SEVERAL APPLICATIONS FOR A LOCAL YOUNG-TYPE INEQUALITY

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Abstract. In this paper we will obtain a local Young-type inequality for three positive variables and then several applications for isotonic linear functional and, for selfadjoint operators in Hilbert spaces will be given.

1. Introduction

The classical inequality of Young is

$$a^{\nu}b^{1-\nu} < \nu a + (1-\nu)b,$$

where a and b are distinct positive real numbers and $0 < \nu < 1$, see [16].

In the paper [1] are proven new inequalities which extend many generalizations of Young 's inequality given in recent years. Many generalizations and refinements of Young's inequality are stated also in [6], [5], [7], [10] and references therein.

Theorem 1. *([1]) Let λ, ν and τ be real numbers with $\lambda \geq 1$ and $0 < \nu < \tau < 1$. Then*

$$\left(\frac{\nu}{\tau}\right)^{\lambda} < \frac{A_{\nu}(a,b)^{\lambda} - G_{\nu}(a,b)^{\lambda}}{A_{\tau}(a,b)^{\lambda} - G_{\tau}(a,b)^{\lambda}} < \left(\frac{1-\nu}{1-\tau}\right)^{\lambda},$$

for all positive and distinct real numbers a and b. Moreover, both bounds are sharp.

We suppose that $a,\ b,\ c > 0$ are three distinct numbers and $p_1,\ p_2,\ p_3 > 0$, $p_1^{'},\ p_2^{'},\ p_3^{'} > 0$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $\frac{1}{p_1^{'}} + \frac{1}{p_2^{'}} + \frac{1}{p_3^{'}} = 1$. We take into account the three variables function

$$f(a,b,c) = \frac{1}{p_1}a^{p_1} + \frac{1}{p_2}b^{p_2} + \frac{1}{p_3}c^{p_3} - abc - \frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}a^{p_1} + \frac{1}{p_2^{'}}b^{p_2} + \frac{1}{p_3^{'}}c^{p_3} - a^{\frac{p_1}{p_1^{'}}}b^{\frac{p_2}{p_2^{'}}}c^{\frac{p_3}{p_3^{'}}}\right)$$

which have the stationary points $A(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c)$ with $c > 0$, $c \neq 1$.

Theorem 2. *([4]) The local extreme points of the above function are $A(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c)$. If the following conditions are satisfied*

$$\frac{p_1}{p_1^{'}} - 1 \geq \max\{\frac{1}{p_2}|\frac{p_1}{p_1^{'}} - \frac{p_2}{p_2^{'}}|, \frac{1}{p_3}|\frac{p_1}{p_1^{'}} - \frac{p_3}{p_3^{'}}|\},$$

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$$1 \geq \frac{1}{p_2} \frac{\frac{p_1}{p_1^{'}} - \frac{p_2^2}{(p_2^{'})^2}}{\frac{p_1}{p_1^{'}} - \frac{p_2}{p_2^{'}}} + \frac{1}{p_3} \frac{\frac{p_1}{p_1^{'}} - \frac{p_3^2}{(p_3^{'})^2}}{\frac{p_1}{p_1^{'}} - \frac{p_3}{p_3^{'}}} + \frac{p_1}{p_1^{'}} \frac{1}{p_2 p_3} \frac{\left(\frac{p_2}{p_2^{'}} - \frac{p_3}{p_3^{'}}\right)^2}{\left(\frac{p_1}{p_1^{'}} - \frac{p_2}{p_2^{'}}\right)\left(\frac{p_1}{p_1^{'}} - \frac{p_3}{p_3^{'}}\right)}.$$

then these points are local minimum points for the function f.

We also need to recall, for the first section, the definition of isotonic linear functional. This definition can be also find in papers as [2], [3], [8], [12] and the references therein.

Definition 1. *Let E be a nonempty set and L be a linear class of real-valued functions $f, g : E \to \mathbb{R}$ having the following properties:*

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha,\ \beta \in \mathbb{R}$.

(L2) $1 \in L$, i.e., if $f_0(t) = 1,\ \forall\ t \in E$, then $f_0 \in L$.

An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having the following properties:

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $\alpha,\ \beta \in \mathbb{R}$;

(A2) If $f \in L$ and $f(t) \geq 0$ then $A(f) \geq 0$.

The mapping A is said to be normalised if

(A3) $A(\mathbf{1}) = 1$.

For several classical examples of isotonic linear functionals, see [3], Example 3.3.

