GENERALIZED OSTROWSKI TYPE NORM INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\gamma \subset G$ is a piecewise smooth path parametrized by $\lambda(t)$, $t \in [0,1]$ from $\lambda(0) = \alpha$ to $\lambda(1) = \beta$. If $f : G \to \mathbb{C}$ is analytic on G, then by using the analytic functional calculus we obtain among others the following result

$$\left\| \int_\gamma f(z)\, dz - \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(a) \left[(\beta - a)^{k+1} + (-1)^k (a - \alpha)^{k+1} \right] \right\| \leq \frac{1}{(n+1)!} K_{a,\gamma}^{(n+1)} \int_\gamma \|z - a\|^{n+1} |dz|$$

provided

$$K_{a,\gamma}^{(n+1)} := \sup_{(s,z) \in [0,1] \times \gamma} \left\| f^{(n+1)} [(1-s) a + sz] \right\| < \infty, \; n \geq 0.$$

Applications for the exponential function of elements in Banach algebras are also given.

1. Introduction

In 1938, A. Ostrowski [24], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a} \int_a^b f(t)\, dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1 (Ostrowski, 1938 [24])**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

For a recent survey on Ostrowski's inequality for scalar functions and Lebesgue integral see [12].

In order to extend Ostrowski's inequality for analytic functions defined on Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \leq \|a\| \|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*. We assume

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that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a, b \in \operatorname{Inv}(\mathcal{B})$ then $ab \in \operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \operatorname{Inv}(\mathcal{B})$;
(ii) $\{a \in \mathcal{B}: \ \|1 - b\| < 1\} \subset \operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} : \ z - a \in \operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \operatorname{Inv}(\mathcal{B})$, $R_a(z) := (z-a)^{-1}$. For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z - w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \le \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_{\delta} f(\xi)(\xi - a)^{-1} d\xi, \tag{1.2}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [7, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.3}$$

holds.

Let $\mathfrak{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\mathfrak{Hol}(a)$ is an algebra where if f, $g \in \mathfrak{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f + g$ have domain $D(f) \cap D(g)$. $\mathfrak{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [7, p. 201-203]:

Theorem 2. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) *The map $f \mapsto f(a)$ of $\mathfrak{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.*
(b) *If $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ has radius of convergence $r > \nu(a)$, then $f \in \mathfrak{Hol}(a)$ and $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.*
(c) *If $f(z) \equiv 1$, then $f(a) = 1$.*
(d) *If $f(z) = z$ for all z, $f(a) = a$.*
(e) *If $f, f_1, ..., f_n...$ are analytic on G, $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\|f_n(a) - f(a)\| \to 0$ as $n \to \infty$.*
(f) *The Riesz Functional Calculus is unique and if a, b are commuting elements in $\mathcal{B}$ and $f \in \mathfrak{Hol}(a)$, then $f(a) b = bf(a)$.*

For some recent norm inequalities for functions on Banach algebras, see [4]-[5] and [10]-[16].

2. Some Identities

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and z, $x \in D$, then we have the following Taylor's expansion with integral remainder

$$f(z) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\xi)(z-\xi)^k + \frac{1}{n!}(z-\xi)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\xi + sz](1-s)^n ds \tag{2.1}$$

for $n \geq 0$, see for instance [26].

Consider the function $f(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi < \operatorname{Arg}(z) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C} \setminus \{x + iy : x \leq 0,\ y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{z^k},\ k \geq 1,\ z \in \mathbb{C}_\ell.$$

Using the representation (2.1) we then have

$$\operatorname{Log}(z) = \operatorname{Log}(\xi) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{z-\xi}{\xi}\right)^k + (-1)^n (z-\xi)^{n+1} \int_0^1 \frac{(1-s)^n ds}{[(1-s)\xi + sz]^{n+1}} \tag{2.2}$$

for all z, $\xi \in \mathbb{C}_\ell$ with $(1-s)\xi + sz \in \mathbb{C}_\ell$ for $s \in [0,1]$.

Consider the complex exponential function $f(z) = \exp(z)$, then by (2.1) we get

$$\exp(z) = \sum_{k=0}^{n} \frac{1}{k!}(z-\xi)^k \exp(\xi) + \frac{1}{n!}(z-\xi)^{n+1} \int_0^1 (1-s)^n \exp[(1-s)\xi + sz] ds \tag{2.3}$$

for all z, $\xi \in \mathbb{C}$.

For various inequalities related to Taylor's expansions for real functions see [1]-[3], [17]-[23] and [25].

We have the following identity for functions in Banach algebras. This is a generalization of the scalar case for functions of real variable established in [6].

Theorem 3. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [0,1]$ with $z(0) = \alpha$ and $z(1) = \beta$ where α, $\beta \in D$ and $n \geq 0$, then we have*

$$\int_\gamma f(z)\,dz - \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(a) \left[(\beta - a)^{k+1} + (-1)^k (a - \alpha)^{k+1} \right] \tag{2.4}$$
$$= \frac{1}{n!} \int_\gamma (z-a)^{n+1} \left(\int_0^1 f^{(n+1)}[(1-s)a + sz](1-s)^n\,ds \right) dz$$
$$= \frac{1}{n!} \int_0^1 \left(\int_\gamma (z-a)^{n+1} f^{(n+1)}[(1-s)a + sz]\,dz \right) (1-s)^n\,ds.$$

Proof. If we take the integral over z on the path $\gamma = \gamma_{\alpha,\beta}$ in the equality (2.1), then we get for all $\xi \in D$ that

