Complex Korovkin Theory

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Abstract

Let K be a compact convex subspace of $\mathbb{C}$ and $C(K, \mathbb{C})$ the space of continuous functions from K into $\mathbb{C}$. We consider bounded linear functionals from $C(K, \mathbb{C})$ into $\mathbb{C}$ and bounded linear operators from $C(K, \mathbb{C})$ into itself. We assume that these are bounded by companion real positive linear entities, respectively. We study quantitatively the rate of convergence of the approximation of these linearities to the corresponding unit elements. Our results are inequalities of Korovkin type involving the complex modulus of continuity and basic test functions.

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1 Introduction

The study of the convergence of positive linear operators became more intensive and attractive when P. Korovkin (1953) proved his famous theorem (see [7], p. 14).

Korovkin's First Theorem. Let $[a, b]$ be a compact interval in $\mathbb{R}$ and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators L_n mapping $C([a, b])$ into itself. Assume that $(L_n f)$ converges uniformly to f for the three test functions $f = 1, x, x^2$. Then $(L_n f)$ converges uniformly to f on $[a, b]$ for all functions of $f \in C([a, b])$.

So a lot of authors since then have worked on the theoretical aspects of the above convergence. But R. A. Mamedov (1959) (see [8]) was the first to put Korovkin's theorem in a quantitative scheme.

Mamedov's Theorem. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators in the space $C([a, b])$, for which $L_n 1 = 1$, $L_n(t, x) = x + \alpha_n(x)$, $L_n(t^2, x) = x^2 + \beta_n(x)$. Then it holds

$$\|L_n(f, x) - f(x)\|_\infty \leq 3\omega_1\left(f, \sqrt{d_n}\right),$$

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where ω_1 is the first modulus of continuity and $d_n = \|\beta_n(x) - 2x\alpha_n(x)\|_\infty$.

An improvement of the last result was the following.

Shisha and Mond's Theorem. (1968, see [10]). Let $[a,b] \subset \mathbb{R}$ be a compact interval. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators acting on $C([a,b])$. For $n = 1, 2, ...,$ suppose $L_n(1)$ is bounded. Let $f \in C([a,b])$. Then for $n = 1, 2, ...,$ it holds

$$\|L_n f - f\|_\infty \le \|f\|_\infty \cdot \|L_n 1 - 1\|_\infty + \|L_n(1) + 1\|_\infty \cdot \omega_1(f, \mu_n),$$

where

$$\mu_n := \left\|\left(L_n\left((t-x)^2\right)\right)(x)\right\|_\infty^{\frac{1}{2}}.$$

Shisha-Mond inequality generated and inspired a lot of research done by many authors worldwide on the rate of convergence of a sequence of positive linear operators to the unit operator, always producing similar inequalities however in many different directions, e.g., see the important work of H. Censka of 1983 in [6], etc.

The author (see [1]) in his 1993 research monograph, produces in many directions best upper bounds for $|(L_n f)(x_0) - f(x_0)|$, $x_0 \in Q \subseteq \mathbb{R}^n$, $n \ge 1$, compact and convex, which lead for the first time to sharp/attained inequalities of Shisha-Mond type. The method of proving is probabilistic from the theory of moments. His pointwise approach is closely related to the study of the weak convergence with rates of a sequence of finite positive measures to the unit measure at a specific point.

The author in [3], pp. 383-412 continued this work in an abstract setting: Let X be a normed vector space, Y be a Banach lattice; $M \subset X$ is a compact and convex subset. Consider the space of continuous functions from M into Y, denoted by $C(M,Y)$; also consider the space of bounded functions $B(M,Y)$. He studied the rate of the uniform convergence of lattice homomorphisms $T : C(M,Y) \to C(M,Y)$ or $T : C(M,Y) \to B(M,Y)$ to the unit operator I. See also [2].

Also the author in [4], pp. 175-188 continued the last abstract work for bounded linear operators that are bounded by companion real positive linear operators. Here the invoved functions are from $[a,b] \subset \mathbb{R}$ into $(X, \|\cdot\|)$ a Banach space.

All the above have inspired and motivated the work of this article. Our results are of Shisha-Mond type, i.e., of Korovkin type.

Namely here let K be a convex and compact subset of $\mathbb{C}$ and l be a linear functional from $C(K,\mathbb{C})$ into $\mathbb{C}$, and let $\widetilde{l}$ be a positive linear functional from $C(K,\mathbb{R})$ into $\mathbb{R}$, such that $|l(f)| \le \widetilde{l}(|f|)$, $\forall\, f \in C(K,\mathbb{C})$.

Clearly then l is a bounded linear functional. Initially we create a quantitative Korovkin type theory over the last described setting, then we transfer these results to related bounded linear operators with similar properties.

2 Background

We need

Theorem 1 *Let $K \subseteq (\mathbb{C}, |\cdot|)$ and f a function from K into $\mathbb{C}$. Consider the first complex modulus of continuity*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in K \\ |x-y|<\delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{1}$$

We have:

(1)' If K is open convex or compact convex, then $\omega_1(f,\delta) < \infty$, $\forall\ \delta > 0$, where $f \in UC(K,\mathbb{C})$ (uniformly continuous functions).

(2)' If K is open convex or compact convex, then $\omega_1(f,\delta)$ is continuous on $\mathbb{R}_+$ in δ, for $f \in UC(K,\mathbb{C})$.

(3)' If K is convex, then

$$\omega_1(f, t_1 + t_2) \leq \omega_1(f, t_1) + \omega_1(f, t_2), \quad t_1, t_2 > 0, \tag{2}$$

that is the subadditivity property is true. Also it holds

$$\omega_1(f, n\delta) \leq n\omega_1(f,\delta) \tag{3}$$

and

$$\omega_1(f, \lambda\delta) \leq \lceil\lambda\rceil\, \omega_1(f,\delta) \leq (\lambda+1)\,\omega_1(f,\delta), \tag{4}$$

where $n \in \mathbb{N}$, $\lambda > 0$, $\delta > 0$, $\lceil\cdot\rceil$ is the ceiling of the number.