Moreover, for the second section it is necessary to recall some basic things about the functional calculus with continuous functions on spectrum. As in [11], we recall that for selfadjoint operators $A, B \in B(\mathcal{H})$ we write $A \leq B$ (or $B \geq A$) if $< Ax, x > \leq < Bx, x >$ for every vector $x \in \mathcal{H}$. We will consider for beginning A as being a selfadjoint linear operator on a complex Hilbert space $(\mathcal{H}; < ., . >)$. The *Gelfand map* establishes a $*$- isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*- algebra $C^*(A)$ generated by A and the identity operator $1_{\mathcal{H}}$ on $\mathcal{H}$ as follows: For any $f, f \in C(Sp(A))$ and for any $\alpha, \beta \in \mathbf{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f) = \Phi(f^*)$;

(iii) $||\Phi(f)|| = ||f|| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A.)$

Using this notation, as in [11] for example, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A. It is known that if A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a *positive operator* on $\mathcal{H}$. In addition, if and f and g are real valued functions on $Sp(A)$ then the following property holds:

$$f(t) \geq g(t) \quad \text{for any} \quad t \in Sp(A) \quad \text{implies that} \quad f(A) \geq g(A) \tag{1}$$

in the operator order of $B(\mathcal{H})$.

2. **A local Young-type inequality and some applications for isotonic linear functionals**

The following result is an immediate consequence of previous Theorem 2, by using the definition of the local minimum points.

Proposition 1. *For any* $p_1,\ p_2,\ p_3>0,\ p_1^{'},\ p_2^{'},\ p_3^{'}>0$ *with* $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$ *and* $\frac{1}{p_1^{'}}+\frac{1}{p_2^{'}}+\frac{1}{p_3^{'}}=1$ *which satisfy the conditions*

$$\frac{p_1}{p_1^{'}}-1\geq \max\{\frac{1}{p_2}|\frac{p_1}{p_1^{'}}-\frac{p_2}{p_2^{'}}|,\frac{1}{p_3}|\frac{p_1}{p_1^{'}}-\frac{p_3}{p_3^{'}}|\},$$

$$1\geq \frac{1}{p_2}\frac{\frac{p_1}{p_1^{'}}-\frac{p_2^2}{(p_2^{'})^2}}{\frac{p_1}{p_1^{'}}-\frac{p_2}{p_2^{'}}}+\frac{1}{p_3}\frac{\frac{p_1}{p_1^{'}}-\frac{p_3^2}{(p_3^{'})^2}}{\frac{p_1}{p_1^{'}}-\frac{p_3}{p_3^{'}}}+\frac{p_1}{p_1^{'}}\frac{1}{p_2p_3}\frac{\left(\frac{p_2}{p_2^{'}}-\frac{p_3}{p_3^{'}}\right)^2}{\left(\frac{p_1}{p_1^{'}}-\frac{p_2}{p_2^{'}}\right)\left(\frac{p_1}{p_1^{'}}-\frac{p_3}{p_3^{'}}\right)},$$

and for any $d>0$, *there is* $r_d>0$ *so that for any* $c\in(d-r_d,d+r_d)$, $b\in(d^{\frac{p_3}{p_2}}-r_d,d^{\frac{p_3}{p_2}}+r_d)$ *and* $a\in(d^{\frac{p_3}{p_1}}-r_d,d^{\frac{p_3}{p_1}}+r_d)$ *it is true the inequality:*

$$\frac{1}{p_1}a^{p_1}+\frac{1}{p_2}b^{p_2}+\frac{1}{p_3}c^{p_3}-abc\geq \frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}a^{p_1}+\frac{1}{p_2^{'}}b^{p_2}+\frac{1}{p_3^{'}}c^{p_3}-a^{\frac{p_1}{p_1^{'}}}b^{\frac{p_2}{p_2^{'}}}c^{\frac{p_3}{p_3^{'}}}\right).$$

Now we will use this inequality in order to establish several Young-type inequalities for normalised isotonic linear functionals.

