alpha)^2\int_0^1\left[f'((1-t)a+t\alpha) - f'(\alpha)\right]dt\right\|$$
$$\leq \frac{1}{|\beta-\alpha|}\left\|(\beta-a)^2\right\|\left\|\int_0^1\left[f'((1-t)a+t\beta) - f'(\beta)\right]dt\right\|$$
$$+ \frac{1}{|\beta-\alpha|}\left\|(a-\alpha)^2\right\|\left\|\int_0^1\left[f'((1-t)a+t\alpha) - f'(\alpha)\right]dt\right\|$$

$$\leq \frac{1}{|\beta-\alpha|}\|\beta-a\|^2\int_0^1\|f'((1-t)a+t\beta)-f'(\beta)\|dt$$
$$+\frac{1}{|\beta-\alpha|}\|a-\alpha\|^2\int_0^1\|f'((1-t)a+t\alpha)-f'(\alpha)\|dt$$
$$\leq \frac{1}{|\beta-\alpha|}\|\beta-a\|^3 L_{\beta,a}\int_0^1(1-t)\,dt+\frac{1}{|\beta-\alpha|}\|a-\alpha\|^3\int_0^1(1-t)\,dt$$
$$=\frac{1}{2|\beta-\alpha|}\left[\|\beta-a\|^3 L_{\beta,a}+\|a-\alpha\|^3 L_{\alpha,a}\right]$$
$$\leq \frac{1}{2|\beta-\alpha|}\max\{L_{\beta,a},L_{\alpha,a}\}\left[\|\beta-a\|^3+\|a-\alpha\|^3\right],$$

which proves (3.7).

The inequality (3.8) follows by (2.15). □

Theorem 7. *With the assumptions of Theorem 7 we have*

$$\left\|\widetilde{\Phi}_f(a;\alpha,\beta)-2(\beta-\alpha)\left(\frac{\beta+\alpha}{2}-a\right)f'(a)\right\| \tag{3.9}$$
$$\leq \frac{1}{2|\beta-\alpha|}\left[\|\beta-a\|^3 L_{\beta,a}+\|a-\alpha\|^3 L_{\alpha,a}\right]$$
$$\leq \frac{1}{2|\beta-\alpha|}\max\{L_{\beta,a},L_{\alpha,a}\}\left[\|\beta-a\|^3+\|a-\alpha\|^3\right]$$

and

$$\|\Phi_f(a;\alpha,\beta)\|\leq\frac{\|(\beta-a)(a-\alpha)\|}{2|\beta-\alpha|}\left[\|\beta-a\|L_{\beta,a}+\|a-\alpha\|L_{\alpha,a}\right] \tag{3.10}$$
$$\leq\frac{\|(\beta-a)(a-\alpha)\|}{2|\beta-\alpha|}\max\{L_{\beta,a},L_{\alpha,a}\}\left[\|\beta-a\|+\|a-\alpha\|\right].$$

Proof. By using the identity (2.16), we get after several steps, that

$$\left\|\widetilde{\Phi}_f(a;\alpha,\beta)+2(\beta-\alpha)\left(a-\frac{\beta+\alpha}{2}\right)f'(a)\right\|$$
$$\leq \frac{1}{|\beta-\alpha|}\|\beta-a\|^2\int_0^1\|f'((1-t)a+t\beta)-f'(a)\|dt$$
$$+\frac{1}{|\beta-\alpha|}\|a-\alpha\|^2\int_0^1\|f'((1-t)a+t\alpha)-f'(a)\|dt$$
$$\leq \frac{1}{2|\beta-\alpha|}\left[\|\beta-a\|^3 L_{\beta,a}+\|a-\alpha\|^3 L_{\alpha,a}\right]$$
$$\leq \frac{1}{2|\beta-\alpha|}\max\{L_{\beta,a},L_{\alpha,a}\}\left[\|\beta-a\|^3+\|a-\alpha\|^3\right],$$

which proves (3.9).