(4)' Clearly in general $\omega_1(f,\delta) \geq 0$ and is increasing in $\delta > 0$ and $\omega_1(f,0) = 0$.

(5)' If K is open or compact, then $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, iff $f \in UC(K,\mathbb{C})$.

(6)' It holds

$$\omega_1(f+g,\delta) \leq \omega_1(f,\delta) + \omega_1(g,\delta), \tag{5}$$

for $\delta > 0$, any $f,g : K \to \mathbb{C}$, $K \subset \mathbb{C}$ is arbitrary.

Proof. (1)' Here K is open convex. Let here $f \in UC(K,\mathbb{C})$, iff $\forall\ \varepsilon > 0$, $\exists\ \delta > 0 : |x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Let $\varepsilon_0 > 0$ then $\exists\ \delta_0 > 0 : |x-y| \leq \delta_0$ with $|f(x) - f(y)| < \varepsilon_0$, hence $\omega_1(f,\delta_0) \leq \varepsilon_0 < \infty$.

Let $\delta > 0$ arbitrary and $x, y \in K : |x-y| \leq \delta$. Choose $n \in \mathbb{N} : n\delta_0 > \delta$, and set $x_i = x + \frac{i}{n}(y-x)$, $0 \leq i \leq n$. Notice that all $x_i \in K$. Then

$$|f(x) - f(y)| = \left|\sum_{i=0}^{n-1}\left(f(x_i) - f(x_{i+1})\right)\right| \leq$$

$$|f(x) - f(x_1)| + |f(x_1) - f(x_2)| + |f(x_2) - f(x_3)| + ... + |f(x_{n-1}) - f(y)| \leq$$

$$n\omega_1(f,\delta_0) \leq n\varepsilon_0 < \infty,$$

since $|x_i - x_{i+1}| = \frac{1}{n}|x-y| \leq \frac{1}{n}\delta < \delta_0$.

Thus $\omega_1(f,\delta) \leq n\varepsilon_0 < \infty$, proving the claim. If K is compact convex, then claim is obvious.

(2)' Let $x, y \in K$ and let $|x-y| \leq t_1 + t_2$, then there exists a point $z \in \overline{xy}$, $z \in K : |x-z| \leq t_1$ and $|y-z| \leq t_2$, where $t_1, t_2 > 0$.

Notice that

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq \omega_1(f,t_1) + \omega_1(f,t_2).$$

Hence

$$\omega_1(f, t_1 + t_2) \leq \omega_1(f,t_1) + \omega_1(f,t_2),$$

proving (3)'. Then by the obvious property (4)' we get

$$0 \leq \omega_1(f, t_1 + t_2) - \omega_1(f,t_1) \leq \omega_1(f,t_2),$$

and

$$|\omega_1(f, t_1 + t_2) - \omega_1(f,t_1)| \leq \omega_1(f,t_2).$$

Let $f \in UC(K,\mathbb{C})$, then $\lim_{t_2 \downarrow 0} \omega_1(f,t_2) = 0$, by property (5)'. Hence $\omega_1(f,\cdot)$ is continuous on $\mathbb{R}_+$.

(5)' ($\Rightarrow$) Let $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ with $\omega_1(f,\delta) \leq \varepsilon$. I.e. $\forall\, x, y \in K : |x-y| \leq \delta$ we get $|f(x) - f(y)| \leq \varepsilon$. That is $f \in UC(K,\mathbb{C})$.

($\Leftarrow$) Let $f \in UC(K,\mathbb{C})$. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$: whenever $|x-y| \leq \delta$, $x, y \in K$, it implies $|f(x) - f(y)| \leq \varepsilon$. I.e. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0 : \omega_1(f,\delta) \leq \varepsilon$. That is $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$.

(6)' Notice that

$$|(f(x) + g(x)) - (f(y) + g(y))| \leq |f(x) - f(y)| + |g(x) - g(y)|.$$

That is property (6)' now is clear. ■

We need

Theorem 2 *([1], p. 208) Let $(V_1, \|\cdot\|)$, $(V_2, \|\cdot\|)$ be real normed vector spaces and $Q \subseteq V_1$ which is star- shaped relative to the fixed point x_0. Consider $f : Q \to V_2$ with the properties:*

$$f(x_0) = 0, \text{ and } \|s-t\| \leq h \text{ implies } \|f(s) - f(t)\| \leq w; \quad w, h > 0. \tag{6}$$

Then, there exists a maximal such function Φ, namely

$$\Phi(t) := \left\lceil \frac{\|t - x_0\|}{h} \right\rceil \cdot w \cdot \overrightarrow{i}, \tag{7}$$

where $\overrightarrow{i}$ is any unit vector in V_2.

That is

$$\|f(t)\| \leq \|\Phi(t)\|, \text{ all } t \in Q. \tag{8}$$

Corollary 3 *Let $K \subseteq (\mathbb{C}, |\cdot|)$ be a compact convex subset, and $f \in C(K, \mathbb{C})$. Then*

$$|f(x) - f(x_0)| \leq \omega_1(f, \delta) \left\lceil \frac{|x - x_0|}{\delta} \right\rceil, \quad \delta > 0, \tag{9}$$

$\forall\ x, x_0 \in K$.