Theorem 3. *Let* $p_1,\ p_2,\ p_3>0,\ p_1^{'},\ p_2^{'},\ p_3^{'}>0$ *with* $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$ *and* $\frac{1}{p_1^{'}}+\frac{1}{p_2^{'}}+\frac{1}{p_3^{'}}=1$ *which satisfy the conditions*

$$\frac{p_1}{p_1^{'}}-1\geq \max\{\frac{1}{p_2}|\frac{p_1}{p_1^{'}}-\frac{p_2}{p_2^{'}}|,\frac{1}{p_3}|\frac{p_1}{p_1^{'}}-\frac{p_3}{p_3^{'}}|\},$$

$$1\geq \frac{1}{p_2}\frac{\frac{p_1}{p_1^{'}}-\frac{p_2^2}{(p_2^{'})^2}}{\frac{p_1}{p_1^{'}}-\frac{p_2}{p_2^{'}}}+\frac{1}{p_3}\frac{\frac{p_1}{p_1^{'}}-\frac{p_3^2}{(p_3^{'})^2}}{\frac{p_1}{p_1^{'}}-\frac{p_3}{p_3^{'}}}+\frac{p_1}{p_1^{'}}\frac{1}{p_2p_3}\frac{\left(\frac{p_2}{p_2^{'}}-\frac{p_3}{p_3^{'}}\right)^2}{\left(\frac{p_1}{p_1^{'}}-\frac{p_2}{p_2^{'}}\right)\left(\frac{p_1}{p_1^{'}}-\frac{p_3}{p_3^{'}}\right)},$$

and let $A:L\to\mathbb{R}$ *be a normalised isotonic linear functional.*

If $f,g,h:E\to\mathbb{R}$, $f,g,h>0$, $f^{p_1},g^{p_2},h^{p_3},fgh,f^{\frac{p_1}{p_1^{'}}}g^{\frac{p_2}{p_2^{'}}}h^{\frac{p_3}{p_3^{'}}}\in L$ *and* $A(f^{p_1}),A(g^{p_2}),A(h^{p_3})>0$ *then for any* $d>0$ *there is* $r_d>0$ *so that*

(a) If, in addition, $d-r_d\leq h\leq d+r_d$, $d^{\frac{p_3}{p_2}}-r_d\leq g\leq d^{\frac{p_3}{p_2}}+r_d$, *and* $d^{\frac{p_3}{p_1}}-r_d\leq f\leq d^{\frac{p_3}{p_1}}+r_d$ *it is true the following inequality:*

$$\frac{1}{p_1}A(f^{p_1})+\frac{1}{p_2}A(g^{p_2})+\frac{1}{p_3}A(h^{p_3})-A(fgh)\geq$$
$$\geq \frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}A(f^{p_1})+\frac{1}{p_2^{'}}A(g^{p_2})+\frac{1}{p_3^{'}}A(h^{p_3})-A(f^{\frac{p_1}{p_1^{'}}}g^{\frac{p_2}{p_2^{'}}}h^{\frac{p_3}{p_3^{'}}})\right).$$

(b) If, in addition, $(d-r_d)^{p_3}A(h)\leq h\leq A(h)(d+r_d)^{p_3}$, $(d^{\frac{p_3}{p_2}}-r_d)^{p_2}A(g)\leq g\leq A(g)(d^{\frac{p_3}{p_2}}+r_d)^{p_2}$, *and* $(d^{\frac{p_3}{p_1}}-r_d)^{p_1}A(f)\leq f\leq A(f)(d^{\frac{p_3}{p_1}}+r_d)^{p_1}$ *it is true the following inequality:*

$$1-\frac{A(f^{\frac{1}{p_1}}g^{\frac{1}{p_2}}h^{\frac{1}{p_3}})}{A^{\frac{1}{p_1}}(f)A^{\frac{1}{p_2}}(g)A^{\frac{1}{p_3}}(h)}\geq\frac{p_1^{'}}{p_1}\left(1-\frac{A(f^{\frac{1}{p_1^{'}}}g^{\frac{1}{p_2^{'}}}h^{\frac{1}{p_3^{'}}})}{A^{\frac{1}{p_1^{'}}}(f)A^{\frac{1}{p_2^{'}}}(g)A^{\frac{1}{p_3^{'}}}(h)}\right).$$

Proof. (a) We put in inequality from Proposition 1, $a=f$, $b=g$ and $c=h$ because the hypothesis of this proposition are satisfied ($d-r_d\leq h\leq d+r_d$, $d^{\frac{p_3}{p_2}}-r_d\leq g\leq d^{\frac{p_3}{p_2}}+r_d$, $d^{\frac{p_3}{p_1}}-r_d\leq f\leq d^{\frac{p_3}{p_1}}+r_d$). We will have,

$$\frac{1}{p_1}f^{p_1}+\frac{1}{p_2}g^{p_2}+\frac{1}{p_3}h^{p_3}-fgh\geq\frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}f^{p_1}+\frac{1}{p_2^{'}}g^{p_2}+\frac{1}{p_3^{'}}h^{p_3}-f^{\frac{p_1}{p_1^{'}}}g^{\frac{p_2}{p_2^{'}}}h^{\frac{p_3}{p_3^{'}}}\right).$$