The inequality (3.10) follows by the identity (2.17) and we omit the details. □

4. Some Examples for Exponential

For the exponential function on the Banach algebra $\mathcal{B}$ we have

$$\Phi_{\exp}(a;\alpha,\beta):=\frac{\exp(\alpha)(\beta-a)+\exp(\beta)(a-\alpha)}{\beta-\alpha}-\exp(a)$$

and

$$\widetilde{\Phi}_{\exp}(a;\alpha,\beta):=\frac{\exp(\alpha)(a-\alpha)+\exp(\beta)(\beta-a)}{\beta-\alpha}-\exp(a),$$

where $a\in\mathcal{B}$, α, $\beta\in\mathbb{C}$ and $\alpha\neq\beta$.

From the first inequality in (3.1) we have

$$\left\|\widetilde{\Phi}_{\exp}(a;\alpha,\beta)\right\|\leq\frac{1}{|\beta-\alpha|}\|\beta-a\|^2\int_0^1\|\exp((1-t)a+t\beta)\|\,dt \\ +\frac{1}{|\beta-\alpha|}\|a-\alpha\|^2\int_0^1\|\exp((1-t)a+t\alpha)\|\,dt. \tag{4.1}$$

Observe that

$$\begin{aligned}\|\exp(ta+(1-t)\mu)\|&=\|\exp[(1-t)\mu]\exp(ta)\|=|\exp[(1-t)\mu]|\,\|\exp(ta)\|\\&=\exp[(1-t)\operatorname{Re}\mu]\,\|\exp(ta)\|\leq\exp[(1-t)\operatorname{Re}\mu]\exp(t\|a\|)\\&=\exp[(1-t)\operatorname{Re}\mu+t\|a\|]\end{aligned}$$

for all $t\in[0,1]$, $\mu\in\mathbb{C}$ and $a\in\mathcal{B}$.

This implies that

$$\begin{aligned}\int_0^1\|\exp(ta+(1-t)\mu)\|\,dt&\leq\int_0^1\exp[(1-t)\operatorname{Re}\mu+t\|a\|]\,dt\\&=\begin{cases}\frac{\exp(\|a\|)-\exp(\operatorname{Re}\mu)}{\|a\|-\operatorname{Re}\mu} & \text{if } \operatorname{Re}\mu\neq\|a\|,\\[2ex] \exp(\|a\|) & \text{if } \operatorname{Re}\mu=\|a\|.\end{cases}\end{aligned}$$

Therefore, by (4.1) we get

$$\begin{aligned}&\left\|\widetilde{\Phi}_{\exp}(a;\alpha,\beta)\right\|\\&\leq\frac{1}{|\beta-\alpha|}\|\beta-a\|^2\begin{cases}\frac{\exp(\|a\|)-\exp(\operatorname{Re}\beta)}{\|a\|-\operatorname{Re}\beta} & \text{if } \operatorname{Re}\beta\neq\|a\|\\[2ex] \exp(\|a\|) & \text{if } \operatorname{Re}\beta=\|a\|\end{cases}\\&+\frac{1}{|\beta-\alpha|}\|a-\alpha\|^2\begin{cases}\frac{\exp(\|a\|)-\exp(\operatorname{Re}\alpha)}{\|a\|-\operatorname{Re}\alpha} & \text{if } \operatorname{Re}\alpha\neq\|a\|\\[2ex] \exp(\|a\|) & \text{if } \operatorname{Re}\alpha=\|a\|.\end{cases}\end{aligned}\tag{4.2}$$

Using the first inequality in (3.2) we have

$$\begin{aligned}&\|\Phi_{\exp}(a;\alpha,\beta)\|\\&\leq\frac{\|(\beta-a)(a-\alpha)\|}{|\beta-\alpha|}\int_0^1\|\exp((1-t)a+t\beta)-\exp((1-t)a+t\alpha)\|\,dt.\end{aligned}\tag{4.3}$$

Observe that

$$\exp((1-t)a+t\beta)-\exp((1-t)a+t\alpha)=[\exp(t\beta)-\exp(t\alpha)][\exp(1-t)a],$$

which implies that

$$\left\|\exp\left((1-t)\,a+t\beta\right)-\exp\left((1-t)\,a+t\alpha\right)\right\| \tag{4.4}$$
$$=\left|\exp\left(t\beta\right)-\exp\left(t\alpha\right)\right|\left\|\exp\left[(1-t)\,a\right]\right\|\leq\left|\exp\left(t\beta\right)-\exp\left(t\alpha\right)\right|\exp\left[(1-t)\left\|a\right\|\right]$$