We make

Remark 4 *Let $K \subseteq (\mathbb{C}, |\cdot|)$ be a compact subset and $g \in C(K, \mathbb{R})$.*

A linear functional I from $C(K, \mathbb{R})$ into $\mathbb{R}$ is positive, iff $I(g_1) \geq I(g_2)$, whenever $g_1 \geq g_2$, where $g_1, g_2 \in C(K, \mathbb{R})$.

Let us assume that I is a positive linear functional. Then by Riesz representation theorem, [9], p. 304, there exists a unique Borel measure μ on K such that

$$I(g) = \int_K g(t)\, d\mu(t), \tag{10}$$

$\forall\ g \in C(K, \mathbb{R})$.

We make

Remark 5 *Here initially we follow [5].*

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$.

We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt. \tag{11}$$

By triangle inequality we have

$$\left| \int_\gamma f(z)\, dz \right| = \left| \int_a^b f(z(t))\, z'(t)\, dt \right| \leq \int_a^b |f(z(t))|\, |z'(t)|\, dt := \int_\gamma |f(z)|\, |dz|. \tag{12}$$

Inequalities (12) provide a typical example on linear functionals: clearly $\int_\gamma f(z)\, dz$ induces a linear functional from $C(\gamma, \mathbb{C})$ into $\mathbb{C}$, and $\int_\gamma |f(z)|\, |dz|$ involves a positive linear functional from $C(\gamma, \mathbb{R})$ into $\mathbb{R}$.

Thus, be given K a convex and compact subset of $\mathbb{C}$ and l be a linear functional from $C(K, \mathbb{C})$ into $\mathbb{C}$, it is not strange to assume that there exists a positive linear functional $\widetilde{l}$ from $C(K, \mathbb{R})$ into $\mathbb{R}$, such that

$$|l(f)| \leq \widetilde{l}(|f|), \quad \forall\ f \in C(K, \mathbb{C}). \tag{13}$$

Furthermore, we may assume that $\widetilde{l}(1(\cdot)) = 1$, where $1(t) = 1$, $\forall\ t \in K$, $l(c(\cdot)) = c, \forall c \in \mathbb{C}$ where $c(t) = c$, $\forall\ t \in K\cdot$

We call $\widetilde{l}$ the companion functional to l.

Here $\mathbb{C}$ is a vector space over the field of reals. The functional l is linear over $\mathbb{R}$ and the functional $\widetilde{l}$ is linear over $\mathbb{R}$.

Next we study approximation properties of $\left(l_n, \widetilde{l}_n\right)$ pairs, $n \in \mathbb{N}$.

3 Main Results - I

First about linear functionals:

We present the following quantitative approximation result of Korovkin type.

Theorem 6 *Here K is a convex and compact subset of $\mathbb{C}$ and l_n is a sequence of linear functionals from $C(K,\mathbb{C})$ into $\mathbb{C}$, $n \in \mathbb{N}$. There is a sequence of companion positive linear functionals $\widetilde{l}_n$ from $C(K,\mathbb{R})$ into $\mathbb{R}$, such that*

$$|l_n(f)| \leq \widetilde{l}_n(|f|), \quad \forall\ f \in C(K,\mathbb{C}), \ \forall\ n \in \mathbb{N}. \tag{14}$$

Additionally, we assume that $\widetilde{l}_n(1(\cdot)) = 1$ and $l_n(c(\cdot)) = c, \forall c \in \mathbb{C}\ \forall\ n \in \mathbb{N}$.

Then

$$|l_n(f) - f(x_0)| \leq 2\omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right), \quad \forall\ n \in \mathbb{N},\ \forall\ x_0 \in K, \tag{15}$$

$\forall\ f \in C(K,\mathbb{C})$.

Proof. We notice that

$$|l_n(f) - f(x_0)| = |l_n(f) - l_n(f(x_0)(\cdot))| =$$

$$|l_n(f(\cdot) - f(x_0)(\cdot))| \overset{(14)}{\leq} \widetilde{l}_n(|f(\cdot) - f(x_0)(\cdot)|) \overset{(\text{by } \delta>0,\ (9))}{\leq}$$

$$\widetilde{l}_n\left(\omega_1(f,\delta)\left\lceil \frac{|\cdot - x_0|}{\delta}\right\rceil\right) \leq \omega_1(f,\delta)\,\widetilde{l}_n\left(1(\cdot) + \frac{|\cdot - x_0|}{\delta}\right) =$$

$$\omega_1(f,\delta)\left[\widetilde{l}_n(1(\cdot)) + \frac{1}{\delta}\widetilde{l}_n(|\cdot - x_0|)\right] =$$

$$\omega_1(f,\delta)\left[1 + \frac{1}{\delta}\widetilde{l}_n(|\cdot - x_0|)\right] = 2\omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right), \tag{16}$$

by choosing

$$\delta := \widetilde{l}_n(|\cdot - x_0|),$$

if $\widetilde{l}_n(|\cdot - x_0|) > 0$, that is proving (15).

Next, we consider the case of $\widetilde{l}_n(|\cdot - x_0|) = 0$. By Riesz representation theorem, see (10) there exists a probability measure μ such that

$$\widetilde{l}_n(g) = \int_K g(t)\,d\mu(t), \quad \forall\ g \in C(K,\mathbb{R}). \tag{17}$$

That is, here it holds

$$\int_K |t - x_0|\,d\mu(t) = 0,$$

which implies $|t - x_0| = 0$, a.e, hence $t - x_0 = 0$, a.e, and $t = x_0$, a.e. Consequently $\mu(\{t \in K : t \neq x_0\}) = 0$. Hence $\mu = \delta_{x_0}$, the Dirac measure with support only $\{x_0\}$.

