SOME DISCRETE INEQUALITIES FOR CONVEX FUNCTIONS DEFINED ON LINEAR SPACES

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Abstract. In this paper we provide some discrete inequalities related to the Hermite-Hadamard result for convex functions defined on convex subsets in a linear space. Applications for norms and univariate real functions with an example for the logarithm, are also given.

1. Introduction

Let X be a real linear space, $a,b\in X$, $a\neq b$ and let $[a,b]:=\{(1-\lambda)a+\lambda b,\ \lambda\in[0,1]\}$ be the *segment* generated by a and b. We consider the function $f:[a,b]\to\mathbb{R}$ and the attached function $g(a,b):[0,1]\to\mathbb{R}$, $g(a,b)(t):=f[(1-t)a+tb]$, $t\in[0,1]$.

It is well known that f is convex on $[a,b]$ iff $g(a,b)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}(a,b)(s)=(\bigtriangledown_{\pm}f[(1-s)a+sb])(b-a)$, $s\in[0,1)$
(ii) $g'_{+}(a,b)(0)=(\bigtriangledown_{+}f(a))(b-a)$
(iii) $g'_{-}(a,b)(1)=(\bigtriangledown_{-}f(b))(b-a)$

where $(\bigtriangledown_{\pm}f(x))(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\begin{aligned}(\bigtriangledown_{+}f(x))(y) &:= \lim_{h\to0+}\left[\frac{f(x+hy)-f(x)}{h}\right],\\ (\bigtriangledown_{-}f(x))(y) &:= \lim_{k\to0-}\left[\frac{f(x+ky)-f(x)}{k}\right],\ \ x,\ y\in X.\end{aligned}$$

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[a,b]\subset X$:

$$f\left(\frac{a+b}{2}\right)\le\int_0^1 f[(1-t)a+tb]\,dt\le\frac{f(a)+f(b)}{2}, \tag{HH}$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $g(a,b):[0,1]\to\mathbb{R}$

$$g(a,b)\left(\frac{1}{2}\right)\le\int_0^1 g(a,b)(t)\,dt\le\frac{g(a,b)(0)+g(a,b)(1)}{2}.$$

For other related results see the monograph on line [4].

We have the following result [2] related to the first Hermite-Hadamard inequality in (HH):

1991 *Mathematics Subject Classification.* 26D15, 46B25.

Key words and phrases. Convex functions, Linear spaces, Jensen's inequality, Hermite-Hadamard inequality, Norm inequalities.

RGMIA Res. Rep. Coll. 22 (2019), Art. 31, 10 pp. Received 21/03/19

Theorem 1. *Let X be a linear space, $a,b\in X$, $a\neq b$ and $f:[a,b]\subset X\to\mathbb{R}$ be a convex function on the segment $[a,b]$. Then for any $s\in(0,1)$ one has the inequality*

$$\begin{aligned}(1.1)\quad &\frac{1}{2}\left[(1-s)^2\left(\triangledown_{+}f\left[(1-s)a+sb\right]\right)(b-a)-s^2\left(\triangledown_{-}f\left[(1-s)a+sb\right]\right)(b-a)\right]\\ &\leq\int_0^1 f\left[(1-t)a+tb\right]dt-f\left[(1-s)a+sb\right]\\ &\leq\frac{1}{2}\left[(1-s)^2\left(\triangledown_{-}f(b)\right)(b-a)-s^2\left(\triangledown_{+}f(a)\right)(b-a)\right].\end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $s=0$ or $s=1$.

If $f:[a,b]\to\mathbb{R}$ is as in Theorem 1 and Gâteaux differentiable in $c:=(1-\lambda)a+\lambda b$, $\lambda\in(0,1)$ along the direction $b-a$, then we have the inequality:

$$(1.2)\qquad \left(\frac{1}{2}-\lambda\right)\left(\triangledown f(c)\right)(b-a)\leq\int_0^1 f\left[(1-t)a+tb\right]dt-f(c).$$