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\xi) \int_{\gamma_{\alpha,\beta}} (z-\xi)^k\,dz$$
$$+ \frac{1}{n!} \int_{\gamma_{\alpha,\beta}} (z-\xi)^{n+1} \left(\int_0^1 f^{(n+1)}[(1-s)\xi + sz](1-s)^n\,ds \right) dz$$
$$= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\xi) \frac{(\beta-\xi)^{k+1} - (\alpha-\xi)^{k+1}}{k+1}$$
$$+ \frac{1}{n!} \int_{\gamma_{\alpha,\beta}} (z-\xi)^{n+1} \left(\int_0^1 f^{(n+1)}[(1-s)\xi + sz](1-s)^n\,ds \right) dz$$
$$= \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(\xi) \left[(\beta-\xi)^{k+1} + (-1)^{k+2} (\xi-\alpha)^{k+1} \right]$$
$$+ \frac{1}{n!} \int_{\gamma_{\alpha,\beta}} (z-\xi)^{n+1} \left(\int_0^1 f^{(n+1)}[(1-s)\xi + sz](1-s)^n\,ds \right) dz,$$

which proves the identity

$$\int_\gamma f(z)\,dz - \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(\xi) \left[(\beta-\xi)^{k+1} + (-1)^k (\xi-\alpha)^{k+1} \right] \tag{2.5}$$
$$= \frac{1}{n!} \int_\gamma (z-\xi)^{n+1} \left(\int_0^1 f^{(n+1)}[(1-s)\xi + sz](1-s)^n\,ds \right) dz$$
$$= \frac{1}{n!} \int_0^1 \left(\int_\gamma (z-\xi)^{n+1} f^{(n+1)}[(1-s)\xi + sz]\,dz \right) (1-s)^n\,ds$$

for all $\xi \in D$, where for the second equality we used Fubini's theorem.

Assume that $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$. By using the analytic functional calculus (1.2) and the equality

(2.5) we obtain

$$
\begin{aligned}
(2.6)\quad & \int_\gamma f(z)\,dz\left(\frac{1}{2\pi i}\int_\delta (\xi-a)^{-1}\,d\xi\right)\\
&-\sum_{k=0}^{n}\frac{1}{(k+1)!}\frac{1}{2\pi i}\int_\delta f^{(k)}(\xi)\left[(\beta-\xi)^{k+1}+(-1)^k(\xi-\alpha)^{k+1}\right](\xi-a)^{-1}\,d\xi\\
&=\frac{1}{n!}\frac{1}{2\pi i}\\
&\times\int_\delta\left(\int_\gamma (z-\xi)^{n+1}\left(\int_0^1 f^{(n+1)}\left[(1-s)\xi+sz\right](1-s)^n\,ds\right)dz\right)(\xi-a)^{-1}\,d\xi\\
&=\frac{1}{n!}\\
&\times\int_\gamma\left(\int_0^1\left(\frac{1}{2\pi i}\int_\delta (z-\xi)^{n+1}f^{(n+1)}\left[(1-s)\xi+sz\right](\xi-a)^{-1}\,d\xi\right)(1-s)^n\,ds\right)dz,
\end{aligned}
$$

where for the last equality in (2.6) we also used Fubini's theorem.

By using the functional calculus for the analytic functions

$$G\ni\xi\mapsto f^{(k)}(\xi)\left[(\beta-\xi)^{k+1}+(-1)^k(\xi-\alpha)^{k+1}\right]\in\mathbb{C}$$

and

$$G\ni\xi\mapsto (z-\xi)^{n+1}f^{(n+1)}\left[(1-s)\xi+sz\right]\in\mathbb{C}$$

where $k=0,...,n,\ z\in\gamma$ and $s\in[0,1]$, then we obtain

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_\delta f^{(k)}(\xi)\left[(\beta-\xi)^{k+1}+(-1)^k(\xi-\alpha)^{k+1}\right](\xi-a)^{-1}\,d\xi\\
&\qquad=f^{(k)}(a)\left[(\beta-a)^{k+1}+(-1)^k(a-\alpha)^{k+1}\right],
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_\delta (z-\xi)^{n+1}f^{(n+1)}\left[(1-s)\xi+sz\right](\xi-a)^{-1}\,d\xi\\
&\qquad=(z-a)^{n+1}f^{(n+1)}\left[(1-s)a+sz\right]
\end{aligned}
$$

and since

$$\frac{1}{2\pi i}\int_\delta (\xi-a)^{-1}\,d\xi=1,$$

then by (2.6) we get the first equality in (2.4).

The second part of (2.4) follows by Fubini's theorem. □

Remark 1. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic on G, $\gamma\subset D$ is a smooth path parametrized by $z(t)$, $t\in[0,1]$ with $z(0)=\alpha$ and $z(1)=\beta$ where $\alpha,\beta\in D$.*

If we take $n=0$ in (2.4), then we obtain the Ostrowski type equality

$$
\begin{aligned}
(2.7)\quad \int_\gamma f(z)\,dz-(\beta-\alpha)f(a)&=\int_\gamma (z-a)\left(\int_0^1 f'\left[(1-s)a+sz\right]ds\right)dz\\
&=\int_0^1\left(\int_\gamma (z-a)f'\left[(1-s)a+sz\right]dz\right)ds.
\end{aligned}
$$

For $n=1$ in (2.4) we get the perturbed Ostrowski's equality

$$\begin{aligned}\int_{\gamma} f(z)\,dz-(\beta-\alpha)f(a)-(\beta-\alpha)f'(a)\left(\frac{\beta+\alpha}{2}-a\right) &= \int_{\gamma}(z-a)^{2}\left(\int_{0}^{1} f''[(1-s)a+sz](1-s)\,ds\right)dz \\ &= \int_{0}^{1}\left(\int_{\gamma}(z-a)^{2} f''[(1-s)a+sz]\,dz\right)(1-s)\,ds. \end{aligned} \tag{2.8}$$

