TWO POINTS AND n-TH DERIVATIVES NORM INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$, α, $\beta \in G$ and $f : G \to \mathbb{C}$ is analytic on G. By using the analytic functional calculus we obtain among others the following result

$$\left\| f(a) - \frac{1}{2}\sum_{k=0}^{n} \frac{1}{k!}\left[f^{(k)}(\alpha)(a-\alpha)^k + (-1)^k f^{(k)}(\beta)(\beta - a)^k \right] \right\|$$
$$\leq \frac{1}{2(n+1)!}\left[\|a-\alpha\|^{n+1} + \|\beta - a\|^{n+1} \right]$$
$$\times \max\left\{ \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\|, \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)a + s\beta] \right\| \right\}.$$

Some examples for the exponential function on Banach algebras are also given.

1. Introduction

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \leq \|a\| \|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\mathrm{Inv}(\mathcal{B})$. If $a, b \in \mathrm{Inv}(\mathcal{B})$ then $ab \in \mathrm{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \mathrm{Inv}(\mathcal{B})$;
(ii) $\{a \in \mathcal{B}: \|1 - b\| < 1\} \subset \mathrm{Inv}(\mathcal{B})$;
(iii) $\mathrm{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\mathrm{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \mathrm{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} : z - a \in \mathrm{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \mathrm{Inv}(\mathcal{B})$, $R_a(z) := (z-a)^{-1}$. For each z, $w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z - w) R_a(z) R_a(w).$$

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We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \leq \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_\delta f(\xi)(\xi - a)^{-1} d\xi, \tag{1.1}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [6, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.2}$$

holds.

Let $\mathfrak{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\mathfrak{Hol}(a)$ is an algebra where if f, $g \in \mathfrak{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f+g$ have domain $D(f) \cap D(g)$. $\mathfrak{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [6, p. 201-203]:

Theorem 1. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) *The map $f \mapsto f(a)$ of $\mathfrak{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.*
(b) *If $f(z) = \sum_{k=0}^\infty \alpha_k z^k$ has radius of convergence $r > \nu(a)$, then $f \in \mathfrak{Hol}(a)$ and $f(a) = \sum_{k=0}^\infty \alpha_k a^k$.*
(c) *If $f(z) \equiv 1$, then $f(a) = 1$.*
(d) *If $f(z) = z$ for all z, $f(a) = a$.*
(e) *If $f, f_1, ..., f_n...$ are analytic on G, $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\|f_n(a) - f(a)\| \to 0$ as $n \to \infty$.*
(f) *The Riesz Functional Calculus is unique and if a, b are commuting elements in $\mathcal{B}$ and $f \in \mathfrak{Hol}(a)$, then $f(a)b = bf(a)$.*

For some recent norm inequalities for functions on Banach algebras, see [4]-[5] and [9]-[15].

By using the analytic functional calculus in Banach algebra $\mathcal{B}$ and function $f \in \mathfrak{Hol}(a)$ we establish in this paper some norm error estimates in approximation the element $f(a)$ by some simpler expressions such as

$$(1-\lambda) f(\alpha) + \lambda f(\beta) + \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda) f^{(k)}(\alpha)(a-\alpha)^k + (-1)^k \lambda f^{(k)}(\beta)(\beta - a)^k\right],$$

$$\frac{1}{\beta-\alpha}\left[(\beta-a) f(\alpha)+(a-\alpha) f(\beta)\right]+\frac{(\beta-a)(a-\alpha)}{\beta-\alpha}$$
$$\times \sum_{k=1}^{n} \frac{1}{k!}\left\{(a-\alpha)^{k-1} f^{(k)}(\alpha)+(-1)^{k}(\beta-a)^{k-1} f^{(k)}(\beta)\right\}$$

and

$$\frac{1}{\beta-\alpha}\left[(a-\alpha) f(\alpha)+(\beta-a) f(\beta)\right]$$
$$+\frac{1}{\beta-\alpha} \sum_{k=1}^{n} \frac{1}{k!}\left\{(a-\alpha)^{k+1} f^{(k)}(\alpha)+(-1)^{k}(\beta-a)^{k+1} f^{(k)}(\beta)\right\}$$

where α, $\beta \in D$ and $\lambda \in \mathbb{C}$.

2. Scalar Identities

Let $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the convex domain D and ξ, $\alpha \in D$, then we have the following Taylor's expansion with integral remainder

$$f(\xi)=\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\alpha)(\xi-\alpha)^{k} + \frac{1}{n!}(\xi-\alpha)^{n+1} \int_{0}^{1} f^{(n+1)}[(1-s) \alpha+s \xi](1-s)^{n} d s \tag{2.1}$$

for $n \geq 0$, see for instance [24].

Consider the function $f(\xi) = \operatorname{Log}(\xi)$ where $\operatorname{Log}(\xi) = \ln|\xi| + i \operatorname{Arg}(\xi)$ and $\operatorname{Arg}(\xi)$ is such that $-\pi < \operatorname{Arg}(\xi) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_{\ell} := \mathbb{C} \backslash \{\alpha + i\beta : \alpha \leq 0,\ \beta = 0\}$ and

$$f^{(k)}(\xi)=\frac{(-1)^{k-1}(k-1)!}{\xi^{k}},\ k \geq 1,\ \xi \in \mathbb{C}_{\ell}.$$

Using the representation (2.1) we then have

$$\operatorname{Log}(\xi)=\operatorname{Log}(\alpha)+\sum_{k=0}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{\xi-\alpha}{\alpha}\right)^{k} +(-1)^{n}(\xi-\alpha)^{n+1} \int_{0}^{1} \frac{(1-s)^{n} d s}{[(1-s) \alpha+s \xi]^{n+1}} \tag{2.2}$$

for all ξ, $\alpha \in \mathbb{C}_{\ell}$ with $(1-s)\alpha + s\xi \in \mathbb{C}_{\ell}$ for $s \in [0,1]$.