Therefore in that case $\widetilde{l}_n(g) = g(x_0)$, $\forall\ g \in C(K,\mathbb{R})$. Thus, it holds $\omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right) = \omega_1(f,0) = 0$, and $\widetilde{l}_n(|f(\cdot) - f(x_0)(\cdot)|) = |f(x_0) - f(x_0)| = 0$, giving $|l_n(f) - f(x_0)| = 0$. That is (15) is again true. ∎

Remark 7 *We have that*

$$\widetilde{l}_n\left(|\cdot - x_0|\right) = \int_K |t - x_0|\, d\mu\left(t\right)$$

(by Schwarz's inequality)

$$\leq \left(\int_K 1 d\mu\left(t\right)\right)^{\frac{1}{2}} \left(\int_K |t - x_0|^2\, d\mu\left(t\right)\right)^{\frac{1}{2}} =$$

$$\left(\widetilde{l}_n\left(1\right)\right)^{\frac{1}{2}} \left(\int_K |t - x_0|^2\, d\mu\left(t\right)\right)^{\frac{1}{2}} = \left(\widetilde{l}_n\left(|\cdot - x_0|^2\right)\right)^{\frac{1}{2}}. \tag{18}$$

We give

Corollary 8 *All as in Theorem 6. Then*

$$|l_n\left(f\right) - f\left(x_0\right)| \leq 2\omega_1\left(f, \left(\widetilde{l}_n\left(|\cdot - x_0|^2\right)\right)^{\frac{1}{2}}\right), \quad \forall\ n \in \mathbb{N},\ \forall\ x_0 \in K. \tag{19}$$

Conclusion 9 *All as in Theorem 6. By (15) and/or (19), as* $\widetilde{l}_n\left(|\cdot - x_0|\right) \to 0$*, or* $\widetilde{l}_n\left(|\cdot - x_0|^2\right) \to 0$*, as* $n \to +\infty$*, we obtain that* $l_n\left(f\right) \to f\left(x_0\right)$ *with rates,* $\forall$ $x_0 \in K$*.*

Next comes a more general quantitative approximation result of Korovkin type.

Theorem 10 *Here* K *is a convex and compact subset of* $\mathbb{C}$ *and* l_n *is a sequence of linear functionals from* $C\left(K, \mathbb{C}\right)$ *into* $\mathbb{C}$*,* $n \in \mathbb{N}$*. There is a sequence of companion positive linear functionals* $\widetilde{l}_n$ *from* $C\left(K, \mathbb{R}\right)$ *into* $\mathbb{R}$*, such that*

$$|l_n\left(f\right)| \leq \widetilde{l}_n\left(|f|\right), \quad \forall\ f \in C\left(K, \mathbb{C}\right),\ \forall\ n \in \mathbb{N}. \tag{20}$$

Additionally, we assume that

$$l_n\left(cg\right) = c\widetilde{l}_n\left(g\right), \quad \forall\ g \in C\left(K, \mathbb{R}\right),\ \forall\ c \in \mathbb{C}. \tag{21}$$

Then, for any $f \in C\left(K, \mathbb{C}\right)$*, we have*

$$|l_n\left(f\right) - f\left(x_0\right)| \leq |f\left(x_0\right)| \left|\widetilde{l}_n\left(1\left(\cdot\right)\right) - 1\right| + \left(\widetilde{l}_n\left(1\left(\cdot\right)\right) + 1\right)\omega_1\left(f, \widetilde{l}_n\left(|\cdot - x_0|\right)\right), \tag{22}$$

$\forall$ $x_0 \in K$, $\forall$ $n \in \mathbb{N}$.

(Notice if $\widetilde{l}_n\left(1\left(\cdot\right)\right) = 1$*, then (22) collapses to (15). So Theorem 10 generalizes Theorem 6).*

By (22), as $\widetilde{l}_n\left(1\left(\cdot\right)\right) \to 1$ *and* $\widetilde{l}_n\left(|\cdot - x_0|\right) \to 0$*, then* $l_n\left(f\right) \to f\left(x_0\right)$*, as* $n \to +\infty$*, with rates, and as here* $\widetilde{l}_n\left(1\left(\cdot\right)\right)$ *is bounded.*

Proof. We observe that

$$
\begin{gathered}
\left|l_n(f)-f(x_0)\right|=\left|l_n(f)-l_n(f(x_0)(\cdot))+l_n(f(x_0)(\cdot))-f(x_0)\right|\leq \\
\left|l_n(f)-l_n(f(x_0)(\cdot))\right|+\left|f(x_0)\widetilde{l}_n(1(\cdot))-f(x_0)\right|= \\
\left|l_n(f(\cdot)-f(x_0)(\cdot))\right|+\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right|\leq \qquad (23)\\
\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right|+\widetilde{l}_n\left(\left|f(\cdot)-f(x_0)(\cdot)\right|\right)\leq \\
\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right|+\widetilde{l}_n\left(\omega_1(f,\delta)\left\lceil\frac{|\cdot-x_0|}{\delta}\right\rceil\right)\leq \\
\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right|+\widetilde{l}_n(\omega_1(f,\delta))\left(1(\cdot)+\frac{|\cdot-x_0|}{\delta}\right)= \\
\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right|+\omega_1(f,\delta)\left[\widetilde{l}_n(1(\cdot))+\frac{1}{\delta}\widetilde{l}_n(|\cdot-x_0|)\right]= \\
\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right|+\left(\widetilde{l}_n(1(\cdot))+1\right)\omega_1\left(f,\widetilde{l}_n(|\cdot-x_0|)\right),
\end{gathered}
$$

by choosing

$$\delta:=\widetilde{l}_n(|\cdot-x_0|), \tag{24}$$

if $\widetilde{l}_n(|\cdot-x_0|)>0$.