Corollary 1. *With the assumptions of Theorem 3 and for $b\in\mathcal{B}$ we have the general perturbed identity*

$$\begin{aligned}&\int_{\gamma} f(z)\,dz-\sum_{k=0}^{n}\frac{1}{(k+1)!}f^{(k)}(a)\left[(\beta-a)^{k+1}+(-1)^{k}(a-\alpha)^{k+1}\right] \\ &\qquad -\frac{1}{(n+2)!}\left[(\beta-a)^{n+2}+(-1)^{n+1}(a-\alpha)^{n+2}\right]b \\ &=\frac{1}{n!}\int_{\gamma}(z-a)^{n+1}\left(\int_{0}^{1}\left(f^{(n+1)}[(1-s)a+sz]-b\right)(1-s)^{n}\,ds\right)dz \\ &=\frac{1}{n!}\int_{0}^{1}\left(\int_{\gamma}(z-a)^{n+1}\left(f^{(n+1)}[(1-s)a+sz]-b\right)dz\right)(1-s)^{n}\,ds. \end{aligned} \tag{2.9}$$

Proof. Observe that

$$\begin{aligned}&\int_{\gamma}(z-a)^{n+1}\left(\int_{0}^{1}\left(f^{(n+1)}[(1-s)a+sz]-b\right)(1-s)^{n}\,ds\right)dz \\ &=\int_{\gamma}(z-a)^{n+1}\left(\int_{0}^{1}\left(f^{(n+1)}[(1-s)a+sz]\right)(1-s)^{n}\,ds\right)dz \\ &\qquad -\left(\int_{0}^{1}(1-s)^{n}\,ds\right)\left(\int_{\gamma}(z-a)^{n+1}\,dz\right)b \\ &=\int_{\gamma}(z-a)^{n+1}\left(\int_{0}^{1}\left(f^{(n+1)}[(1-s)a+sz]\right)(1-s)^{n}\,ds\right)dz \\ &\qquad -\frac{1}{(n+1)(n+2)}\left[(\beta-a)^{n+2}-(\alpha-a)^{n+2}\right]b \\ &=\int_{\gamma}(z-a)^{n+1}\left(\int_{0}^{1}\left(f^{(n+1)}[(1-s)a+sz]\right)(1-s)^{n}\,ds\right)dz \\ &\qquad -\frac{1}{(n+1)(n+2)}\left[(\beta-a)^{n+2}+(-1)^{n+1}(a-\alpha)^{n+2}\right]b, \end{aligned}$$

and by (2.4) we get (2.9). □

Remark 2. *With the assumptions of Theorem 3, we have the identities*

$$\int_\gamma f(z)\,dz - \sum_{k=0}^{n+1} \frac{1}{(k+1)!} f^{(k)}(a) \left[(\beta - a)^{k+1} + (-1)^k (a-\alpha)^{k+1} \right] \tag{2.10}$$
$$= \frac{1}{n!} \int_\gamma (z-a)^{n+1} \left(\int_0^1 \left(f^{(n+1)}[(1-s)a + sz] - f^{(n+1)}(a) \right) (1-s)^n \, ds \right) dz$$
$$= \frac{1}{n!} \int_0^1 \left(\int_\gamma (z-a)^{n+1} \left(f^{(n+1)}[(1-s)a + sz] - f^{(n+1)}(a) \right) dz \right) (1-s)^n \, ds.$$

3. Norm Inequalities

We start to the following result:

Theorem 4. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [0,1]$ with $z(0) = \alpha$ and $z(1) = \beta$ where $\alpha, \beta \in D$ and $n \geq 0$, then we have*

$$\left\| \int_\gamma f(z)\,dz - \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(a) \left[(\beta - a)^{k+1} + (-1)^k (a-\alpha)^{k+1} \right] \right\| \tag{3.1}$$
$$\leq \frac{1}{n!} \int_\gamma \|z-a\|^{n+1} \left\| \int_0^1 f^{(n+1)}[(1-s)a + sz] (1-s)^n \, ds \right\| |dz|$$
$$\leq \frac{1}{n!} \int_\gamma \|z-a\|^{n+1} \left(\int_0^1 \left\| f^{(n+1)}[(1-s)a + sz] \right\| (1-s)^n \, ds \right) |dz|$$
$$\leq \begin{cases} \frac{1}{(n+1)!} \int_\gamma \|z-a\|^{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)a + sz] \right\| |dz| \\ \\ \frac{1}{(qn+1)^{1/q}} \frac{1}{n!} \int_\gamma \|z-a\|^{n+1} \left(\int_0^1 \left\| f^{(n+1)}[(1-s)a + sz] \right\|^p ds \right)^{1/p} |dz| \\ \text{for } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{1}{n!} \int_\gamma \|z-a\|^{n+1} \left(\int_0^1 \left\| f^{(n+1)}[(1-s)a + sz] \right\| ds \right) |dz| . \end{cases}$$

Proof. Using the first identity in (2.4) we have

$$\left\| \int_\gamma f(z)\,dz - \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(a) \left[(\beta - a)^{k+1} + (-1)^k (a-\alpha)^{k+1} \right] \right\| \tag{3.2}$$
$$\leq \frac{1}{n!} \int_\gamma \left\| (z-a)^{n+1} \left(\int_0^1 f^{(n+1)}[(1-s)a + sz] (1-s)^n \, ds \right) \right\| |dz|$$
$$\leq \frac{1}{n!} \int_\gamma \left\| (z-a)^{n+1} \right\| \left\| \int_0^1 f^{(n+1)}[(1-s)a + sz] (1-s)^n \, ds \right\| |dz|$$
$$\leq \frac{1}{n!} \int_\gamma \|z-a\|^{n+1} \left\| \int_0^1 f^{(n+1)}[(1-s)a + sz] (1-s)^n \, ds \right\| |dz|$$
$$\leq \frac{1}{n!} \int_\gamma \|z-a\|^{n+1} \left(\int_0^1 \left\| f^{(n+1)}[(1-s)a + sz] \right\| (1-s)^n \, ds \right) |dz| =: A,$$

which proves the first part of (3.1).