If f is as in Theorem 1, then

$$\begin{aligned}(1.3)\qquad 0&\leq\frac{1}{8}\left[\triangledown_{+}f\left(\frac{a+b}{2}\right)(b-a)-\triangledown_{-}f\left(\frac{a+b}{2}\right)(b-a)\right]\\ &\leq\int_0^1 f\left[(1-t)a+tb\right]dt-f\left(\frac{a+b}{2}\right)\\ &\leq\frac{1}{8}\left[\left(\triangledown_{-}f(b)\right)(b-a)-\left(\triangledown_{+}f(a)\right)(b-a)\right].\end{aligned}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

Also we have the following result [3] related to the second Hermite-Hadamard inequality in (HH):

Theorem 2. *Let X be a linear space, $a,b\in X$, $a\neq b$ and $f:[a,b]\subset X\to\mathbb{R}$ be a convex function on the segment $[a,b]$. Then for any $s\in(0,1)$ one has the inequality*

$$\begin{aligned}(1.4)\quad &\frac{1}{2}\left[(1-s)^2\left(\triangledown_{+}f\left[(1-s)a+sb\right]\right)(b-a)-s^2\left(\triangledown_{-}f\left[(1-s)a+sb\right]\right)(b-a)\right]\\ &\leq(1-s)f(a)+sf(b)-\int_0^1 f\left[(1-t)a+tb\right]dt\\ &\leq\frac{1}{2}\left[(1-s)^2\left(\triangledown_{-}f(b)\right)(b-a)-s^2\left(\triangledown_{+}f(a)\right)(b-a)\right].\end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $s=0$ or $s=1$.

If $f:[a,b]\to\mathbb{R}$ is as in Theorem 2 and Gâteaux differentiable in $c:=(1-\lambda)a+\lambda b$, $\lambda\in(0,1)$ along the direction $b-a$, then we have the inequality:

$$(1.5)\quad \left(\frac{1}{2}-\lambda\right)\left(\triangledown f(c)\right)(b-a)\leq(1-\lambda)f(a)+\lambda f(b)-\int_0^1 f\left[(1-t)a+tb\right]dt.$$

If f is as in Theorem 2, then

$$0 \le \frac{1}{8}\left[\nabla_+ f\left(\frac{a+b}{2}\right)(b-a) - \nabla_- f\left(\frac{a+b}{2}\right)(b-a)\right] \tag{1.6}$$
$$\le \frac{f(a)+f(b)}{2} - \int_0^1 f[(1-t)a+tb]\,dt$$
$$\le \frac{1}{8}\left[(\nabla_- f(b))(b-a) - (\nabla_+ f(a))(b-a)\right].$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

2. The Results

Let $f : C \subset X \to \mathbb{R}$ be a convex function on C. We define the function $F_f : C \times C \to \mathbb{R}$ by

$$F_f(x,y) := \int_0^1 f((1-t)x+ty)\,dt. \tag{2.1}$$

Theorem 3. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on C. Then the function F_f is convex on $C \times C$ and if $x_i,\ y_i \in C$ and $p_i \ge 0$ for $i = 1,...,n$ with $\sum_{i=1}^n p_i = 1$, then we have the inequalities*

$$\sum_{i=1}^n p_i \int_0^1 f((1-t)x_i+ty_i)\,dt \ge \int_0^1 f\left((1-t)\sum_{i=1}^n p_i x_i + t\sum_{i=1}^n p_i y_i\right)dt \tag{2.2}$$
$$\ge f\left(\sum_{i=1}^n p_i\left(\frac{x_i+y_i}{2}\right)\right),$$

$$\sum_{i=1}^n p_i\left(\frac{f(x_i)+f(y_i)}{2}\right) \ge \sum_{i=1}^n p_i \int_0^1 f((1-t)x_i+ty_i)\,dt \tag{2.3}$$
$$\ge \sum_{i=1}^n p_i f\left(\frac{x_i+y_i}{2}\right) \ge f\left(\sum_{i=1}^n p_i\left(\frac{x_i+y_i}{2}\right)\right)$$

and

$$\sum_{i=1}^n p_i\left(\frac{f(x_i)+f(y_i)}{2}\right) \ge \frac{1}{2}\left[f\left(\sum_{i=1}^n p_i x_i\right) + f\left(\sum_{i=1}^n p_i y_i\right)\right] \tag{2.4}$$
$$\ge \int_0^1 f\left((1-t)\sum_{i=1}^n p_i x_i + t\sum_{i=1}^n p_i y_i\right)dt.$$