Next we consider the case of

$$\widetilde{l}_n(|\cdot-x_0|)=0. \tag{25}$$

By Riesz representation theorem there exists a positive finite measure μ such that

$$\widetilde{l}_n(g)=\int_K g(t)\,d\mu(t),\ \ \forall\ g\in C(K,\mathbb{R}). \tag{26}$$

That is

$$\int_K |t-x_0|\,d\mu(t)=0, \tag{27}$$

which implies $|t-x_0|=0$, a.e., hence $t-x_0=0$, a.e, and $t=x_0$, a.e. on K. Consequently $\mu(\{t\in K:t\neq x_0\})=0$. That is $\mu=\delta_{x_0}M$ (where $0<M:=\mu(K)=\widetilde{l}_n(1(\cdot))$). Hence, in that case $\widetilde{l}_n(g)=g(x_0)M$. Consequently it holds $\omega_1\left(f,\widetilde{l}_n(|\cdot-x_0|)\right)=0$, and the right hand side of (22) equals $|f(x_0)||M-1|$. Also, it is $\widetilde{l}_n(|f(\cdot)-f(x_0)(\cdot)|)=|f(x_0)-f(x_0)|M=0$. Hence from the first part of this proof we get $|l_n(f)-l_n(f(x_0)(\cdot))|=0$, and $l_n(f)=l_n(f(x_0)(\cdot))=f(x_0)\widetilde{l}_n(1(\cdot))=Mf(x_0)$.

Consequently the left hand side of (22) becomes

$$|l_n(f)-f(x_0)|=|Mf(x_0)-f(x_0)|=|f(x_0)||M-1|.$$

So that (22) becomes an equality, and both sides equal $|f(x_0)||M-1|$ in the extreme case of $\widetilde{l}_n(|\cdot-x_0|)=0$. Thus inequality (22) is proved completely in all cases. ■

We make

Remark 11 *By Schwartz's inequality we get*

$$\widetilde{l}_n\left(|\cdot - x_0|\right) \leq \left(\widetilde{l}_n\left(|\cdot - x_0|^2\right)\right)^{\frac{1}{2}}\left(\widetilde{l}_n\left(1\left(\cdot\right)\right)\right)^{\frac{1}{2}}. \tag{28}$$

We give

Corollary 12 *All as in Theorem 10. Then*

$$|l_n(f) - f(x_0)| \leq |f(x_0)|\left|\widetilde{l}_n\left(1\left(\cdot\right)\right) - 1\right| +$$

$$\left(\widetilde{l}_n\left(1\left(\cdot\right)\right) + 1\right)\omega_1\left(f, \left(\widetilde{l}_n\left(1\left(\cdot\right)\right)\right)^{\frac{1}{2}}\left(\widetilde{l}_n\left(|\cdot - x_0|^2\right)\right)^{\frac{1}{2}}\right), \tag{29}$$

$\forall\ x_0 \in K$, $\forall\ n \in \mathbb{N}$.

Next we give another version of our Korovkin type result.

Theorem 13 *Here all are as in Theorem 10. Then, for any $f \in C(K, \mathbb{C})$, we have*

$$|l_n(f) - f(x_0)| \leq |f(x_0)|\left|\widetilde{l}_n\left(1\left(\cdot\right)\right) - 1\right| + \left(\widetilde{l}_n\left(1\left(\cdot\right)\right) + 1\right)\omega_1\left(f, \left(\widetilde{l}_n\left(|\cdot - x_0|^2\right)\right)^{\frac{1}{2}}\right), \tag{30}$$

$\forall\ x_0 \in K$, $\forall\ n \in \mathbb{N}$.

By (30), as $\widetilde{l}_n(1(\cdot)) \to 1$ and $\widetilde{l}_n\left(|\cdot - x_0|^2\right) \to 0$, then $l_n(f) \to f(x_0)$, as $n \to +\infty$, with rates, and as here $\widetilde{l}_n(1(\cdot))$ is bounded.

Proof. Let $t, x_0 \in K$ and $\delta > 0$. If $|t - x_0| > \delta$, then

$$|f(t) - f(x_0)| \leq \omega_1(f, |t - x_0|) = \omega_1\left(f, |t - x_0|\,\delta^{-1}\delta\right) \leq \tag{31}$$

$$\left(1 + \frac{|t - x_0|}{\delta}\right)\omega_1(f, \delta) \leq \left(1 + \frac{|t - x_0|^2}{\delta^2}\right)\omega_1(f, \delta).$$

The estimate

$$|f(t) - f(x_0)| \leq \left(1 + \frac{|t - x_0|^2}{\delta^2}\right)\omega_1(f, \delta) \tag{32}$$

also holds trivially when $|t - x_0| \leq \delta$.

So (32) is true always, $\forall\ t \in K$, for any $x_0 \in K$.

We can rewrite

$$|f(\cdot) - f(x_0)| \leq \left(1 + \frac{|\cdot - x_0|^2}{\delta^2}\right)\omega_1(f, \delta). \tag{33}$$

As in the proof of Theorem 10 we have

$$|l_n(f) - f(x_0)| \leq \ldots \leq |f(x_0)|\left|\widetilde{l}_n\left(1\left(\cdot\right)\right) - 1\right| +$$

$$\widetilde{l}_n\left(\omega_1(f,\delta)\left(1(\cdot)+\frac{|\cdot-x_0|^2}{\delta^2}\right)\right)=$$

$$|f(x_0)|\left|\widetilde{l}_n(1(\cdot))-1\right|+\omega_1(f,\delta)\left[\widetilde{l}_n(1(\cdot))+\frac{1}{\delta^2}\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right]= \tag{34}$$

$$|f(x_0)|\left|\widetilde{l}_n(1(\cdot))-1\right|+\omega_1\left(f,\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right)\left(\widetilde{l}_n(1(\cdot))+1\right),$$

by choosing

$$\delta:=\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}, \tag{35}$$

if $\widetilde{l}_n\left(|\cdot-x_0|^2\right)>0$.