Using Hölder's integral, we have

$$\int_0^1 \left\| f^{(n+1)} [(1-s) a + sz] \right\| (1-s)^n ds$$
$$\leq \begin{cases} \sup_{s\in[0,1]} \left\| f^{(n+1)} [(1-s) a + sz] \right\| \int_0^1 (1-s)^n ds \\ \\ \left(\int_0^1 \left\| f^{(n+1)} [(1-s) a + sz] \right\|^p ds \right)^{1/p} \left(\int_0^1 (1-s)^{qn} ds \right)^{1/q} \\ \text{for } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \int_0^1 \left\| f^{(n+1)} [(1-s) a + sz] \right\| ds \end{cases}$$
$$= \begin{cases} \frac{1}{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)} [(1-s) a + sz] \right\| \\ \\ \frac{1}{(qn+1)^{1/q}} \left(\int_0^1 \left\| f^{(n+1)} [(1-s) a + sz] \right\|^p ds \right)^{1/p} \\ \text{for } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \int_0^1 \left\| f^{(n+1)} [(1-s) a + sz] \right\| ds, \end{cases}$$

which implies that

$$A \leq \begin{cases} \frac{1}{n+1} \frac{1}{n!} \int_\gamma \|z-a\|^{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)} [(1-s) a + sz] \right\| |dz| \\ \\ \frac{1}{(qn+1)^{1/q}} \frac{1}{n!} \int_\gamma \|z-a\|^{n+1} \left(\int_0^1 \left\| f^{(n+1)} [(1-s) a + sz] \right\|^p ds \right)^{1/p} |dz| \\ \text{for } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{1}{n!} \int_\gamma \|z-a\|^{n+1} \left(\int_0^1 \left\| f^{(n+1)} [(1-s) a + sz] \right\| ds \right) |dz|, \end{cases}$$

which proves the last part of (3.1). □

Remark 3. *We observe that, if*

$$K_{a,\gamma}^{(n+1)} := \sup_{(s,z)\in[0,1]\times\gamma} \left\| f^{(n+1)} [(1-s) a + sz] \right\| < \infty, \ n \geq 0,$$

then from the first branch of (3.1) we get the simpler error estimate

$$\left\| \int_\gamma f(z) dz - \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(a) \left[(\beta - a)^{k+1} + (-1)^k (a-\alpha)^{k+1} \right] \right\| \leq \frac{1}{(n+1)!} K_{a,\gamma}^{(n+1)} \int_\gamma \|z-a\|^{n+1} |dz|. \tag{3.3}$$

Using Hölder's inequality we also have

$$\begin{aligned}
&\int_\gamma \|z-a\|^{n+1}\left(\int_0^1 \left\|f^{(n+1)}\left[(1-s)a+sz\right]\right\|^p ds\right)^{1/p}|dz| \\
&\leq \left(\int_\gamma \left[\left(\int_0^1 \left\|f^{(n+1)}\left[(1-s)a+sz\right]\right\|^p ds\right)^{1/p}\right]^p |dz|\right)^{1/p} \\
&\qquad\times\left(\int_\gamma \|z-a\|^{q(n+1)}|dz|\right)^{1/q} \\
&=\left(\int_\gamma \left(\int_0^1 \left\|f^{(n+1)}\left[(1-s)a+sz\right]\right\|^p ds\right)|dz|\right)^{1/p}\left(\int_\gamma \|z-a\|^{q(n+1)}|dz|\right)^{1/q}
\end{aligned}$$

and by the second branch of (3.1) we get

$$\begin{aligned}
(3.4)\quad &\left\|\int_\gamma f(z)\,dz-\sum_{k=0}^{n}\frac{1}{(k+1)!}f^{(k)}(a)\left[(\beta-a)^{k+1}+(-1)^k(a-\alpha)^{k+1}\right]\right\| \\
&\leq \frac{1}{(qn+1)^{1/q}}\frac{1}{n!}\left(\int_\gamma \|z-a\|^{q(n+1)}|dz|\right)^{1/q} \\
&\qquad\times\left(\int_\gamma\left(\int_0^1 \left\|f^{(n+1)}\left[(1-s)a+sz\right]\right\|^p ds\right)|dz|\right)^{1/p}
\end{aligned}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