Consider the complex exponential function $f(\xi) = \exp(\xi)$, then by (2.1) we get

$$\exp(\xi)=\sum_{k=0}^{n} \frac{1}{k!}(\xi-\alpha)^{k} \exp(\alpha) +\frac{1}{n!}(\xi-\alpha)^{n+1} \int_{0}^{1}(1-s)^{n} \exp[(1-s) \alpha+s \xi] d s \tag{2.3}$$

for all ξ, $\alpha \in \mathbb{C}$.

For various inequalities related to Taylor's expansions for real functions see [1]-[3], [16]-[22] and [23].

We have:

Lemma 1. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and ξ, α, $\beta \in D$, then for all $\lambda \in \mathbb{C}$ and $n \geq 1$ we have*

$$\begin{aligned} f(\xi) &= (1-\lambda) f(\alpha) + \lambda f(\beta) \\ &\quad + \sum_{k=1}^{n} \frac{1}{k!} \left[(1-\lambda) f^{(k)}(\alpha)(\xi-\alpha)^k + (-1)^k \lambda f^{(k)}(\beta)(\beta-\xi)^k \right] \\ &\quad + S_{n,\lambda}(\xi,\alpha,\beta), \end{aligned} \tag{2.4}$$

where the remainder $S_{n,\lambda}(\xi,\alpha,\beta)$ is given by

$$\begin{aligned} &S_{n,\lambda}(\xi,\alpha,\beta) \\ &:= \frac{1}{n!} \left[(1-\lambda)(\xi-\alpha)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\alpha + s\xi](1-s)^n ds \right. \\ &\left. + (-1)^{n+1} \lambda (\beta-\xi)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\xi + s\beta] s^n ds \right]. \end{aligned} \tag{2.5}$$

Proof. If we replace in (2.1) α by β, then we get

$$\begin{aligned} f(\xi) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\alpha)(\xi-\beta)^k \\ &\quad + \frac{1}{n!}(\xi-\beta)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\beta + s\xi](1-s)^n ds \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(\alpha)(\beta-\xi)^k \\ &\quad + \frac{(-1)^{n+1}}{n!}(\beta-\xi)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\beta + s\xi](1-s)^n ds \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(\alpha)(\beta-\xi)^k \\ &\quad + \frac{(-1)^{n+1}}{n!}(\beta-\xi)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\xi + s\beta] s^n ds. \end{aligned} \tag{2.6}$$

Assume that $\lambda \neq 1, 0$. If we multiply (2.1) by $1-\lambda$ and (2.6) by λ we get the desired representation (2.4) with the remainder $S_{n,\lambda}(\xi,\alpha,\beta)$ given by (2.5).

If either $\lambda = 1$ or $\lambda = 0$, then the theorem also holds by the use of Taylor's usual expansion. □

Remark 1. *We observe that for $n = 0$ the representation from Lemma 1 becomes*

$$f(\xi) = (1-\lambda) f(\alpha) + \lambda f(\beta) + S_\lambda(\xi,\alpha,\beta), \tag{2.7}$$

where the remainder $S_\lambda(\xi,\alpha,\beta)$ is given by

$$\begin{aligned} S_\lambda(\xi,\alpha,\beta) &:= (1-\lambda)(\xi-\alpha) \int_0^1 f'((1-s)\alpha + s\xi) ds \\ &\quad - \lambda(\beta-\xi) \int_0^1 f'((1-s)\xi + s\beta) ds. \end{aligned} \tag{2.8}$$

Corollary 1. *With the assumptions in Lemma 1 we have for each distinct* ξ, α, $\beta \in D$ *with* $\beta \neq \alpha$

$$f(\xi) = \frac{1}{\beta-\alpha}\left[(\beta-\xi) f(\alpha) + (\xi-\alpha) f(\beta)\right] + \frac{(\beta-\xi)(\xi-\alpha)}{\beta-\alpha} \times \sum_{k=1}^{n} \frac{1}{k!}\left\{(\xi-\alpha)^{k-1} f^{(k)}(\alpha) + (-1)^{k}(\beta-\xi)^{k-1} f^{(k)}(\beta)\right\} + L_n(\xi,\alpha,\beta), \tag{2.9}$$

where

$$L_n(\xi,\alpha,\beta) := \frac{(\beta-\xi)(\xi-\alpha)}{n!(\beta-\alpha)}\left[(\xi-\alpha)^{n}\int_0^1 f^{(n+1)}((1-s)\alpha+s\xi)(1-s)^{n}ds + (-1)^{n+1}(\beta-\xi)^{n}\int_0^1 f^{(n+1)}((1-s)\xi+s\beta)s^{n}ds\right]$$

and

$$f(\xi) = \frac{1}{\beta-\alpha}\left[(\xi-\alpha) f(\alpha) + (\beta-\xi) f(\beta)\right] + \frac{1}{\beta-\alpha}\sum_{k=1}^{n} \frac{1}{k!}\left\{(\xi-\alpha)^{k+1} f^{(k)}(\alpha) + (-1)^{k}(\beta-\xi)^{k+1} f^{(k)}(\beta)\right\} + P_n(\xi,\alpha,\beta), \tag{2.10}$$

where

$$P_n(\xi,\alpha,\beta) := \frac{1}{n!(\beta-\alpha)}\left[(\xi-\alpha)^{n+2}\int_0^1 f^{(n+1)}((1-s)\alpha+s\xi)(1-s)^{n}ds + (-1)^{n+1}(\beta-\xi)^{n+2}\int_0^1 f^{(n+1)}((1-s)\xi+s\beta)s^{n}ds\right],$$

respectively.

The proof is obvious, by choosing $\lambda = (\xi-\alpha)/(\beta-\alpha)$ and $\lambda = (\beta-\xi)/(\beta-\alpha)$, respectively, in Lemma 1. The details are omitted.