Now taking into account that A is a normalised isotonic linear functional, previous inequality can be written:

$$\frac{1}{p_1}A(f^{p_1})+\frac{1}{p_2}A(g^{p_2})+\frac{1}{p_3}A(h^{p_3})-A(fgh)\geq$$

$$\geq\frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}A(f^{p_1})+\frac{1}{p_2^{'}}A(g^{p_2})+\frac{1}{p_3^{'}}A(h^{p_3})-A(f^{\frac{p_1}{p_1^{'}}}g^{\frac{p_2}{p_2^{'}}}h^{\frac{p_3}{p_3^{'}}})\right).$$

(b) This time we choose $a=\frac{f^{\frac{1}{p_1}}}{A^{\frac{1}{p_1}}(f)}$, $b=\frac{g^{\frac{1}{p_2}}}{A^{\frac{1}{p_2}}(g)}$ and $c=\frac{h^{\frac{1}{p_3}}}{A^{\frac{1}{p_3}}(h)}$ in inequality from Proposition 1 and we get:

$$\frac{1}{p_1}\frac{f}{A(f)}+\frac{1}{p_2}\frac{g}{A(g)}+\frac{1}{p_3}\frac{h}{A(h)}-\frac{f^{\frac{1}{p_1}}g^{\frac{1}{p_2}}h^{\frac{1}{p_3}}}{A^{\frac{1}{p_1}}(f)A^{\frac{1}{p_2}}(g)A^{\frac{1}{p_3}}(h)}\geq$$

$$\geq\frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}\frac{f}{A(f)}+\frac{1}{p_2^{'}}\frac{g}{A(g)}+\frac{1}{p_3^{'}}\frac{h}{A(h)}-\frac{f^{\frac{1}{p_1^{'}}}g^{\frac{1}{p_2^{'}}}h^{\frac{1}{p_3^{'}}}}{A^{\frac{1}{p_1^{'}}}(f)A^{\frac{1}{p_2^{'}}}(g)A^{\frac{1}{p_3^{'}}}(h)}\right).$$

Using again the normalised isotonic linear functional A, we have,

$$\frac{1}{p_1}\frac{A(f)}{A(f)}+\frac{1}{p_2}\frac{A(g)}{A(g)}+\frac{1}{p_3}\frac{A(h)}{A(h)}-\frac{A(f^{\frac{1}{p_1}}g^{\frac{1}{p_2}}h^{\frac{1}{p_3}})}{A^{\frac{1}{p_1}}(f)A^{\frac{1}{p_2}}(g)A^{\frac{1}{p_3}}(h)}\geq$$

$$\geq\frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}\frac{A(f)}{A(f)}+\frac{1}{p_2^{'}}\frac{A(g)}{A(g)}+\frac{1}{p_3^{'}}\frac{A(h)}{A(h)}-\frac{A(f^{\frac{1}{p_1^{'}}}g^{\frac{1}{p_2^{'}}}h^{\frac{1}{p_3^{'}}})}{A^{\frac{1}{p_1^{'}}}(f)A^{\frac{1}{p_2^{'}}}(g)A^{\frac{1}{p_3^{'}}}(h)}\right),$$

and by hypothesis that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$ and $\frac{1}{p_1^{'}}+\frac{1}{p_2^{'}}+\frac{1}{p_3^{'}}=1$ we get

$$1-\frac{A(f^{\frac{1}{p_1}}g^{\frac{1}{p_2}}h^{\frac{1}{p_3}})}{A^{\frac{1}{p_1}}(f)A^{\frac{1}{p_2}}(g)A^{\frac{1}{p_3}}(h)}\geq\frac{p_1^{'}}{p_1}\left(1-\frac{A(f^{\frac{1}{p_1^{'}}}g^{\frac{1}{p_2^{'}}}h^{\frac{1}{p_3^{'}}})}{A^{\frac{1}{p_1^{'}}}(f)A^{\frac{1}{p_2^{'}}}(g)A^{\frac{1}{p_3^{'}}}(h)}\right),$$

and here is the desired inequality from (b).

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3. Some inequalities for positive operators on Hilbert spaces

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, <.,>)$ and $A, B, C \in \mathcal{B}(\mathcal{H})$ be three positive operators.