In the recent paper [11] we obtained the following norm inequality for the exponential function

$$\left\|\exp y-\exp x\right\|\leq\left\|y-x\right\|\int_0^1\exp\left(\left\|(1-s)\,x+sy\right\|\right)ds. \tag{4.5}$$

This implies that

$$\begin{aligned}\left|\exp\left(t\beta\right)-\exp\left(t\alpha\right)\right| &\leq t\left|\beta-\alpha\right|\int_0^1\exp\left[t\left|(1-s)\,\beta+s\alpha\right|\right]ds \\ &\leq t\left|\beta-\alpha\right|\int_0^1\exp\left[\left((1-s)\,t\left|\beta\right|+st\left|\alpha\right|\right)\right]ds \\ &= t\left|\beta-\alpha\right|\frac{\exp t\left|\beta\right|-\exp t\left|\alpha\right|}{t\left|\beta\right|-t\left|\alpha\right|}=\frac{\left|\beta-\alpha\right|}{\left|\beta\right|-\left|\alpha\right|}\left[\exp\left(t\left|\beta\right|\right)-\exp\left(t\left|\alpha\right|\right)\right]\end{aligned} \tag{4.6}$$

and by (4.4) we get

$$\begin{aligned}&\left\|\exp\left((1-t)\,a+t\beta\right)-\exp\left((1-t)\,a+t\alpha\right)\right\| \\ &\qquad\leq\exp\left[(1-t)\left\|a\right\|\right]\frac{\left|\beta-\alpha\right|}{\left|\beta\right|-\left|\alpha\right|}\left[\exp\left(t\left|\beta\right|\right)-\exp\left(t\left|\alpha\right|\right)\right] \\ &\qquad=\frac{\left|\beta-\alpha\right|}{\left|\beta\right|-\left|\alpha\right|}\left[\exp\left((1-t)\left\|a\right\|+t\left|\beta\right|\right)-\exp\left((1-t)\left\|a\right\|+t\left|\alpha\right|\right)\right].\end{aligned}$$

By integrating this inequality in $[0,1]$ we get

$$\begin{aligned}&\int_0^1\left\|\exp\left((1-t)\,a+t\beta\right)-\exp\left((1-t)\,a+t\alpha\right)\right\|dt \\ &\leq\frac{\left|\beta-\alpha\right|}{\left|\beta\right|-\left|\alpha\right|}\left[\int_0^1\exp\left((1-t)\left\|a\right\|+t\left|\beta\right|\right)dt-\int_0^1\exp\left((1-t)\left\|a\right\|+t\left|\alpha\right|\right)dt\right] \\ &\qquad=\frac{\left|\beta-\alpha\right|}{\left|\beta\right|-\left|\alpha\right|}\left[\frac{\exp\left\|a\right\|-\exp\left|\beta\right|}{\left\|a\right\|-\left|\beta\right|}-\frac{\exp\left\|a\right\|-\exp\left|\alpha\right|}{\left\|a\right\|-\left|\alpha\right|}\right],\end{aligned}$$

provided $\left\|a\right\|\neq\left|\beta\right|,\left\|a\right\|\neq\left|\alpha\right|.$

Therefore by (4.3) we obtain:

$$\begin{aligned}&\left\|\Phi_{\exp}\left(a;\alpha,\beta\right)\right\| \\ &\qquad\leq\frac{\left\|\left(\beta-a\right)\left(a-\alpha\right)\right\|}{\left|\beta\right|-\left|\alpha\right|}\left[\frac{\exp\left|\beta\right|-\exp\left\|a\right\|}{\left|\beta\right|-\left\|a\right\|}-\frac{\exp\left\|a\right\|-\exp\left|\alpha\right|}{\left\|a\right\|-\left|\alpha\right|}\right]\end{aligned} \tag{4.7}$$

provided $a\in\mathcal{B}$, $\alpha,\ \beta\in\mathbb{C}$ and $\alpha\neq\beta$, $\left\|a\right\|\neq\left|\beta\right|,\ \left\|a\right\|\neq\left|\alpha\right|.$

Similar inequalities may be obtained by employing the other general results above, however the details are not presented here.

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