The case $n = 0$ produces the following simple identities for each distinct ξ, α, $\beta \in D$

$$f(\xi) = \frac{1}{\beta-\alpha}\left[(\beta-\xi) f(\alpha) + (\xi-\alpha) f(\beta)\right] + L(\xi,\alpha,\beta), \tag{2.11}$$

where

$$L(\xi,\alpha,\beta) := \frac{(\beta-\xi)(\xi-\alpha)}{\beta-\alpha}\left[\int_0^1 f'((1-s)\alpha+s\xi)ds - \int_0^1 f'((1-s)\xi+s\beta)ds\right] \tag{2.12}$$

and

$$f(\xi) = \frac{1}{\beta-\alpha}\left[(\xi-\alpha) f(\alpha) + (\beta-\xi) f(\beta)\right] + P(\xi,\alpha,\beta), \tag{2.13}$$

where

$$P(\xi,\alpha,\beta) \tag{2.14}$$
$$:= \frac{1}{\beta-\alpha}\left[(\xi-\alpha)^2\int_0^1 f'((1-s)\alpha+s\xi)\,ds-(\beta-\xi)^2\int_0^1 f'((1-s)\xi+s\beta)\,ds\right].$$

3. Identities in Banach Algebras

We have the following two point representation of an analytic function on Banach algebras:

Theorem 2. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$ and $\alpha,\ \beta\in D$. If $f:G\to\mathbb{C}$ is analytic on G, then for all $\lambda\in\mathbb{C}$ and $n\geq 1$ we have*

$$\begin{aligned} f(a) &= (1-\lambda)f(\alpha)+\lambda f(\beta) \\ &+\sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)f^{(k)}(\alpha)(a-\alpha)^k+(-1)^k\lambda f^{(k)}(\beta)(\beta-a)^k\right] \\ &+S_{n,\lambda}(a,\alpha,\beta), \end{aligned} \tag{3.1}$$

where the remainder $S_{n,\lambda}(a,\alpha,\beta)$ is given by

$$\begin{aligned} &S_{n,\lambda}(a,\alpha,\beta) \\ &:=\frac{1}{n!}\left[(1-\lambda)(a-\alpha)^{n+1}\int_0^1 f^{(n+1)}[(1-s)\alpha+sa](1-s)^n\,ds\right. \\ &\left.+(-1)^{n+1}\lambda(\beta-a)^{n+1}\int_0^1 f^{(n+1)}[(1-s)a+s\beta]s^n ds\right]. \end{aligned} \tag{3.2}$$

In particular, for $\lambda=\frac{1}{2}$, we have the trapezoid type identity

$$\begin{aligned} f(a) &= \frac{f(\alpha)+f(\beta)}{2} \\ &+\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k!}\left[f^{(k)}(\alpha)(a-\alpha)^k+(-1)^k f^{(k)}(\beta)(\beta-a)^k\right] \\ &+T_n(a,\alpha,\beta), \end{aligned} \tag{3.3}$$

where the remainder $T_n(a,\alpha,\beta)$ is given by

$$\begin{aligned} &T_n(a,\alpha,\beta) \\ &:=\frac{1}{2n!}\left[(a-\alpha)^{n+1}\int_0^1 f^{(n+1)}[(1-s)\alpha+sa](1-s)^n\,ds\right. \\ &\left.+(-1)^{n+1}(\beta-a)^{n+1}\int_0^1 f^{(n+1)}[(1-s)a+s\beta]s^n ds\right]. \end{aligned} \tag{3.4}$$

Proof. Assume that $\delta\subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a)\subset\operatorname{ins}(\delta)$. By using the analytic functional calculus (1.1) and Lemma 1, we get

$$\frac{1}{2\pi i}\int_\delta f(\xi)(\xi-a)^{-1}d\xi = [(1-\lambda)f(\alpha)+\lambda f(\beta)]\frac{1}{2\pi i}\int_\delta(\xi-a)^{-1}d\xi \tag{3.5}$$
$$+\sum_{k=1}^{n}\frac{1}{k!}(1-\lambda)f^{(k)}(\alpha)\frac{1}{2\pi i}\int_\delta(\xi-\alpha)^k(\xi-a)^{-1}d\xi$$
$$+\sum_{k=1}^{n}\frac{1}{k!}(-1)^k\lambda f^{(k)}(\beta)\frac{1}{2\pi i}\int_\delta(\beta-\xi)^k(\xi-a)^{-1}d\xi$$
$$+\frac{1}{n!}(1-\lambda)\frac{1}{2\pi i}\int_\delta(\xi-\alpha)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)\alpha+s\xi](1-s)^n ds\right)(\xi-a)^{-1}d\xi$$
$$+\frac{1}{n!}(-1)^{n+1}\lambda\frac{1}{2\pi i}\int_\delta(\beta-\xi)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)\xi+s\beta]s^n ds\right)(\xi-a)^{-1}d\xi$$
$$=\frac{1}{n!}(1-\lambda)\int_0^1\left(\frac{1}{2\pi i}\int_\delta(\xi-\alpha)^{n+1}f^{(n+1)}[(1-s)\alpha+s\xi](\xi-a)^{-1}d\xi\right)(1-s)^n ds$$
$$+\frac{1}{n!}(-1)^{n+1}\lambda\int_0^1\left(\frac{1}{2\pi i}\int_\delta(\beta-\xi)^{n+1}f^{(n+1)}[(1-s)\xi+s\beta](\xi-a)^{-1}d\xi\right)s^n ds,$$

where for the last equality we used Fubini's theorem.

By using the functional calculus for the analytic functions

$$G\ni\xi\mapsto(\xi-\alpha)^{n+1}f^{(n+1)}[(1-s)\alpha+s\xi]\in\mathbb{C}$$

and

$$G\ni\xi\mapsto(\beta-\xi)^{n+1}f^{(n+1)}[(1-s)\xi+s\beta]\in\mathbb{C}$$

we get

$$\frac{1}{2\pi i}\int_\delta(\xi-\alpha)^{n+1}f^{(n+1)}[(1-s)\alpha+s\xi](\xi-a)^{-1}d\xi$$
$$=(a-\alpha)^{n+1}f^{(n+1)}[(1-s)\alpha+sa]$$

and

$$\frac{1}{2\pi i}\int_\delta(\beta-\xi)^{n+1}f^{(n+1)}[(1-s)\xi+s\beta](\xi-a)^{-1}d\xi$$
$$=(\beta-a)^{n+1}f^{(n+1)}[(1-s)a+s\beta].$$