Proof. Let $(x,y), (u,v) \in C \times C$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned}
F_f(\alpha(x,y) + \beta(u,v)) &= F_f(\alpha x + \beta u, \alpha y + \beta v) \\
&= \int_0^1 f((1-t)(\alpha x + \beta u) + t(\alpha y + \beta v))\,dt \\
&= \int_0^1 f(\alpha[(1-t)x + ty] + \beta[(1-t)u + tv])\,dt \\
&\leq \int_0^1 [\alpha f((1-t)x + ty) + \beta f((1-t)u + tv)]\,dt \\
&= \alpha \int_0^1 f((1-t)x + ty)\,dt + \beta \int_0^1 f((1-t)u + tv)\,dt \\
&= \alpha F_f(x,y) + \beta F_f(u,v),
\end{aligned}$$

which proves the joint convexity of the function F_f.

By Jensen's inequality for the convex function F_f we have

$$\sum_{i=1}^n p_i F_f(x_i, y_i) \geq F_f\left(\sum_{i=1}^n p_i (x_i, y_i)\right) = F_f\left(\sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i\right),$$

which is equivalent to the first inequality in (2.2).

By Hermite-Hadamard inequality (HH) we have

$$\begin{aligned}
\int_0^1 f\left((1-t)\sum_{i=1}^n p_i x_i + t\sum_{i=1}^n p_i y_i\right) dt &\geq f\left(\frac{\sum_{i=1}^n p_i x_i + \sum_{i=1}^n p_i y_i}{2}\right) dt \\
&= f\left(\sum_{i=1}^n p_i \left(\frac{x_i + y_i}{2}\right)\right)
\end{aligned}$$

and the second part of (2.2) is proved.

From (HH) we also have for each $i \in \{1, ..., n\}$ that

$$\frac{f(x_i) + f(y_i)}{2} \geq \int_0^1 f[(1-t)x_i + ty_i]\,dt \geq f\left(\frac{x_i + y_i}{2}\right).$$

If we multiply this inequality by $p_i \geq 0$ and sum over i from 1 to n we get the first and second inequality in (2.3).

The last part in (2.3) follows by Jensen's inequality.

Let $u := \sum_{i=1}^n p_i x_i$ and $v := \sum_{i=1}^n p_i y_i$. By Hermite-Hadamard inequality (HH) we also have

$$\frac{f(u) + f(v)}{2} \geq \int_0^1 f[(1-t)u + tv]\,dt,$$

which produces the second inequality in (2.4).

By Jensen's inequality for f we have

$$\sum_{i=1}^n p_i f(x_i) \geq f\left(\sum_{i=1}^n p_i x_i\right)$$

and

$$\sum_{i=1}^n p_i f(y_i) \geq f\left(\sum_{i=1}^n p_i y_i\right).$$

If we sum these two inequalities and divide by 2 we get the first inequality in (2.4). □

The following result also holds:

Theorem 4. *With the assumptions of Theorem 3 we have*

$$
\begin{aligned}
(2.5)\quad 0 &\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i \left(\nabla_+ f\left(\frac{x_i+y_i}{2}\right)(y_i - x_i)\right)\right. \\
&\qquad \left. - \sum_{i=1}^{n} p_i \left(\nabla_- f\left(\frac{x_i+y_i}{2}\right)(y_i - x_i)\right)\right] \\
&\leq \sum_{i=1}^{n} p_i \int_0^1 f\left[(1-t)\,x_i + t y_i\right] dt - \sum_{i=1}^{n} p_i f\left(\frac{x_i+y_i}{2}\right) \\
&\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i \left(\nabla_- f(y_i)\right)(y_i - x_i) - \sum_{i=1}^{n} p_i \left(\nabla_+ f(x_i)\right)(y_i - x_i)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
(2.6)\quad 0 &\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i \left(\nabla_+ f\left(\frac{x_i+y_i}{2}\right)(y_i - x_i)\right)\right. \\
&\qquad \left. - \sum_{i=1}^{n} p_i \left(\nabla_- f\left(\frac{x_i+y_i}{2}\right)(y_i - x_i)\right)\right] \\
&\leq \sum_{i=1}^{n} p_i \left(\frac{f(x_i) + f(y_i)}{2}\right) - \sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)\,x_i + t y_i\right) dt \\
&\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i \left(\nabla_- f(y_i)\right)(y_i - x_i) - \sum_{i=1}^{n} p_i \left(\nabla_+ f(x_i)\right)(y_i - x_i)\right].
\end{aligned}
$$