Next we consider the case of

$$\widetilde{l}_n\left(|\cdot-x_0|^2\right)=0. \tag{36}$$

By Riesz representation theorem there exists a positive finite measure μ such that

$$\widetilde{l}_n(g)=\int_K g(t)\,d\mu(t),\ \ \forall\ g\in C(K,\mathbb{R}). \tag{37}$$

That is

$$\int_K |t-x_0|^2\,d\mu(t)=0,$$

which implies $|t-x_0|^2=0$, a.e., hence $t-x_0=0$, a.e, and $t=x_0$, a.e. on K. Consequently $\mu(\{t\in K: t\neq x_0\})=0$. That is $\mu=\delta_{x_0}M$ (where $0<M:=\mu(K)=\widetilde{l}_n(1(\cdot))$). Hence, in that case $\widetilde{l}_n(g)=g(x_0)M$. Consequently it holds $\omega_1\left(f,\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right)=0$, and the right hand side of (30) equals $|f(x_0)|\,|M-1|$.

Also, it is $\widetilde{l}_n(|f(\cdot)-f(x_0)(\cdot)|)=|f(x_0)-f(x_0)|M=0$. Hence from the first part of this proof we get: $|l_n(f)-l_n(f(x_0)(\cdot))|=0$, and $l_n(f)=l_n(f(x_0)(\cdot))=f(x_0)\widetilde{l}_n(1(\cdot))=Mf(x_0)$.

Consequently the left hand side of (30) becomes

$$|l_n(f)-f(x_0)|=|f(x_0)|\,|M-1|.$$

So that (30) is true again. The proof of the theorem is now complete. ■

Corollary 14 *Here all are as in Theorem 10. Then*

$$|l_n(f)-f(x_0)|\leq|f(x_0)|\left|\widetilde{l}_n(1(\cdot))-1\right|+\left(\widetilde{l}_n(1(\cdot))+1\right)\cdot$$

$$\min\left\{\omega_1\left(f,\left(\widetilde{l}_n(1(\cdot))\right)^{\frac{1}{2}}\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right),\omega_1\left(f,\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right)\right\}, \tag{38}$$

$\forall\ x_0\in K$, $\forall\ n\in\mathbb{N}$.

Proof. By (29) and (30). ■

So (29) is better that (30) only if $\widetilde{l}_n(1(\cdot)) < 1$.

We need

Theorem 15 *Let $K \subseteq \mathbb{C}$ convex, $x_0 \in K^0$ (interior of K) and $f : K \to \mathbb{R}$ such that $|f(t) - f(x_0)|$ is convex in $t \in K$. Furthermore let $\delta > 0$ so that the closed disk $D(x_0, \delta) \subset K$. Then*

$$|f(t) - f(x_0)| \leq \frac{\omega_1(f, \delta)}{\delta} |t - x_0|, \quad \forall\, t \in K. \tag{39}$$

Proof. Let $g(t) := |f(t) - f(x_0)|$, $t \in K$, which is convex in $t \in K$ and $g(x_0) = 0$.

Then by Lemma 8.1.1, p. 243 of [1], we obtain

$$g(t) \leq \frac{\omega_1(g, \delta)}{\delta} |t - x_0|, \quad \forall\, t \in K. \tag{40}$$

We notice the following

$$|f(t_1) - f(x_0)| = |f(t_1) - f(t_2) + f(t_2) - f(x_0)| \leq$$

$$|f(t_1) - f(t_2)| + |f(t_2) - f(x_0)|,$$

hence

$$|f(t_1) - f(x_0)| - |f(t_2) - f(x_0)| \leq |f(t_1) - f(t_2)|. \tag{41}$$

Similarly, it holds

$$|f(t_2) - f(x_0)| - |f(t_1) - f(x_0)| \leq |f(t_1) - f(t_2)|. \tag{42}$$

Therefore for any $t_1, t_2 \in K : |t_1 - t_2| \leq \delta$ we get

$$\left| |f(t_1) - f(x_0)| - |f(t_2) - f(x_0)| \right| \leq |f(t_1) - f(t_2)| \leq \omega_1(f, \delta). \tag{43}$$

That is

$$\omega_1(g, \delta) \leq \omega_1(f, \delta). \tag{44}$$

The last and (40) imply

$$|f(t) - f(x_0)| \leq \frac{\omega_1(f, \delta)}{\delta} |t - x_0|, \quad \forall\, t \in K, \tag{45}$$

proving (39). ■

We continue with a convex Korovkin type result:

Theorem 16 *All as in Theorem 10. Let $x_0 \in K^0$ and assume that $|f(t) - f(x_0)|$ is convex in $t \in K$. Let $\delta > 0$,such that the closed disk $D(x_0, \delta) \subset K$. Then*

$$|l_n(f) - f(x_0)| \leq |f(x_0)| \left| \widetilde{l}_n(1(\cdot)) - 1 \right| + \omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right), \quad \forall\, n \in \mathbb{N}. \tag{46}$$

Proof. As in the proof Theorem 10 we have

$$\left|l_n(f) - f(x_0)\right| \leq ... \leq |f(x_0)| \left|\widetilde{l}_n(1(\cdot)) - 1\right| + \widetilde{l}_n(|f(\cdot) - f(x_0)(\cdot)|) \overset{(39)}{\leq} \tag{47}$$

$$|f(x_0)| \left|\widetilde{l}_n(1(\cdot)) - 1\right| + \frac{\omega_1(f,\delta)}{\delta} \widetilde{l}_n(|\cdot - x_0|) =$$

$$|f(x_0)| \left|\widetilde{l}_n(1(\cdot)) - 1\right| + \omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right),$$

by choosing

$$\delta := \widetilde{l}_n(|\cdot - x_0|) > 0,$$

if the last is positive. The case of $\widetilde{l}_n(|\cdot - x_0|) = 0$ is treated similarly as in the proof of Theorem 10. The theorem is proved. ■

Theorem 17 *All as in Theorem 16. Inequality (46) is sharp, in fact it is attained by* $f^*(t) = \overrightarrow{j}|t - x_0|$, *where* $\overrightarrow{j}$ *is a unit vector of* $(\mathbb{C}, |\cdot|)$; $t, x_0 \in K$.