Therefore

$$\int_0^1\left(\frac{1}{2\pi i}\int_\delta(\xi-\alpha)^{n+1}f^{(n+1)}[(1-s)\alpha+s\xi](\xi-a)^{-1}d\xi\right)(1-s)^n ds$$
$$=\int_0^1(a-\alpha)^{n+1}f^{(n+1)}[(1-s)\alpha+sa](1-s)^n ds$$
$$=(a-\alpha)^{n+1}\int_0^1 f^{(n+1)}[(1-s)\alpha+sa](1-s)^n ds$$

and

$$\begin{aligned}
&\int_0^1 \left(\frac{1}{2\pi i} \int_\delta (\beta - \xi)^{n+1} f^{(n+1)} [(1-s)\xi + s\beta] (\xi - a)^{-1} d\xi \right) s^n ds \\
&= \int_0^1 (\beta - a)^{n+1} f^{(n+1)} [(1-s) a + s\beta] s^n ds \\
&= (\beta - a)^{n+1} \int_0^1 f^{(n+1)} [(1-s) a + s\beta] s^n ds.
\end{aligned}$$

Since

$$\frac{1}{2\pi i} \int_\delta (\xi - a)^{-1} d\xi = 1, \ \frac{1}{2\pi i} \int_\delta (\xi - \alpha)^k (\xi - a)^{-1} d\xi = (a - \alpha)^k$$

and

$$\frac{1}{2\pi i} \int_\delta (\beta - \xi)^k (\xi - a)^{-1} d\xi = (\beta - a)^k$$

for $k = 1, ..., n$, hence by (3.5) we get the representation (3.1) with the remainder (3.2). □

Remark 2. *Withe the assumptions from Theorem 2 and by using the scalar identity (2.7) we have, for $n = 0$, that*

$$f(a) = (1-\lambda) f(\alpha) + \lambda f(\beta) + S_\lambda(a, \alpha, \beta), \tag{3.6}$$

where the remainder $S_\lambda(a, \alpha, \beta)$ is given by

$$\begin{aligned}
S_\lambda(a, \alpha, \beta) &:= (1-\lambda)(a-\alpha) \int_0^1 f'((1-s)\alpha + sa) ds \\
&\qquad - \lambda (\beta - a) \int_0^1 f'((1-s) a + s\beta) ds.
\end{aligned} \tag{3.7}$$

In particular, we have

$$f(a) = \frac{f(\alpha) + f(\beta)}{2} + T(a, \alpha, \beta), \tag{3.8}$$

where the remainder $T(a, \alpha, \beta)$ is given by

$$\begin{aligned}
&T(a, \alpha, \beta) \\
&:= \frac{1}{2} \left[(a-\alpha) \int_0^1 f'((1-s)\alpha + sa) ds - (\beta - a) \int_0^1 f'((1-s) a + s\beta) ds \right].
\end{aligned} \tag{3.9}$$

We also have:

Theorem 3. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and α, $\beta \in D$ with $\alpha \neq \beta$. If $f : G \to \mathbb{C}$ is analytic on G, then*

$$\begin{aligned}
f(a) &= \frac{1}{\beta - \alpha} [f(\alpha)(\beta - a) + f(\beta)(a - \alpha)] + \frac{(\beta - a)(a - \alpha)}{\beta - \alpha} \\
&\quad \times \sum_{k=1}^n \frac{1}{k!} \left\{ f^{(k)}(\alpha)(a-\alpha)^{k-1} + (-1)^k f^{(k)}(\beta)(\beta - a)^{k-1} \right\} \\
&\quad + L_n(a, \alpha, \beta),
\end{aligned} \tag{3.10}$$

where

$$L_n(a,\alpha,\beta) := \frac{(\beta-a)(a-\alpha)}{n!(\beta-\alpha)}\left[(a-\alpha)^n \int_0^1 f^{(n+1)}((1-s)\alpha+sa)(1-s)^n ds \right. \\ \left. + (-1)^{n+1}(\beta-a)^n \int_0^1 f^{(n+1)}((1-s)a+s\beta)s^n ds\right]$$

and

$$f(a) = \frac{1}{\beta-\alpha}[f(\alpha)(a-\alpha)+f(\beta)(\beta-a)] \\ + \frac{1}{\beta-\alpha}\sum_{k=1}^{n}\frac{1}{k!}\left\{f^{(k)}(\alpha)(a-\alpha)^{k+1}+(-1)^k f^{(k)}(\beta)(\beta-a)^{k+1}\right\} \\ + P_n(a,\alpha,\beta), \tag{3.11}$$

where

$$P_n(a,\alpha,\beta) := \frac{1}{n!(\beta-\alpha)}\left[(a-\alpha)^{n+2}\int_0^1 f^{(n+1)}((1-s)\alpha+sa)(1-s)^n ds \right. \\ \left. + (-1)^{n+1}(\beta-a)^{n+2}\int_0^1 f^{(n+1)}((1-s)a+s\beta)s^n ds\right].$$

The proof follows in a similar way to the one from Theorem 2 by utilising the functional calculus for analytic functions (1.1) and the scalar identities (2.9) and (2.10).