We also have

$$
\begin{aligned}
(2.7)\quad 0 &\leq \frac{1}{8}\left[\nabla_+ f\left(\sum_{i=1}^{n} p_i \left(\frac{x_i+y_i}{2}\right)\right)\left(\sum_{i=1}^{n} p_i (y_i - x_i)\right)\right. \\
&\qquad \left. - \nabla_- f\left(\sum_{i=1}^{n} p_i \left(\frac{x_i+y_i}{2}\right)\right)\left(\sum_{i=1}^{n} p_i (y_i - x_i)\right)\right] \\
&\leq \int_0^1 f\left((1-t)\sum_{i=1}^{n} p_i x_i + t\sum_{i=1}^{n} p_i y_i\right) dt - f\left(\sum_{i=1}^{n} p_i \left(\frac{x_i+y_i}{2}\right)\right) \\
&\leq \frac{1}{8}\left[\left(\nabla_- f\left(\sum_{i=1}^{n} p_i y_i\right)\right)\left(\sum_{i=1}^{n} p_i (y_i - x_i)\right)\right. \\
&\qquad \left. - \left(\nabla_+ f\left(\sum_{i=1}^{n} p_i x_i\right)\right)\left(\sum_{i=1}^{n} p_i (y_i - x_i)\right)\right]
\end{aligned}
$$

and

$$0\leq\frac{1}{8}\left[\bigtriangledown_{+}f\left(\sum_{i=1}^{n}p_{i}\left(\frac{x_{i}+y_{i}}{2}\right)\right)\left(\sum_{i=1}^{n}p_{i}\left(y_{i}-x_{i}\right)\right)-\bigtriangledown_{-}f\left(\sum_{i=1}^{n}p_{i}\left(\frac{x_{i}+y_{i}}{2}\right)\right)\left(\sum_{i=1}^{n}p_{i}\left(y_{i}-x_{i}\right)\right)\right]$$
$$\leq\frac{1}{2}\left[f\left(\sum_{i=1}^{n}p_{i}x_{i}\right)+f\left(\sum_{i=1}^{n}p_{i}y_{i}\right)\right]-\int_{0}^{1}f\left((1-t)\sum_{i=1}^{n}p_{i}x_{i}+t\sum_{i=1}^{n}p_{i}y_{i}\right)dt$$
$$\leq\frac{1}{8}\left[\left(\bigtriangledown_{-}f\left(\sum_{i=1}^{n}p_{i}y_{i}\right)\right)\left(\sum_{i=1}^{n}p_{i}\left(y_{i}-x_{i}\right)\right)-\left(\bigtriangledown_{+}f\left(\sum_{i=1}^{n}p_{i}x_{i}\right)\right)\left(\sum_{i=1}^{n}p_{i}\left(y_{i}-x_{i}\right)\right)\right]. \tag{2.8}$$

Proof. From the inequality (1.3) we have for $a=x_i$ and $b=y_i$, where $i\in\{1,...,n\}$ that

$$0\leq\frac{1}{8}\left[\bigtriangledown_{+}f\left(\frac{x_{i}+y_{i}}{2}\right)(y_{i}-x_{i})-\bigtriangledown_{-}f\left(\frac{x_{i}+y_{i}}{2}\right)(y_{i}-x_{i})\right]$$
$$\leq\int_{0}^{1}f\left[(1-t)x_{i}+ty_{i}\right]dt-f\left(\frac{x_{i}+y_{i}}{2}\right)$$
$$\leq\frac{1}{8}\left[\left(\bigtriangledown_{-}f\left(y_{i}\right)\right)(y_{i}-x_{i})-\left(\bigtriangledown_{+}f\left(x_{i}\right)\right)(y_{i}-x_{i})\right],$$

for any $i\in\{1,...,n\}$.