We observe that, if we take $n=0$ in (3.1) then we get the Ostrowski type inequalities

$$\begin{aligned}
(3.5)\quad &\left\|\int_\gamma f(z)\,dz-(\beta-\alpha)f(a)\right\| \\
&\leq \int_\gamma \|z-a\|\left\|\int_0^1 f'\left[(1-s)a+sz\right]ds\right\||dz| \\
&\leq \int_\gamma \|z-a\|\left(\int_0^1 \left\|f'\left[(1-s)a+sz\right]\right\|ds\right)|dz| \\
&\leq \begin{cases}\int_\gamma \|z-a\|\sup_{s\in[0,1]}\left\|f'\left[(1-s)a+sz\right]\right\||dz| \\[2ex] \int_\gamma \|z-a\|\left(\int_0^1 \left\|f'\left[(1-s)a+sz\right]\right\|^p ds\right)^{1/p}|dz| \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\end{cases}
\end{aligned}$$

while for $n=1$ in (3.1) then we get the Ostrowski type perturbed inequalities

$$\begin{aligned}(3.6)\quad & \left\Vert \int_\gamma f(z)\,dz-(\beta-\alpha)f(a)-(\beta-\alpha)f'(a)\left(\frac{\beta+\alpha}{2}-a\right)\right\Vert \\ & \le \int_\gamma \Vert z-a\Vert^2\left\Vert\int_0^1 f''[(1-s)a+sz](1-s)\,ds\right\Vert |dz| \\ & \le \int_\gamma \Vert z-a\Vert^2\left(\int_0^1 \Vert f''[(1-s)a+sz]\Vert(1-s)\,ds\right)|dz| \\ & \le \begin{cases} \frac{1}{2}\int_\gamma \Vert z-a\Vert^2 \sup_{s\in[0,1]}\Vert f''[(1-s)a+sz]\Vert\,|dz| \\ \frac{1}{(q+1)^{1/q}}\int_\gamma \Vert z-a\Vert^2\left(\int_0^1\Vert f''[(1-s)a+sz]\Vert^p ds\right)^{1/p}|dz| \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \int_\gamma \Vert z-a\Vert^2\left(\int_0^1\Vert f''[(1-s)a+sz]\Vert\,ds\right)|dz|. \end{cases}\end{aligned}$$

If

$$K_{a,\gamma}^{(1)}:=\sup_{(s,z)\in[0,1]\times\gamma}\Vert f'[(1-s)a+sz]\Vert<\infty,\ n\ge 0,$$

then by (3.5) we get

$$(3.7)\qquad \left\Vert\int_\gamma f(z)\,dz-(\beta-\alpha)f(a)\right\Vert\le K_{a,\gamma}^{(1)}\int_\gamma\Vert z-a\Vert\,|dz|.$$

Also, if

$$K_{a,\gamma}^{(2)}:=\sup_{(s,z)\in[0,1]\times\gamma}\Vert f''[(1-s)a+sz]\Vert<\infty,\ n\ge 0,$$

then by (3.6) we get

$$\begin{aligned}(3.8)\quad & \left\Vert \int_\gamma f(z)\,dz-(\beta-\alpha)f(a)-(\beta-\alpha)f'(a)\left(\frac{\beta+\alpha}{2}-a\right)\right\Vert \\ & \le \frac{1}{2}K_{a,\gamma}^{(2)}\int_\gamma\Vert z-a\Vert^2\,|dz|.\end{aligned}$$

We also have:

Theorem 5. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic on G, $\gamma\subset D$ is a smooth path parametrized by $z(t)$, $t\in[0,1]$ with $z(0)=\alpha$ and $z(1)=\beta$ where $\alpha,\ \beta\in D$ and*

$n \geq 0$, *then we have*

$$\left\| \int_\gamma f(z)\,dz - \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(a) \left[(\beta - a)^{k+1} + (-1)^k (a-\alpha)^{k+1} \right] \right\| \tag{3.9}$$
$$\leq \frac{1}{n!} \int_0^1 \left(\int_\gamma \|z-a\|^{n+1} \left\| f^{(n+1)} [(1-s)a + sz] \right\| |dz| \right) (1-s)^n\, ds$$
$$\leq \begin{cases} \frac{1}{n!} \int_\gamma \|z-a\|^{n+1} |dz| \int_0^1 \sup_{z\in\gamma} \left\| f^{(n+1)} [(1-s)a+sz] \right\| (1-s)^n\, ds, \\ \\ \frac{1}{n!} \left(\int_\gamma \|z-a\|^{q(n+1)} |dz| \right)^{1/q} \\ \times \int_0^1 \left(\int_\gamma \left\| f^{(n+1)} [(1-s)a+sz] \right\|^p |dz| \right)^{1/p} (1-s)^n\, ds \\ \text{for } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{n!} \sup_{z\in\gamma} \|z-a\|^{n+1} \int_0^1 \left(\int_\gamma \left\| f^{(n+1)} [(1-s)a+sz] \right\| |dz| \right) (1-s)^n\, ds. \end{cases}$$

Proof. Using the second identity from (3.1), we have

$$\left\| \int_\gamma f(z)\,dz - \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(a) \left[(\beta - a)^{k+1} + (-1)^k (a-\alpha)^{k+1} \right] \right\| \tag{3.10}$$
$$\leq \frac{1}{n!} \int_0^1 \left\| \int_\gamma (z-a)^{n+1} f^{(n+1)} [(1-s)a+sz]\, dz \right\| (1-s)^n\, ds$$
$$\leq \frac{1}{n!} \int_0^1 \left(\int_\gamma \left\| (z-a)^{n+1} f^{(n+1)} [(1-s)a+sz] \right\| |dz| \right) (1-s)^n\, ds$$
$$\leq \frac{1}{n!} \int_0^1 \left(\int_\gamma \left\| (z-a)^{n+1} \right\| \left\| f^{(n+1)} [(1-s)a+sz] \right\| |dz| \right) (1-s)^n\, ds$$
$$\leq \frac{1}{n!} \int_0^1 \left(\int_\gamma \|z-a\|^{n+1} \left\| f^{(n+1)} [(1-s)a+sz] \right\| dz \right) (1-s)^n\, ds =: B.$$