The case $n=0$ produces the following simple identities for each distinct $\alpha, \beta \in D$

$$f(a) = \frac{1}{\beta-\alpha}[f(\alpha)(\beta-a)+f(\beta)(a-\alpha)] + L(a,\alpha,\beta), \tag{3.12}$$

where

$$L(a,\alpha,\beta) \\ := \frac{(\beta-a)(a-\alpha)}{\beta-\alpha}\left[\int_0^1 f'((1-s)\alpha+sa)ds - \int_0^1 f'((1-s)a+s\beta)ds\right]$$

and

$$f(a) = \frac{1}{\beta-\alpha}[f(\alpha)(a-\alpha)+f(\beta)(\beta-a)] + P(a,\alpha,\beta), \tag{3.13}$$

where

$$P(a,\alpha,\beta) \\ := \frac{1}{\beta-\alpha}\left[(a-\alpha)^2\int_0^1 f'((1-s)\alpha+sa)ds - (\beta-a)^2\int_0^1 f'((1-s)a+s\beta)ds\right].$$

4. Norm Inequalities

The following result providing norm error estimates, holds:

Theorem 4. *Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and $\alpha, \beta \in D$. If $f : G \to \mathbb{C}$ is analytic on G, then for all $\lambda \in \mathbb{C}$ and*

$n \geq 1$ we have the representation (2.5) and the remainder $S_{n,\lambda}(a,\alpha,\beta)$ satisfies the norm inequalities

$$
\begin{aligned}
(4.1)\quad & \|S_{n,\lambda}(a,\alpha,\beta)\| \\
&\leq \frac{1}{n!}\left[|1-\lambda|\,\|a-\alpha\|^{n+1}\int_0^1\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\|(1-s)^n\,ds\right. \\
&\qquad\left.+|\lambda|\,\|\beta-a\|^{n+1}\int_0^1\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|s^n ds\right] \\
&\leq \frac{1}{n!}|1-\lambda|\,\|a-\alpha\|^{n+1}\begin{cases}\frac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\| \\ \\ \frac{1}{(nq+1)^{1/q}}\left(\int_0^1\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\|^p ds\right)^{1/p} \\ \text{for } p,q>1,\ \text{where } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\|ds\end{cases} \\
&\quad+\frac{1}{n!}|\lambda|\,\|\beta-a\|^{n+1}\begin{cases}\frac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|, \\ \\ \frac{1}{(nq+1)^{1/q}}\left(\int_0^1\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|^p ds\right)^{1/p} \\ \text{for } p,q>1,\ \text{where } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_0^1\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|ds.\end{cases}
\end{aligned}
$$

In particular, we have the representation (3.3) and the remainder satisfies the norm inequalities

$$
\begin{aligned}
(4.2)\quad & \|T_n(a,\alpha,\beta)\| \\
&\leq \frac{1}{2n!}\left[\|a-\alpha\|^{n+1}\int_0^1\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\|(1-s)^n\,ds\right. \\
&\qquad\left.+\|\beta-a\|^{n+1}\int_0^1\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|s^n ds\right] \\
&\leq \frac{1}{2n!}\|a-\alpha\|^{n+1}\begin{cases}\frac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\| \\ \\ \frac{1}{(nq+1)^{1/q}}\left(\int_0^1\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\|^p ds\right)^{1/p} \\ \text{for } p,q>1,\ \text{where } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\|ds\end{cases} \\
&\quad+\frac{1}{2n!}\|\beta-a\|^{n+1}\begin{cases}\frac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|, \\ \\ \frac{1}{(nq+1)^{1/q}}\left(\int_0^1\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|^p ds\right)^{1/p} \\ \text{for } p,q>1,\ \text{where } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_0^1\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|ds.\end{cases}
\end{aligned}
$$

Proof. Using the representation (2.6) we have

$$\begin{aligned}
(4.3)\quad & \left\| S_{n,\lambda}(a,\alpha,\beta) \right\| \\
& \leq \frac{1}{n!}\left[|1-\lambda| \left\| (a-\alpha)^{n+1} \int_0^1 f^{(n+1)}[(1-s)\alpha + sa](1-s)^n ds \right\| \right. \\
& \qquad \left. + |\lambda| \left\| (\beta - a)^{n+1} \int_0^1 f^{(n+1)}[(1-s)a + s\beta] s^n ds \right\| \right]
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{1}{n!}\left[|1-\lambda| \left\| (a-\alpha)^{n+1} \right\| \left\| \int_0^1 f^{(n+1)}[(1-s)\alpha + sa](1-s)^n ds \right\| \right. \\
& \qquad \left. + |\lambda| \left\| (\beta - a)^{n+1} \right\| \left\| \int_0^1 f^{(n+1)}[(1-s)a + s\beta] s^n ds \right\| \right] \\
& \leq \frac{1}{n!}\left[|1-\lambda| \left\| a-\alpha \right\|^{n+1} \int_0^1 \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\| (1-s)^n ds \right. \\
& \qquad \left. + |\lambda| \left\| \beta - a \right\|^{n+1} \int_0^1 \left\| f^{(n+1)}[(1-s)a + s\beta] \right\| s^n ds \right] =: A.
\end{aligned}$$

This proves the first inequality in (4.1).

Using Hölder's integral inequality, we have

$$\begin{aligned}
& \int_0^1 \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\| (1-s)^n ds \\
& \leq \begin{cases}
\sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\| \int_0^1 (1-s)^n ds, \\
\left(\int_0^1 \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\|^p ds \right)^{1/p} \left(\int_0^1 (1-s)^{qn} ds \right)^{1/q} \\
\text{for } p,q>1, \text{ where } \frac{1}{p}+\frac{1}{q}=1, \\
\sup_{s\in[0,1]} (1-s)^n \int_0^1 \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\| ds,
\end{cases}
\end{aligned}$$

$$= \begin{cases}
\frac{1}{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\|, \\
\frac{1}{(nq+1)^{1/q}} \left(\int_0^1 \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\|^p ds \right)^{1/p} \\
\text{for } p,q>1, \text{ where } \frac{1}{p}+\frac{1}{q}=1, \\
\int_0^1 \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\| ds.
\end{cases}$$

Similarly,

$$\int_0^1 \left\| f^{(n+1)}\left[(1-s)a+s\beta\right]\right\| s^n ds$$

$$\leq \begin{cases} \frac{1}{n+1}\sup_{s\in[0,1]}\left\| f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|, \\ \\ \frac{1}{(nq+1)^{1/q}}\left(\int_0^1 \left\| f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|^p ds\right)^{1/p} \\ \text{for } p,q>1, \text{ where } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_0^1 \left\| f^{(n+1)}\left[(1-s)a+s\beta\right]\right\| ds. \end{cases}$$