Proof. Indeed, f^* here fulfills the assumptions of the theorem. We further notice that $f^*(x_0) = 0$, and $|f^*(t) - f^*(x_0)| = |t - x_0|$ is convex in $t \in K$. The left hand side of (46) is

$$|l_n(f^*) - f^*(x_0)| = |l_n(f^*)| = \left|l_n\left(\overrightarrow{j}|\cdot - x_0|\right)\right| \overset{(21)}{=}$$

$$\left|\overrightarrow{j}\widetilde{l}_n(|\cdot - x_0|)\right| = \left|\widetilde{l}_n(|\cdot - x_0|)\right|. \tag{48}$$

The right hand side of (46) is

$$\omega_1\left(f^*, \widetilde{l}_n(|\cdot - x_0|)\right) = \omega_1\left(\overrightarrow{j}|\cdot - x_0|, \widetilde{l}_n(|\cdot - x_0|)\right) =$$

$$\sup_{\substack{t_1,t_2 \in K \\ |t_1 - t_2| \leq \widetilde{l}_n(|\cdot - x_0|)}} \left|\overrightarrow{j}|t_1 - x_0| - \overrightarrow{j}|t_2 - x_0|\right| =$$

$$\sup_{\substack{t_1,t_2 \in K \\ |t_1 - t_2| \leq \widetilde{l}_n(|\cdot - x_0|)}} \left||t_1 - x_0| - |t_2 - x_0|\right| \leq \tag{49}$$

$$\sup_{\substack{t_1,t_2 \in K \\ |t_1 - t_2| \leq \widetilde{l}_n(|\cdot - x_0|)}} |t_1 - t_2| = \widetilde{l}_n(|\cdot - x_0|).$$

Hence we have found that

$$\omega_1\left(f^*, \widetilde{l}_n(|\cdot - x_0|)\right) \leq \widetilde{l}_n(|\cdot - x_0|). \tag{50}$$

Clearly (46) is attained.

The theorem is proved. ■

4 Main Results - II

Next we give results on linear operators:

Let K be a compact convex subset of $\mathbb{C}$. Consider $L : C(K,\mathbb{C}) \to C(K,\mathbb{C})$ a linear operator and $\widetilde{L} : C(K,\mathbb{R}) \to C(K,\mathbb{R})$ a positve linear operator (i.e. for $f_1.f_2 \in C(K,\mathbb{R})$ with $f_1 \geq f_2$ we get $\widetilde{L}(f_1) \geq \widetilde{L}(f_2)$) both over the field of $\mathbb{R}$.

We assume that

$$|L(f)| \leq \widetilde{L}(|f|), \ \forall f \in C(K,\mathbb{C}),$$

(i.e. $|L(f)(z)| \leq \widetilde{L}(|f|)(z)$, $\forall z \in K$).

We call $\widetilde{L}$ the companion operator of L.

Let $x_0 \in K$. Clearly, then $L(\cdot)(x_0)$ is a linear functional from $C(K,\mathbb{C})$ into $\mathbb{C}$, and $\widetilde{L}(\cdot)(x_0)$ is a positive linear functional from $C(K,\mathbb{R})$ into $\mathbb{R}$. Notice $L(f)(z) \in \mathbb{C}$ and $\widetilde{L}(|f|)(z) \in \mathbb{R}$, $\forall f \in C(K,\mathbb{C})$ (thus $|f| \in C(K,\mathbb{R})$). Here $L(f) \in C(K,\mathbb{C})$, and $\widetilde{L}(|f|) \in C(K,\mathbb{R})$, $\forall f \in C(K,\mathbb{C})$.

Notice that $C(K,\mathbb{C}) = UC(K,\mathbb{C})$, also $C(K,\mathbb{R}) = UC(K,\mathbb{R})$ (uniformly continuous functions).

By [3], p. 388, we have that $\widetilde{L}(|\cdot - x_0|^r)(x_0)$, $r > 0$, is a continuous function in $x_0 \in K$.

After this preparation we transfer the main results from section 3 to linear operators.

We have the following approximation results with rates of Korovkin type.

Theorem 18 *Here K is a convex and compact subset of $\mathbb{C}$ and L_n is a sequence of linear operators from $C(K,\mathbb{C})$ into itself, $n \in \mathbb{N}$. There is a sequence of companion positive linear operators $\widetilde{L}_n$ from $C(K,\mathbb{R})$ into itself, such that*

$$|L_n(f)| \leq \widetilde{L}_n(|f|), \ \forall f \in C(K,\mathbb{C}), \ \forall n \in \mathbb{N} \tag{51}$$

(i.e. $|L_n(f)(x_0)| \leq \left(\widetilde{L}_n(|f|)\right)(x_0)$, $\forall x_0 \in K$).

Additionally, we assume that

$$L_n(cg) = c\widetilde{L}_n(g), \ \ \forall g \in C(K,\mathbb{R}), \ \forall c \in \mathbb{C} \tag{52}$$

(i.e. $(L_n(cg))(x_0) = c\left(\widetilde{L}_n(g)\right)(x_0)$, $\forall x_0 \in K$).