Using the Hölder's integral inequality we have

$$\int_\gamma \|z-a\|^{n+1} \left\| f^{(n+1)} [(1-s)a+sz] \right\| |dz|$$
$$\leq \begin{cases} \sup_{z\in\gamma} \left\| f^{(n+1)} [(1-s)a+sz] \right\| \int_\gamma \|z-a\|^{n+1} |dz| \\ \\ \left(\int_\gamma \left\| f^{(n+1)} [(1-s)a+sz] \right\|^p |dz| \right)^{1/p} \left(\int_\gamma \|z-a\|^{q(n+1)} |dz| \right)^{1/q} \\ \text{for } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1. \\ \\ \sup_{z\in\gamma} \|z-a\|^{n+1} \int_\gamma \left\| f^{(n+1)} [(1-s)a+sz] \right\| |dz|. \end{cases}$$

Therefore,

$$B \le \begin{cases} \frac{1}{n!}\int_0^1 \sup_{z\in\gamma}\left\|f^{(n+1)}\left[(1-s)a+sz\right]\right\|\left(\int_\gamma \|z-a\|^{n+1}|dz|\right)(1-s)^n ds \\ \\ \frac{1}{n!}\int_0^1 \left(\int_\gamma \left\|f^{(n+1)}\left[(1-s)a+sz\right]\right\|^p |dz|\right)^{1/p} \\ \times\left(\int_\gamma \|z-a\|^{q(n+1)}|dz|\right)^{1/q}(1-s)^n ds \\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1. \\ \\ \frac{1}{n!}\int_0^1 \sup_{z\in\gamma}\|z-a\|^{n+1}\left(\int_\gamma \left\|f^{(n+1)}\left[(1-s)a+sz\right]\right\||dz|\right)(1-s)^n ds \end{cases}$$

$$= \begin{cases} \frac{1}{n!}\int_\gamma \|z-a\|^{n+1}|dz|\int_0^1 \sup_{z\in\gamma}\left\|f^{(n+1)}\left[(1-s)a+sz\right]\right\|(1-s)^n ds \\ \\ \frac{1}{n!}\left(\int_\gamma \|z-a\|^{q(n+1)}|dz|\right)^{1/q} \\ \times\int_0^1\left(\int_\gamma \left\|f^{(n+1)}\left[(1-s)a+sz\right]\right\|^p |dz|\right)^{1/p}(1-s)^n ds \\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1. \\ \\ \frac{1}{n!}\sup_{z\in\gamma}\|z-a\|^{n+1}\int_0^1\left(\int_\gamma \left\|f^{(n+1)}\left[(1-s)a+sz\right]\right\||dz|\right)(1-s)^n ds, \end{cases}$$

which proves the last part of (3.9). □

4. Perturbed Norm Inequalities

Let $a\in\mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$ and $\lambda\in G$. We define $G_{\lambda,a}:=\{(1-t)\lambda+ta \mid \text{with } t\in[0,1]\}$. We observe that $G_{\lambda,a}$ is a convex subset in $\mathcal{B}$ for every $\lambda\in G$.

For two distinct elements u, v in the Banach algebra B we say that the function $g:G_{\lambda,a}\to\mathcal{B}$ belongs to the class $\Delta_{u,v}(G_{\lambda,a})$ if it satisfies the boundedness condition

$$\left\|g\left((1-t)\lambda+ta\right)-\frac{u+v}{2}\right\|\le\frac{1}{2}\|v-u\| \tag{4.1}$$

for all $t\in[0,1]$. We write $g\in\Delta_{u,v}(G_{\lambda,a})$. This definition is an extension to Banach algebras valued functions of the scalar case, see [8].

We say that the function $g:G_{\lambda,a}\to B$ is Lipschitzian on $G_{\lambda,a}$ with the constant $L_{\lambda,a}>0$, if for all x, $y\in G_{\lambda,a}$ we have

$$\|g(x)-g(y)\|\le L_{\lambda,a}\|x-y\|.$$

This is equivalent to

$$\|g((1-t)\lambda+ta)-g((1-s)\lambda+sa)\|\le L_{\lambda,a}|t-s|\|a-\lambda\| \tag{4.2}$$

for all t, $s\in[0,1]$. We write this by $g\in\mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$.

Let $h:G\to\mathbb{C}$ be an analytic function on G. For $t\in[0,1]$ and $\lambda\in G$, the auxiliary function $h_{t,\lambda}$ defined on G by $h_{t,\lambda}(\xi):=h((1-t)\lambda+t\xi)$ is also analytic and using the analytic functional calculus (2.1) for the element $a \in \mathcal{B}$, we can define

$$\widetilde{h}\left(\left(1-t\right)\lambda+ta\right):=h_{t,\lambda}\left(a\right)=\frac{1}{2\pi i}\int_{\gamma}h_{t,\lambda}\left(\xi\right)\left(\xi-a\right)^{-1}d\xi \tag{4.3}$$
$$=\frac{1}{2\pi i}\int_{\gamma}h\left(\left(1-t\right)\lambda+t\xi\right)\left(\xi-a\right)^{-1}d\xi.$$

We say that the scalar function $h \in \Delta_{u,v}(G_{\lambda,a})$ if its extension $\widetilde{h} : G_{\lambda,a} \to B$ satisfies the boundedness condition (4.1). Also, we say that the scalar function $h \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$ if its extension $\widetilde{h} : G_{\lambda,a} \to B$ satisfies the Lipschitz condition (4.2).