Therefore

$$A \leq \frac{1}{n!}\left|1-\lambda\right|\left\|a-\alpha\right\|^{n+1}\begin{cases} \frac{1}{n+1}\sup_{s\in[0,1]}\left\| f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\|, \\ \\ \frac{1}{(nq+1)^{1/q}}\left(\int_0^1 \left\| f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\|^p ds\right)^{1/p} \\ \text{for } p,q>1, \text{ where } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_0^1 \left\| f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\| ds. \end{cases}$$

$$+\frac{1}{n!}\left|\lambda\right|\left\|\beta-a\right\|^{n+1}\begin{cases} \frac{1}{n+1}\sup_{s\in[0,1]}\left\| f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|, \\ \\ \frac{1}{(nq+1)^{1/q}}\left(\int_0^1 \left\| f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|^p ds\right)^{1/p} \\ \text{for } p,q>1, \text{ where } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_0^1 \left\| f^{(n+1)}\left[(1-s)a+s\beta\right]\right\| ds. \end{cases}$$

By using (4.3) we get the second part of (4.1). □

Remark 3. *In the case $n=0$ we have the representations (3.6) and (3.8) and the remainders $S_\lambda(a,\alpha,\beta)$ and $T(a,\alpha,\beta)$ satisfy the bounds*

$$\left\|S(a,\alpha,\beta)\right\| \leq \left|1-\lambda\right|\left\|a-\alpha\right\|\int_0^1 \left\|f'\left[(1-s)\alpha+sa\right]\right\| ds + \left|\lambda\right|\left\|\beta-a\right\|\int_0^1 \left\|f'\left[(1-s)a+s\beta\right]\right\| ds \tag{4.4}$$

$$\leq \left|1-\lambda\right|\left\|a-\alpha\right\|\begin{cases} \sup_{s\in[0,1]}\left\|f'\left[(1-s)\alpha+sa\right]\right\| \\ \\ \left(\int_0^1 \left\|f'\left[(1-s)\alpha+sa\right]\right\|^p ds\right)^{1/p} \\ \textit{for } p,q>1, \textit{ where } \frac{1}{p}+\frac{1}{q}=1 \end{cases}$$

$$+\left|\lambda\right|\left\|\beta-a\right\|\begin{cases} \sup_{s\in[0,1]}\left\|f'\left[(1-s)a+s\beta\right]\right\|, \\ \\ \left(\int_0^1 \left\|f\left[(1-s)a+s\beta\right]\right\|^p ds\right)^{1/p} \\ \textit{for } p,q>1, \textit{ where } \frac{1}{p}+\frac{1}{q}=1 \end{cases}$$

and

$$
\begin{aligned}
(4.5)\qquad \|T(a,\alpha,\beta)\| &\le \frac{1}{2}\left[\|a-\alpha\|\int_0^1 \|f[(1-s)\alpha+sa]\|\,ds\right.\\
&\qquad\left.+\|\beta-a\|\int_0^1 \|f'[(1-s)a+s\beta]\|\,ds\right]\\
&\le \frac{1}{2}\|a-\alpha\|\begin{cases}\sup_{s\in[0,1]}\|f'[(1-s)\alpha+sa]\|\\[2ex] \left(\int_0^1\|f'[(1-s)\alpha+sa]\|^p\,ds\right)^{1/p}\\ \text{for } p,q>1,\ \text{where } \frac{1}{p}+\frac{1}{q}=1\end{cases}\\
&\qquad+\frac{1}{2}\|\beta-a\|\begin{cases}\sup_{s\in[0,1]}\|f'[(1-s)a+s\beta]\|,\\[2ex] \left(\int_0^1\|f'[(1-s)a+s\beta]\|^p\,ds\right)^{1/p}\\ \text{for } p,q>1,\ \text{where } \frac{1}{p}+\frac{1}{q}=1.\end{cases}
\end{aligned}
$$

Corollary 2. *With the assumptions of Theorem 4 we have*

$$
\begin{aligned}
(4.6)\qquad \|S_{n,\lambda}(a,\alpha,\beta)\| &\le \frac{1}{(n+1)!}\left[|1-\lambda|\,\|a-\alpha\|^{n+1}+|\lambda|\,\|\beta-a\|^{n+1}\right]\\
&\times\max\left\{\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|,\ \sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)a+s\beta]\right\|\right\}
\end{aligned}
$$

and, in particular,

$$
\begin{aligned}
(4.7)\qquad \|T_n(a,\alpha,\beta)\| &\le \frac{1}{2(n+1)!}\left[\|a-\alpha\|^{n+1}+\|\beta-a\|^{n+1}\right]\\
&\times\max\left\{\sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)\alpha+sa]\right\|,\ \sup_{s\in[0,1]}\left\|f^{(n+1)}[(1-s)a+s\beta]\right\|\right\}.
\end{aligned}
$$

We have the following

Theorem 5. *Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a)\subset G$ and α, $\beta\in D$ with $\alpha\neq\beta$. If $f:G\to\mathbb{C}$ is analytic on G, then for $n\ge 1$ we have the representations (3.10) and (3.11) and the remainders $L_n(a,\alpha,\beta)$*