Next result will extend the inequality from Proposition 1 for the norm of the positive operators A, B and C.

Theorem 4. *Let $p_1,\ p_2,\ p_3 > 0,\ p_1^{'},\ p_2^{'},\ p_3^{'} > 0$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $\frac{1}{p_1^{'}} + \frac{1}{p_2^{'}} + \frac{1}{p_3^{'}} = 1$ which satisfy the conditions*

$$\frac{p_1}{p_1^{'}} - 1 \geq \max\{\frac{1}{p_2}|\frac{p_1}{p_1^{'}} - \frac{p_2}{p_2^{'}}|, \frac{1}{p_3}|\frac{p_1}{p_1^{'}} - \frac{p_3}{p_3^{'}}|\},$$

$$1 \geq \frac{1}{p_2}\frac{\frac{p_1}{p_1^{'}} - \frac{p_2^2}{(p_2^{'})^2}}{\frac{p_1}{p_1^{'}} - \frac{p_2}{p_2^{'}}} + \frac{1}{p_3}\frac{\frac{p_1}{p_1^{'}} - \frac{p_3^2}{(p_3^{'})^2}}{\frac{p_1}{p_1^{'}} - \frac{p_3}{p_3^{'}}} + \frac{p_1}{p_1^{'}}\frac{1}{p_2 p_3}\frac{\left(\frac{p_2}{p_2^{'}} - \frac{p_3}{p_3^{'}}\right)^2}{\left(\frac{p_1}{p_1^{'}} - \frac{p_2}{p_2^{'}}\right)\left(\frac{p_1}{p_1^{'}} - \frac{p_3}{p_3^{'}}\right)},$$

and let $A,\ B,\ C$ be three positive operators in the complex Hilbert space $\mathcal{H}$. For any $d > 0$ if

$(d - r_d)I \leq C \leq (d + r_d)I,\ (d^{\frac{p_3}{p_2}} - r_d)I \leq B \leq (d^{\frac{p_3}{p_2}} + r_d)I,$ and $(d^{\frac{p_3}{p_1}} - r_d)I \leq A \leq (d^{\frac{p_3}{p_1}} + r_d)I$ then the following inequality holds:

$$\frac{1}{p_1}||A^{\frac{p_1}{2}}x||^2||y||^2||z||^2 + \frac{1}{p_2}||B^{\frac{p_2}{2}}y||^2||x||^2||z||^2 + \frac{1}{p_3}||C^{\frac{p_3}{2}}z||^2||x||^2||y||^2 -$$
$$-||A^{\frac{1}{2}}x||^2||B^{\frac{1}{2}}y||^2||C^{\frac{1}{2}}z||^2 \geq$$
$$\geq \frac{p_1^{'}}{p_1}(\frac{1}{p_1^{'}}||A^{\frac{p_1}{2}}x||^2||y||^2||z||^2 + \frac{1}{p_2^{'}}||B^{\frac{p_2}{2}}y||^2||x||^2||z||^2 + \frac{1}{p_3^{'}}||C^{\frac{p_3}{2}}z||^2||x||^2||y||^2 -$$
$$-||A^{\frac{p_1}{2p_1^{'}}}x||^2||B^{\frac{p_2}{2p_2^{'}}}y||^2||C^{\frac{p_3}{2p_3^{'}}}z||^2),$$

for any $x, y, z \in \mathcal{H}$.

Proof. The method used in our proof will be as in [7]. We will use the inequality from Proposition 1,

$$\frac{1}{p_1}a^{p_1} + \frac{1}{p_2}b^{p_2} + \frac{1}{p_3}c^{p_3} - abc \geq \frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}a^{p_1} + \frac{1}{p_2^{'}}b^{p_2} + \frac{1}{p_3^{'}}c^{p_3} - a^{\frac{p_1}{p_1^{'}}}b^{\frac{p_2}{p_2^{'}}}c^{\frac{p_3}{p_3^{'}}}\right),$$

where $c \in (d - r_d, d + r_d)$, $b \in (d^{\frac{p_3}{p_2}} - r_d, d^{\frac{p_3}{p_2}} + r_d)$ and $a \in (d^{\frac{p_3}{p_1}} - r_d, d^{\frac{p_3}{p_1}} + r_d)$. Using the functional calculus with continuous functions for the operator A, we obtain