We have:

Theorem 6. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. Assume also that $f : G \to \mathbb{C}$ is analytic on G and $\lambda \in G$. If there exists $u, v \in \mathcal{B}$ with $u \neq v$ such that $f^{(n+1)} \in \Delta_{u,v}(G_{\lambda,a})$, then*

$$\Bigg\|\int_{\gamma}f\left(z\right)dz-\sum_{k=0}^{n}\frac{1}{\left(k+1\right)!}f^{\left(k\right)}\left(a\right)\left[\left(\beta-a\right)^{k+1}+\left(-1\right)^{k}\left(a-\alpha\right)^{k+1}\right] \tag{4.4}$$
$$-\frac{1}{\left(n+2\right)!}\left[\left(\beta-a\right)^{n+2}+\left(-1\right)^{n+1}\left(a-\alpha\right)^{n+2}\right]\frac{u+v}{2}\Bigg\|$$
$$\leq\frac{1}{2}\frac{1}{\left(n+1\right)!}\left\|v-u\right\|\int_{\gamma}\left\|z-a\right\|^{n+1}\left|dz\right|.$$

Proof. Taking the norm in the equality (2.10) for $b = \frac{u+v}{2}$ and using the fact that $f^{(n+1)} \in \Delta_{u,v}(G_{\lambda,a})$, then we have

$$\Bigg\|\int_{\gamma}f\left(z\right)dz-\sum_{k=0}^{n}\frac{1}{\left(k+1\right)!}f^{\left(k\right)}\left(a\right)\left[\left(\beta-a\right)^{k+1}+\left(-1\right)^{k}\left(a-\alpha\right)^{k+1}\right] \tag{4.5}$$
$$-\frac{1}{\left(n+2\right)!}\left[\left(\beta-a\right)^{n+2}+\left(-1\right)^{n+1}\left(a-\alpha\right)^{n+2}\right]\frac{u+v}{2}\Bigg\|$$
$$\leq\frac{1}{n!}\int_{\gamma}\left\|\left(z-a\right)^{n+1}\left(\int_{0}^{1}\left(f^{\left(n+1\right)}\left[\left(1-s\right)a+sz\right]-\frac{u+v}{2}\right)\left(1-s\right)^{n}ds\right)\right\|\left|dz\right|$$
$$\leq\frac{1}{n!}\int_{\gamma}\left\|z-a\right\|^{n+1}\left\|\left(\int_{0}^{1}\left(f^{\left(n+1\right)}\left[\left(1-s\right)a+sz\right]-\frac{u+v}{2}\right)\left(1-s\right)^{n}ds\right)\right\|\left|dz\right|$$
$$\leq\frac{1}{n!}\int_{\gamma}\left\|z-a\right\|^{n+1}\left(\int_{0}^{1}\left\|f^{\left(n+1\right)}\left[\left(1-s\right)a+sz\right]-\frac{u+v}{2}\right\|\left(1-s\right)^{n}ds\right)\left|dz\right|$$
$$\leq\frac{1}{n!}\frac{1}{2}\left\|v-u\right\|\left(\int_{0}^{1}\left(1-s\right)^{n}ds\right)\int_{\gamma}\left\|z-a\right\|^{n+1}\left|dz\right|$$
$$=\frac{1}{2}\frac{1}{\left(n+1\right)!}\left\|v-u\right\|\int_{\gamma}\left\|z-a\right\|^{n+1}\left|dz\right|,$$

which proves (4.4). □

We have:

Theorem 7. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$. Assume also that $f : G \to \mathbb{C}$ is analytic on G and $\lambda \in G$. If $f^{(n+1)} \in \mathfrak{Lip}_{L_{\lambda,a}}(G_{\lambda,a})$ for some $L_{\lambda,a} > 0$, then*

$$\left\| \int_\gamma f(z)\,dz - \sum_{k=0}^{n+1} \frac{1}{(k+1)!} f^{(k)}(a) \left[(\beta - a)^{k+1} + (-1)^k (a-\alpha)^{k+1} \right] \right\| \leq \frac{1}{(n+2)!} L_{\lambda,a} \int_\gamma \|z-a\|^{n+2} |dz| . \tag{4.6}$$

Proof. By using the identity (2.10), we get

$$\begin{aligned} &\left\| \int_\gamma f(z)\,dz - \sum_{k=0}^{n+1} \frac{1}{(k+1)!} f^{(k)}(a) \left[(\beta - a)^{k+1} + (-1)^k (a-\alpha)^{k+1} \right] \right\| \\ &\leq \frac{1}{n!} \int_\gamma \left\| (z-a)^{n+1} \left(\int_0^1 \left(f^{(n+1)}[(1-s)a + sz] - f^{(n+1)}(a) \right) (1-s)^n ds \right) \right\| |dz| \\ &\leq \frac{1}{n!} \int_\gamma \left\| (z-a)^{n+1} \right\| \left\| \int_0^1 \left(f^{(n+1)}[(1-s)a + sz] - f^{(n+1)}(a) \right) (1-s)^n ds \right\| |dz| \\ &\leq \frac{1}{n!} \int_\gamma \|z-a\|^{n+1} \left(\int_0^1 \left\| f^{(n+1)}[(1-s)a + sz] - f^{(n+1)}(a) \right\| (1-s)^n ds \right) |dz| \\ &\leq \frac{1}{n!} L_{\lambda,a} \int_\gamma \|z-a\|^{n+2} \left(\int_0^1 s(1-s)^n ds \right) |dz| \\ &= \left(\int_0^1 s(1-s)^n ds \right) \frac{1}{n!} L_{\lambda,a} \int_\gamma \|z-a\|^{n+2} |dz| . \end{aligned} \tag{4.7}$$