and $P_n(a,\alpha,\beta)$ satisfy the norm inequalities

$$
\begin{aligned}
(4.8)\quad & \left\| L_n(a,\alpha,\beta)\right\| \\
&\leq \frac{1}{n!\left|\beta-\alpha\right|}\left\|(\beta-a)(a-\alpha)\right\|\left[\left\|a-\alpha\right\|^n\int_0^1\left\|f^{(n+1)}\left((1-s)\alpha+sa\right)\right\|(1-s)^n ds\right. \\
&\qquad\left.+\left\|\beta-a\right\|^n\int_0^1\left\|f^{(n+1)}\left((1-s)a+s\beta\right)\right\|s^n ds\right] \\
&\leq \frac{1}{n!\left|\beta-\alpha\right|}\left\|(\beta-a)(a-\alpha)\right\| \\
&\times\left[\left\|a-\alpha\right\|^n\begin{cases}\frac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\| \\ \\ \frac{1}{(nq+1)^{1/q}}\left(\int_0^1\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\|^p ds\right)^{1/p} \\ \text{for } p,q>1,\ \text{where } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\| ds\end{cases}\right. \\
&\qquad\left.+\left\|\beta-a\right\|^n\begin{cases}\frac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|, \\ \\ \frac{1}{(nq+1)^{1/q}}\left(\int_0^1\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|^p ds\right)^{1/p} \\ \text{for } p,q>1,\ \text{where } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_0^1\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\| ds.\end{cases}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(4.9)\quad & \left\| P_n(a,\alpha,\beta)\right\| \\
&\leq \frac{1}{n!\left|\beta-\alpha\right|}\left[\left\|a-\alpha\right\|^{n+2}\int_0^1\left\|f^{(n+1)}\left((1-s)\alpha+sa\right)\right\|(1-s)^n ds\right. \\
&\qquad\left.+\left\|\beta-a\right\|^{n+2}\int_0^1\left\|f^{(n+1)}\left((1-s)a+s\beta\right)\right\|s^n ds\right] \\
&\leq \frac{1}{n!\left|\beta-\alpha\right|}\left[\left\|a-\alpha\right\|^{n+2}\begin{cases}\frac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\|, \\ \\ \frac{1}{(nq+1)^{1/q}}\left(\int_0^1\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\|^p ds\right)^{1/p} \\ \text{for } p,q>1,\ \text{where } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_0^1\left\|f^{(n+1)}\left[(1-s)\alpha+sa\right]\right\| ds\end{cases}\right. \\
&\qquad\left.+\left\|\beta-a\right\|^{n+2}\begin{cases}\frac{1}{n+1}\sup_{s\in[0,1]}\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|, \\ \\ \frac{1}{(nq+1)^{1/q}}\left(\int_0^1\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\|^p ds\right)^{1/p} \\ \text{for } p,q>1,\ \text{where } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_0^1\left\|f^{(n+1)}\left[(1-s)a+s\beta\right]\right\| ds.\end{cases}\right]
\end{aligned}
$$

Proof. From Theorem 3 we have

$$
\begin{aligned}
&\|L_n(a,\alpha,\beta)\| \\
&\leq \frac{1}{n!\,|\beta-\alpha|}\|(\beta-a)(a-\alpha)\|\left[\left\|(a-\alpha)^n\int_0^1 f^{(n+1)}((1-s)\alpha+sa)(1-s)^n\,ds\right\|\right. \\
&\qquad\left.+\left\|(\beta-a)^n\int_0^1 f^{(n+1)}((1-s)a+s\beta)s^n ds\right\|\right] \\
&\leq \frac{1}{n!\,|\beta-\alpha|}\|(\beta-a)(a-\alpha)\|\left[\|(a-\alpha)^n\|\left\|\int_0^1 f^{(n+1)}((1-s)\alpha+sa)(1-s)^n\,ds\right\|\right. \\
&\qquad\left.+\|(\beta-a)^n\|\left\|\int_0^1 f^{(n+1)}((1-s)a+s\beta)s^n ds\right\|\right] \\
&\leq \frac{1}{n!\,|\beta-\alpha|}\|(\beta-a)(a-\alpha)\|\left[\|a-\alpha\|^n\int_0^1\left\|f^{(n+1)}((1-s)\alpha+sa)\right\|(1-s)^n\,ds\right. \\
&\qquad\left.+\|\beta-a\|^n\int_0^1\left\|f^{(n+1)}((1-s)a+s\beta)\right\|s^n ds\right] =: B,
\end{aligned}
$$

which proves the first inequality in (4.8). The second part follows by Hölder's integral inequality.

The inequality (4.9) can be proved in a similar way. □

Remark 4. *In the case $n=0$ we get*

$$
f(a)=\frac{1}{\beta-\alpha}\left[f(\alpha)(\beta-a)+f(\beta)(a-\alpha)\right]+L(a,\alpha,\beta), \tag{4.10}
$$

where

$$
\begin{aligned}
&\|L(a,\alpha,\beta)\| \\
&\leq \frac{1}{|\beta-\alpha|}\|(\beta-a)(a-\alpha)\|\int_0^1\|f'((1-s)\alpha+sa)\,ds-f'(sa+(1-s)\beta)\|\,ds
\end{aligned} \tag{4.11}
$$

and

$$
f(a)=\frac{1}{\beta-\alpha}\left[f(\alpha)(a-\alpha)+f(\beta)(\beta-a)\right]+P(a,\alpha,\beta), \tag{4.12}
$$

where

$$
\begin{aligned}
&\|P(a,\alpha,\beta)\|\leq\frac{1}{|\beta-\alpha|} \\
&\times\left[\|a-\alpha\|^2\int_0^1\|f'((1-s)\alpha+sa)\|\,ds+\|\beta-a\|^2\int_0^1\|f'((1-s)a+s\beta)\|\,ds\right].
\end{aligned} \tag{4.13}
$$

Moreover, if there exist $L_a>0$ such that

$$
\|f'((1-s)\alpha+sa)\,ds-f'(sa+(1-s)\beta)\|\leq(1-\alpha)L_a|\alpha-\beta|
$$

for all $s\in[0,1]$, then by (4.11) we get

$$
\|L(a,\alpha,\beta)\|\leq\frac{1}{2}L_a\|(\beta-a)(a-\alpha)\|. \tag{4.14}
$$

5. Examples for Exponential Function

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$. Consider the exponential function $f(z) = \exp(z)$, $z \in \mathbb{C}$ and put

$$E_{a,z} := \sup_{s\in[0,1]} \|\exp[(1-s)a + sz]\| < \infty,\ n \geq 0.$$

Observe that

$$\exp((1-t)\lambda + ta) = \exp[(1-t)\lambda]\exp(ta),$$

which gives

$$\begin{aligned}
&\|\exp((1-t)\lambda + ta)\| \\
&\quad = |\exp[(1-t)\lambda]| \, \|\exp(ta)\| = \exp[(1-t)\operatorname{Re}\lambda] \, \|\exp(ta)\| \\
&\quad \leq \exp[(1-t)\operatorname{Re}\lambda]\exp(t\|a\|) = \exp[(1-t)\operatorname{Re}\lambda + t\|a\|] \\
&\qquad\qquad \leq \exp(\max\{\operatorname{Re}\lambda, \|a\|\})
\end{aligned}$$

for any $t \in [0,1]$, $\lambda \in \mathbb{C}$.