If we multiply this inequality by $p_i\geq 0$ and sum over i from 1 to n, then we get

$$0\leq\frac{1}{8}\sum_{i=1}^{n}p_{i}\left[\bigtriangledown_{+}f\left(\frac{x_{i}+y_{i}}{2}\right)(y_{i}-x_{i})-\bigtriangledown_{-}f\left(\frac{x_{i}+y_{i}}{2}\right)(y_{i}-x_{i})\right]$$
$$\leq\sum_{i=1}^{n}p_{i}\int_{0}^{1}f\left[(1-t)x_{i}+ty_{i}\right]dt-\sum_{i=1}^{n}p_{i}f\left(\frac{x_{i}+y_{i}}{2}\right)$$
$$\leq\frac{1}{8}\sum_{i=1}^{n}p_{i}\left[\left(\bigtriangledown_{-}f\left(y_{i}\right)\right)(y_{i}-x_{i})-\left(\bigtriangledown_{+}f\left(x_{i}\right)\right)(y_{i}-x_{i})\right],$$

which is equivalent to (2.5).

The inequality (2.6) follows in a similar way by employing the inequality (1.6).

The inequalities (2.7) and (2.8) follow by taking $a=\sum_{i=1}^{n}p_ix_i$ and $b=\sum_{i=1}^{n}p_iy_i$ in the inequalities (1.3) and (1.6). □

3. Examples for Norms

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s:=\left(\bigtriangledown_{+}f_0(y)\right)(x)=\lim_{t\to0+}\left[\frac{\|y+tx\|^2-\|y\|^2}{2t}\right]$;

(v) $\langle x,y\rangle_i:=\left(\bigtriangledown_{-}f_0(y)\right)(x)=\lim_{s\to0-}\left[\frac{\|y+sx\|^2-\|y\|^2}{2s}\right]$;

for any x, $y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [1] or [5]), assuming that p, $q \in \{s, i\}$ and $p \neq q$:

(a) $\langle x, x\rangle_p = \|x\|^2$ for all $x \in X$;
(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x, y\rangle_p$ if α, $\beta \geq 0$ and x, $y \in X$;
(aaa) $\left|\langle x, y\rangle_p\right| \leq \|x\| \|y\|$ for all x, $y \in X$;
(av) $\langle \alpha x + y, x\rangle_p = \alpha \langle x, x\rangle_p + \langle y, x\rangle_p$ if x, $y \in X$ and $\alpha \in \mathbb{R}$;
(v) $\langle -x, y\rangle_p = -\langle x, y\rangle_q$ for all x, $y \in X$;
(va) $\langle x + y, z\rangle_p \leq \|x\| \|z\| + \langle y, z\rangle_p$ for all x, y, $z \in X$;
(vaa) The mapping $\langle \cdot, \cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);
(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X \setminus \{0\}$ if and only if $\langle y, x_0\rangle_s = \langle y, x_0\rangle_i$ for all $y \in X$; in general $\langle y, x\rangle_i \leq \langle y, x\rangle_s$ for all x, $y \in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot\rangle$, then $\langle y, x\rangle_i = \langle y, x\rangle = \langle y, x\rangle_s$ for all x, $y \in X$.

Applying inequality (HH) for the convex function $f_r(x) = \|x\|^r$, $r \geq 1$ one may deduce the inequality

$$\left\|\frac{x+y}{2}\right\|^r \leq \int_0^1 \|(1-t)x + ty\|^r dt \leq \frac{\|x\|^r + \|y\|^r}{2} \tag{3.1}$$

for any $x, y \in X$.