$$\frac{1}{p_1} < A^{p_1}x, x > + \frac{1}{p_2}b^{p_2} < x, x > + \frac{1}{p_3}c^{p_3} < x, x > - bc < Ax, x > \geq$$
$$\geq \frac{p_1^{'}}{p_1}(\frac{1}{p_1^{'}} < A^{p_1}x, x > + \frac{1}{p_2^{'}}b^{p_2} < x, x > + \frac{1}{p_3^{'}}c^{p_3} < x, x > - b^{\frac{p_2}{p_2^{'}}}c^{\frac{p_3}{p_3^{'}}} < A^{\frac{p_1}{p_1^{'}}}x, x >),$$

for any $x \in \mathcal{H},\ b \in (d^{\frac{p_3}{p_2}} - r_d, d^{\frac{p_3}{p_2}} + r_d)$, $c \in (d - r_d, d + r_d)$ and $p_1,\ p_2,\ p_3,\ p_1^{'},\ p_2^{'},\ p_3^{'}$ as in hypothesis of the theorem.

Now, we use in last inequality the functional calculus with continous functions for the operator B and we get:

$$\frac{1}{p_1} < A^{p_1}x, x > ||y||^2 + \frac{1}{p_2}||x||^2 < B^{p_2}y, y > + \frac{1}{p_3}c^{p_3}||x||^2||y||^2 - c < Ax, x >< By, y > \geq$$

$$\geq \frac{p_1^{'}}{p_1}(\frac{1}{p_1^{'}} < A^{p_1}x, x > ||y||^2 + \frac{1}{p_2^{'}}||x||^2 < B^{p_2}y, y > + \frac{1}{p_3^{'}}c^{p_3}||x||^2||y||^2 -$$
$$-c^{\frac{p_3}{p_3^{'}}} < A^{\frac{p_1}{p_1^{'}}}x, x >< B^{\frac{p_2}{p_2^{'}}}y, y >),$$

for any $x,\ y \in \mathcal{H}$, $c \in (d - r_d, d + r_d)$ and $p_1,\ p_2,\ p_3,\ p_1^{'},\ p_2^{'},\ p_3^{'}$ as in hypothesis of the theorem.

By functional calculus with continuous functions for the operator C, we will have from last inequality that,

$$\frac{1}{p_1} < A^{p_1}x, x > ||y||^2||z||^2 + \frac{1}{p_2}||x||^2||z||^2 < B^{p_2}y, y > + \frac{1}{p_3} < C^{p_3}z, z > ||x||^2||y||^2 -$$
$$- < Cz, z >< Ax, x >< By, y > \geq$$
$$\geq \frac{p_1^{'}}{p_1}(\frac{1}{p_1^{'}} < A^{p_1}x, x > ||y||^2||z||^2 + \frac{1}{p_2^{'}}||x||^2||z||^2 < B^{p_2}y, y > + \frac{1}{p_3^{'}} < C^{p_3}z, z > ||x||^2||y||^2 -$$
$$- < C^{\frac{p_3}{p_3^{'}}}z, z >< A^{\frac{p_1}{p_1^{'}}}x, x >< B^{\frac{p_2}{p_2^{'}}}y, y >),$$

for any $x,\ y,\ z \in \mathcal{H}$ and $p_1,\ p_2,\ p_3,\ p_1^{'},\ p_2^{'},\ p_3^{'}$ as in hypothesis of the theorem.

From here we get the desired inequality.

∎

Consequence 1. *Under previous conditions, for each $x, y, z \in \mathcal{H}$ with $||x|| = ||y|| = ||z|| = 1$ we have the following inequality:*

$$\frac{1}{p_1}||A^{\frac{p_1}{2}}x||^2 + \frac{1}{p_2}||B^{\frac{p_2}{2}}y||^2 + \frac{1}{p_3}||C^{\frac{p_3}{2}}z||^2 - ||A^{\frac{1}{2}}x||^2||B^{\frac{1}{2}}y||^2||C^{\frac{1}{2}}z||^2 \geq$$
$$\geq \frac{p_1^{'}}{p_1}(\frac{1}{p_1^{'}}||A^{\frac{p_1}{2}}x||^2 + \frac{1}{p_2^{'}}||B^{\frac{p_2}{2}}y||^2 + \frac{1}{p_3^{'}}||C^{\frac{p_3}{2}}z||^2 - ||A^{\frac{p_1}{2p_1^{'}}}x||^2||B^{\frac{p_2}{2p_2^{'}}}y||^2||C^{\frac{p_3}{2p_3^{'}}}z||^2).$$

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