Therefore

$$E_{a,z} \leq \exp(\max\{\operatorname{Re} z, \|a\|\}).$$

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$, G be a convex domain of $\mathbb{C}$ with $\sigma(a) \subset G$ and α, $\beta \in D$. If $f : G \to \mathbb{C}$ is analytic on G, then by the inequality (4.1) we get

$$\begin{aligned}
\|T_n(a,\alpha,\beta)\| \leq{}& \frac{1}{2(n+1)!}\|a-\alpha\|^{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)\alpha + sa] \right\| \\
&+ \frac{1}{2(n+1)!}\|\beta - a\|^{n+1} \sup_{s\in[0,1]} \left\| f^{(n+1)}[(1-s)a + s\beta] \right\|.
\end{aligned} \tag{5.1}$$

If we apply the inequality (5.1) for the exponential function, then we get the following norm inequality

$$\begin{aligned}
&\left\| \exp a - \frac{\exp\alpha + \exp\beta}{2} \right. \\
&\qquad \left. - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k!}\left[\exp(\alpha)(a-\alpha)^k + (-1)^k \exp(\beta)(\beta - a)^k\right] \right\| \\
&\qquad\qquad \leq \frac{1}{2(n+1)!} \\
&\times \left[\|a-\alpha\|^{n+1}\exp(\max\{\operatorname{Re}\alpha, \|a\|\}) + \|\beta - a\|^{n+1}\exp(\max\{\operatorname{Re}\beta, \|a\|\})\right] \\
&\quad \leq \frac{1}{2(n+1)!}\left[\|a-\alpha\|^{n+1} + \|\beta - a\|^{n+1}\right]\exp(\max\{\operatorname{Re}\alpha, \operatorname{Re}\beta, \|a\|\}).
\end{aligned} \tag{5.2}$$

Using the inequality (4.8) we have for $\alpha \neq \beta$ that

$$\begin{aligned}
(5.3)\qquad \left\| L_n(a,\alpha,\beta) \right\| \leq & \frac{1}{(n+1)!\left|\beta-\alpha\right|} \left\| (\beta-a)(a-\alpha) \right\| \\
& \times \left[\left\| a-\alpha \right\|^n \sup_{s\in[0,1]} \left\| f^{(n+1)}\left[(1-s)\alpha+sa\right] \right\| \right. \\
& \left. + \left\| \beta-a \right\|^n \sup_{s\in[0,1]} \left\| f^{(n+1)}\left[(1-s)a+s\beta\right] \right\| \right].
\end{aligned}$$

If we apply the inequality (5.3) for the exponential function, we get

$$\begin{aligned}
(5.4)\qquad & \left\| \exp a - \frac{1}{\beta-\alpha}\left[\exp(\alpha)(\beta-a)+\exp(\beta)(a-\alpha)\right] \right. \\
& \left. - \frac{(\beta-a)(a-\alpha)}{\beta-\alpha}\sum_{k=1}^{n}\frac{1}{k!}\left\{\exp(\alpha)(a-\alpha)^{k-1}+(-1)^k\exp(\beta)(\beta-a)^{k-1}\right\} \right\| \\
& \leq \frac{1}{(n+1)!\left|\beta-\alpha\right|}\left\|(\beta-a)(a-\alpha)\right\| \\
& \times \left[\left\|a-\alpha\right\|^n\exp\left(\max\left\{\operatorname{Re}\alpha,\left\|a\right\|\right\}\right)+\left\|\beta-a\right\|^n\exp\left(\max\left\{\operatorname{Re}\beta,\left\|a\right\|\right\}\right)\right] \\
& \leq \frac{1}{(n+1)!\left|\beta-\alpha\right|}\left\|(\beta-a)(a-\alpha)\right\| \\
& \times \left[\left\|a-\alpha\right\|^n+\left\|\beta-a\right\|^n\right]\exp\left(\max\left\{\operatorname{Re}\alpha,\operatorname{Re}\beta,\left\|a\right\|\right\}\right).
\end{aligned}$$

Using the inequality (4.9) we get

$$\begin{aligned}
(5.5)\qquad \left\| P_n(a,\alpha,\beta) \right\| \leq & \frac{1}{(n+1)!\left|\beta-\alpha\right|} \\
& \times \left[\left\| a-\alpha \right\|^{n+2} \sup_{s\in[0,1]} \left\| f^{(n+1)}\left[(1-s)\alpha+sa\right] \right\| \right. \\
& \left. + \left\| \beta-a \right\|^{n+2} \sup_{s\in[0,1]} \left\| f^{(n+1)}\left[(1-s)a+s\beta\right] \right\| \right]
\end{aligned}$$

for $\alpha \neq \beta$.

By writing this inequality (5.5) for the exponential function, we obtain

$$\begin{aligned}
(5.6)\qquad & \left\| \exp(a) - \frac{1}{\beta-\alpha}\sum_{k=0}^{n}\frac{1}{k!}\left\{\exp(\alpha)(a-\alpha)^{k+1}+(-1)^k\exp(\beta)(\beta-a)^{k+1}\right\} \right\| \\
& \leq \frac{1}{(n+1)!\left|\beta-\alpha\right|} \\
& \times \left[\left\|a-\alpha\right\|^{n+2}\exp\left(\max\left\{\operatorname{Re}\alpha,\left\|a\right\|\right\}\right)+\left\|\beta-a\right\|^{n+2}\exp\left(\max\left\{\operatorname{Re}\beta,\left\|a\right\|\right\}\right)\right] \\
& \leq \frac{1}{(n+1)!\left|\beta-\alpha\right|}\left[\left\|a-\alpha\right\|^{n+2}+\left\|\beta-a\right\|^{n+2}\right]\exp\left(\max\left\{\operatorname{Re}\alpha,\operatorname{Re}\beta,\left\|a\right\|\right\}\right).
\end{aligned}$$

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