Let $(X, \|\cdot\|)$ be a normed linear space and $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ be n-tuples of vectors in X, then for the probability distribution $p = (p_1, ..., p_n)$ and $r \geq 1$ we have by Theorem 3 for the convex function $f(x) = \|x\|^r$ that

$$\begin{aligned}\sum_{i=1}^n p_i \int_0^1 \|(1-t)x_i + ty_i\|^r dt &\geq \int_0^1 \left\|(1-t)\sum_{i=1}^n p_i x_i + t\sum_{i=1}^n p_i y_i\right\|^r dt \\ &\geq \left\|\sum_{i=1}^n p_i \left(\frac{x_i + y_i}{2}\right)\right\|^r,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\sum_{i=1}^n p_i \left(\frac{\|x_i\|^r + \|y_i\|^r}{2}\right) &\geq \sum_{i=1}^n \int_0^1 p_i \|(1-t)x_i + ty_i\|^r dt \\ &\geq \sum_{i=1}^n p_i \left\|\frac{x_i + y_i}{2}\right\|^r \geq \left\|\sum_{i=1}^n p_i \left(\frac{x_i + y_i}{2}\right)\right\|^r\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\sum_{i=1}^n p_i \left(\frac{\|x_i\|^r + \|y_i\|^r}{2}\right) &\geq \frac{1}{2}\left[\left\|\sum_{i=1}^n p_i x_i\right\|^r + \left\|\sum_{i=1}^n p_i y_i\right\|^r\right] \\ &\geq \int_0^1 \left\|(1-t)\sum_{i=1}^n p_i x_i + t\sum_{i=1}^n p_i y_i\right\|^r dt.\end{aligned} \tag{3.4}$$

If we use Theorem 4 for the convex function $f(x) = \frac{1}{2}\|x\|^2$ then for $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ n-tuples of vectors in X and for the probability distribution $p = (p_1, ..., p_n)$ we have

$$0 \le \frac{1}{4}\left[\sum_{k=1}^{n} p_k \left\langle y_k - x_k, \frac{x_k + y_k}{2}\right\rangle_s - \sum_{k=1}^{n} p_k \left\langle y_k - x_k, \frac{x_k + y_k}{2}\right\rangle_i\right] \le \sum_{k=1}^{n} p_k \int_0^1 \|(1-t)x_k + ty_k\|^2 dt - \sum_{k=1}^{n} p_k \left\|\frac{x_k + y_k}{2}\right\|^2 \le \frac{1}{4}\left[\sum_{k=1}^{n} p_k \langle y_k - x_k, y_k\rangle_i - \sum_{k=1}^{n} p_k \langle y_k - x_k, x_k\rangle_s\right], \tag{3.5}$$

and

$$0 \le \frac{1}{4}\left[\sum_{k=1}^{n} p_k \left\langle y_k - x_k, \frac{x_k + y_k}{2}\right\rangle_s - \sum_{k=1}^{n} p_k \left\langle y_k - x_k, \frac{x_k + y_k}{2}\right\rangle_i\right] \le \sum_{k=1}^{n} p_k \left(\frac{\|x_k\|^2 + \|y_k\|^2}{2}\right) - \sum_{k=1}^{n} p_k \int_0^1 \|(1-t)x_k + ty_k\|^2 dt \le \frac{1}{4}\left[\sum_{k=1}^{n} p_k \langle y_k - x_k, y_k\rangle_i - \sum_{k=1}^{n} p_k \langle y_k - x_k, x_k\rangle_s\right]. \tag{3.6}$$

We also have

$$0 \le \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_s - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_i\right] \le \int_0^1 \left\|(1-t)\sum_{k=1}^{n} p_k x_k + t\sum_{k=1}^{n} p_k y_k\right\|^2 dt - \left\|\sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\|^2 \le \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k y_k\right\rangle_i - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k x_k\right\rangle_s\right] \tag{3.7}$$

and

$$0 \le \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_s - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_i\right] \le \frac{1}{2}\left[\left\|\sum_{k=1}^{n} p_k x_k\right\|^2 + \left\|\sum_{k=1}^{n} p_k y_k\right\|^2\right] - \int_0^1 \left\|(1-t)\sum_{k=1}^{n} p_k x_k + t\sum_{k=1}^{n} p_k y_k\right\|^2 dt \le \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k y_k\right\rangle_i - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k x_k\right\rangle_s\right]. \tag{3.8}$$