Then, for any $f \in C(K,\mathbb{C})$, we have

$$|(L_n(f))(x_0) - f(x_0)| \leq |f(x_0)| \left|\widetilde{L}_n(1(\cdot))(x_0) - 1\right| +$$
$$\left(\widetilde{L}_n(1(\cdot))(x_0) + 1\right) \omega_1\left(f, \widetilde{L}_n(|\cdot - x_0|)(x_0)\right), \tag{53}$$

$\forall x_0 \in K$, $\forall n \in \mathbb{N}$.

Proof. By Theorem 10. ■

Corollary 19 *All as in Theorem 18. Then*

$$\left\|L_n(f)-f\right\|_{\infty,K}\leq\|f\|_{\infty,K}\left\|\widetilde{L}_n(1(\cdot))-1\right\|_{\infty,K}+$$

$$\left\|\widetilde{L}_n(1(\cdot))+1\right\|_{\infty,K}\omega_1\left(f,\left\|\widetilde{L}_n(|\cdot-x_0|)(x_0)\right\|_{\infty,K}\right), \tag{54}$$

$\forall\ n\in\mathbb{N}$.

If $\widetilde{L}_n(1(\cdot))=1$, $\forall\ n\in\mathbb{N}$, *then*

$$\left\|L_n(f)-f\right\|_{\infty,K}\leq 2\omega_1\left(f,\left\|\widetilde{L}_n(|\cdot-x_0|)(x_0)\right\|_{\infty,K}\right), \tag{55}$$

$\forall\ n\in\mathbb{N}$.

As $\widetilde{L}_n(1(\cdot))\xrightarrow{u}1$, $\left\|\widetilde{L}_n(|\cdot-x_0|)(x_0)\right\|_{\infty,K}\xrightarrow{u}0$, *then (by (54))* $L_n(f)\xrightarrow{u}f$, *as* $n\to+\infty$, *where* u *means uniformly. Notice* $\widetilde{L}_n(1(\cdot))$ *is bounded, and all the suprema in (54) are finite.*

We continue with

Theorem 20 *Here all as in Theorem 18. Then, for any* $f\in C(K,\mathbb{C})$, *we have*

$$\left|(L_n(f))(x_0)-f(x_0)\right|\leq|f(x_0)|\left|\widetilde{L}_n(1(\cdot))(x_0)-1\right|+$$

$$\left(\widetilde{L}_n(1(\cdot))(x_0)+1\right)\omega_1\left(f,\left(\widetilde{L}_n\left(|\cdot-x_0|^2\right)(x_0)\right)^{\frac{1}{2}}\right), \tag{56}$$

$\forall\ x_0\in K$, $\forall\ n\in\mathbb{N}$.

Proof. By Theorem 13. ■

Corollary 21 *All as in Theorem 18. Then, for any* $f\in C(K,\mathbb{C})$, *we have*

$$\left\|L_n(f)-f\right\|_{\infty,K}\leq\|f\|_{\infty,K}\left\|\widetilde{L}_n(1(\cdot))-1\right\|_{\infty,K}+$$

$$\left\|\widetilde{L}_n(1(\cdot))+1\right\|_{\infty,K}\omega_1\left(f,\left\|\widetilde{L}_n\left(|\cdot-x_0|^2\right)(x_0)\right\|_{\infty,K}^{\frac{1}{2}}\right), \tag{57}$$

$\forall\ n\in\mathbb{N}$.

If $\widetilde{L}_n(1(\cdot))=1$, *then*

$$\left\|L_n(f)-f\right\|_{\infty,K}\leq 2\omega_1\left(f,\left\|\widetilde{L}_n\left(|\cdot-x_0|^2\right)(x_0)\right\|_{\infty,K}^{\frac{1}{2}}\right), \tag{58}$$

$\forall\ n\in\mathbb{N}$.

As $\widetilde{L}_n(1(\cdot))\xrightarrow{u}1$, $\left\|\widetilde{L}_n\left(|\cdot-x_0|^2\right)(x_0)\right\|_{\infty,K}\xrightarrow{u}0$, *then (by (57))* $L_n(f)\xrightarrow{u}f$, *as* $n\to+\infty$.

We continue with a convex Korovkin type result:

Theorem 22 *All as in Theorem 18. Let a fixed $x_0^* \in K^0$ and assume that $|f(t) - f(x_0^*)|$ is convex in $t \in K$. Let $\delta > 0$,such that the closed disk $D(x_0^*, \delta) \subset K$. Then*

$$|(L_n(f))(x_0^*) - f(x_0^*)| \leq |f(x_0^*)| \left| \widetilde{L}_n(1(\cdot))(x_0^*) - 1 \right|$$
$$+\omega_1\left(f, \widetilde{L}_n(|\cdot - x_0^*|)(x_0^*)\right), \quad \forall\, n \in \mathbb{N}. \tag{59}$$

As $\widetilde{L}_n(1(\cdot))(x_0^*) \to 1$, and $\widetilde{L}_n(|\cdot - x_0^*|)(x_0^*) \to 0$, we get that $(L_n(f))(x_0^*) \to f(x_0^*)$, as $n \to +\infty$, a pointwise convergence.

Proof. By Theorem 16. ■

Note: Theorem 22 goes throw if (51), (52) are valid only for the particular x_0^*.

We finish with

Proposition 23 *All as in Theorem 22. Inequality (59) is sharp, in fact it is attained by $\overline{f}(t) = \overrightarrow{j}|t - x_0^*|$, where $\overrightarrow{j}$ is a unit vector of $\mathbb{C}$; $x_0^*, t \in K$.*

Proof. By Theorem 17. ■

Note: Let K be a convex compact subset of a real normed vector space $(V, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ is a Banach space. We can consider bounded linear functionals and bounded operators on $C(K, X)$. This paper's methodology can be applied to this more general setting and produce a similar Korovkin theory in full strength.

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