Since

$$\begin{aligned} \int_0^1 s(1-s)^n ds &= \int_0^1 (1-s) s^n ds = \int_0^1 \left(s^n - s^{n+1} \right) ds \\ &= \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}, \end{aligned}$$

hence by (4.7) we get (4.6). □

5. Applications for Exponential Function

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, $\gamma \subset D$ is a *smooth path* parametrized by $z(t)$, $t \in [0,1]$ with $z(0) = \alpha$ and $z(1) = \beta$ where α, $\beta \in \mathbb{C}$. Consider the exponential function $f(z) = \exp(z)$, $z \in \mathbb{C}$ and put

$$E_{a,\gamma} := \sup_{(s,z) \in [0,1] \times \gamma} \|\exp[(1-s)a + sz]\| < \infty, \ n \geq 0.$$

Observe that

$$\exp((1-t)\lambda + ta) = \exp[(1-t)\lambda] \exp(ta),$$

which gives

$$\begin{aligned}
&\|\exp((1-t)\lambda + ta)\| \\
&= |\exp[(1-t)\lambda]| \, \|\exp(ta)\| = \exp[(1-t)\operatorname{Re}\lambda] \, \|\exp(ta)\| \\
&\leq \exp[(1-t)\operatorname{Re}\lambda] \exp(t\|a\|) = \exp[(1-t)\operatorname{Re}\lambda + t\|a\|] \\
&\leq \exp(\max\{\operatorname{Re}\lambda, \|a\|\})
\end{aligned}$$

for any $t \in [0,1]$, $\lambda \in \mathbb{C}$.

Therefore

$$E_{a,\gamma} \leq \sup_{z\in\gamma} \exp(\max\{\operatorname{Re} z, \|a\|\}) = \exp\left(\max\left\{\max_{z\in\gamma} \operatorname{Re} z, \|a\|\right\}\right).$$

By utilising (3.3) we then get

$$\left\|\exp\beta - \exp\alpha - \sum_{k=0}^{n} \frac{1}{(k+1)!}\left[(\beta-a)^{k+1} + (-1)^k (a-\alpha)^{k+1}\right]\exp a\right\| \leq \frac{1}{(n+1)!}\exp\left(\max\left\{\max_{z\in\gamma}\operatorname{Re} z, \|a\|\right\}\right)\int_\gamma \|z-a\|^{n+1}|dz|. \tag{5.1}$$

Now, if we define

(5.2)

$$\begin{aligned}
E_n(\lambda, a) &:= \int_0^1 \exp[(1-s)\operatorname{Re}\lambda + s\|a\|](1-s)^n ds \\
&= \int_0^1 \exp[s\operatorname{Re}\lambda + (1-s)\|a\|] s^n ds = \int_0^1 \exp[\|a\| + s(\operatorname{Re}\lambda - \|a\|)] s^n ds
\end{aligned}$$

then by using the integration by parts and assuming that $\operatorname{Re}\lambda \neq \|a\|$ we have

$$\begin{aligned}
&\int_0^1 \exp[\|a\| + s(\operatorname{Re}\lambda - \|a\|)] s^n ds \\
&= \frac{1}{\operatorname{Re}\lambda - \|a\|}\int_0^1 s^n d(\exp[\|a\| + s(\operatorname{Re}\lambda - \|a\|)]) \\
&= \frac{1}{\operatorname{Re}\lambda - \|a\|} \\
&\times \left[s^n \exp[\|a\| + s(\operatorname{Re}\lambda - \|a\|)]\Big|_0^1 - n\int_0^1 s^{n-1}\exp[\|a\| + s(\operatorname{Re}\lambda - \|a\|)] ds\right] \\
&= \frac{1}{\operatorname{Re}\lambda - \|a\|}\left[\exp(\operatorname{Re}\lambda) - n\int_0^1 s^{n-1}\exp[\|a\| + s(\operatorname{Re}\lambda - \|a\|)] ds\right],
\end{aligned}$$

which gives the recursive relation

$$E_n(\lambda, a) = \frac{1}{\operatorname{Re}\lambda - \|a\|}[\exp(\operatorname{Re}\lambda) - nE_{n-1}(\lambda, a)], \; n \geq 1 \tag{5.3}$$

with

$$E_0(\lambda, a) = \frac{\exp(\operatorname{Re}\lambda) - \exp(\|a\|)}{\operatorname{Re}\lambda - \|a\|}. \tag{5.4}$$

For $\operatorname{Re}\lambda = \|a\|$ we have $E_n(\lambda, a) = \frac{1}{n+1}\exp(\|a\|)$.

From the inequality (3.1) we get

$$\left\|\exp\beta-\exp\alpha-\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[(\beta-a)^{k+1}+(-1)^{k}(a-\alpha)^{k+1}\right]\exp a\right\| \tag{5.5}$$
$$\leq\frac{1}{n!}\int_{\gamma}\|z-a\|^{n+1}\left(\int_{0}^{1}\|\exp[(1-s)a+sz]\|(1-s)^{n}ds\right)|dz|$$
$$\leq\frac{1}{n!}\int_{\gamma}\|z-a\|^{n+1}\left(\int_{0}^{1}\exp[(1-s)\operatorname{Re}z+s\|a\|](1-s)^{n}ds\right)|dz|$$

and since, as shown by (5.2)

$$\int_{0}^{1}\exp[(1-s)\operatorname{Re}z+s\|a\|](1-s)^{n}ds=E_{n}(z,a),$$

hence we get the inequality

$$\left\|\exp\beta-\exp\alpha-\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[(\beta-a)^{k+1}+(-1)^{k}(a-\alpha)^{k+1}\right]\exp a\right\| \tag{5.6}$$
$$\leq\frac{1}{n!}\int_{\gamma}\|z-a\|^{n+1}E_{n}(z,a)|dz|.$$

The inequality (5.6) is sharper than (5.1) but the upper bound from (5.6) is much more difficult to calculate for a given path γ than the one from 5.1.

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