4. Examples for Functions of a Real Variable

If $f : I \to \mathbb{R}$ is convex on the interval I and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then

$$\sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)x_i + ty_i\right) dt \geq \int_0^1 f\left((1-t)\sum_{i=1}^{n} p_i x_i + t\sum_{i=1}^{n} p_i y_i\right) dt \tag{4.1}$$

$$\sum_{i=1}^{n} p_i \left(\frac{f(x_i) + f(y_i)}{2}\right) \geq \sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)x_i + ty_i\right) dt. \tag{4.2}$$

If $f : I \to \mathbb{R}$ is convex and differentiable on the interior of $\mathring{I}$ then for all $x_i \in \mathring{I}$ and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then by (2.5) and (2.6) we get

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i \int_0^1 f\left[(1-t)x_i + ty_i\right] dt - \sum_{i=1}^{n} p_i f\left(\frac{x_i + y_i}{2}\right) \\ &\leq \frac{1}{8}\sum_{i=1}^{n} p_i \left[f'(y_i) - f'(x_i)\right](y_i - x_i), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i \left(\frac{f(x_i) + f(y_i)}{2}\right) - \sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)x_i + ty_i\right) dt \\ &\leq \frac{1}{8}\sum_{i=1}^{n} p_i \left[f'(y_i) - f'(x_i)\right](y_i - x_i). \end{aligned} \tag{4.4}$$

If $f(t) = \frac{1}{t}$ with $t > 0$, then for $y_i \neq x_i$, $i \in \{1, ..., n\}$ we have

$$\int_0^1 f\left((1-t)x_i + ty_i\right) dt = \int_0^1 \frac{1}{(1-t)x_i + ty_i} dt = \frac{\ln y_i - \ln x_i}{y_i - x_i}$$

and

$$\int_0^1 \left((1-t)\sum_{i=1}^{n} p_i x_i + t\sum_{i=1}^{n} p_i y_i\right)^{-1} dt = \frac{\ln\left(\sum_{i=1}^{n} p_i x_i\right) - \ln\left(\sum_{i=1}^{n} p_i y_i\right)}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i},$$

provided $\sum_{i=1}^{n} p_i x_i \neq \sum_{i=1}^{n} p_i y_i$.

From (4.1) we get

$$\sum_{i=1}^{n} p_i \frac{\ln y_i - \ln x_i}{y_i - x_i} \geq \frac{\ln\left(\sum_{i=1}^{n} p_i x_i\right) - \ln\left(\sum_{i=1}^{n} p_i y_i\right)}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i}$$

that is equivalent to

$$\ln\left(\prod_{i=1}^{n} \left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i - x_i}}\right) \geq \ln\left[\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i y_i}\right)^{\frac{1}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i}}\right]$$

and to

$$\prod_{i=1}^{n} \left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i - x_i}} \geq \left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i y_i}\right)^{\frac{1}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i}}. \tag{4.5}$$

From (4.2) we get in a similar way that

$$\exp\left[\sum_{i=1}^{n} p_i \left(\frac{x_i + y_i}{2x_i y_i}\right)\right] \geq \prod_{i=1}^{n} \left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i - x_i}}, \tag{4.6}$$

from (4.3) we get

$$1 \leq \frac{\prod_{i=1}^{n} \left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i - x_i}}}{\exp\left[\sum_{i=1}^{n} p_i \left(\frac{2}{x_i + y_i}\right)\right]} \leq \exp\left(\frac{1}{8}\sum_{i=1}^{n} p_i \frac{(x_i + y_i)(y_i - x_i)^2}{x_i^2 y_i^2}\right) \tag{4.7}$$

and from (4.4) we get

$$1 \leq \frac{\exp\left[\sum_{i=1}^{n} p_i \left(\frac{x_i + y_i}{2x_i y_i}\right)\right]}{\prod_{i=1}^{n} \left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i - x_i}}} \leq \exp\left(\frac{1}{8}\sum_{i=1}^{n} p_i \frac{(x_i + y_i)(y_i - x_i)^2}{x_i^2 y_i^2}\right). \tag{4.8}$$

The interested reader may apply some of the above inequalities for other instances of convex functions such as $f(t) = -\ln t$, $t \ln t$, $\exp t$ etc... and we omit the details.

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