SOME INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS DEFINED ON LINEAR SPACES

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Abstract. In this paper we provide some new integral inequalities related to the Hermite-Hadamard result for convex functions defined on convex subsets in a linear space. Applications for norms and multivariate real functions are also given.

1. Introduction

Let X be a real linear space, $a, b \in X$, $a \neq b$ and let $[a,b] := \{(1-\lambda) a + \lambda b,\ \lambda \in [0,1]\}$ be the *segment* generated by a and b. We consider the function $f : [a,b] \to \mathbb{R}$ and the attached function $g(a,b) : [0,1] \to \mathbb{R}$, $g(a,b)(t) := f[(1-t)a + tb]$, $t \in [0,1]$.

It is well known that f is convex on $[a,b]$ iff $g(a,b)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}(a,b)(s) = (\triangledown_{\pm} f[(1-s)a + sb])(b-a)$, $s \in [0,1)$
(ii) $g'_{+}(a,b)(0) = (\triangledown_{+} f(a))(b-a)$
(iii) $g'_{-}(a,b)(1) = (\triangledown_{-} f(b))(b-a)$

where $(\triangledown_{\pm} f(x))(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\begin{aligned}
(\triangledown_{+} f(x))(y) &:= \lim_{h\to 0+}\left[\frac{f(x+hy)-f(x)}{h}\right],\\
(\triangledown_{-} f(x))(y) &:= \lim_{k\to 0-}\left[\frac{f(x+ky)-f(x)}{k}\right],\quad x,\ y \in X.
\end{aligned}$$

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[a,b] \subset X$:

$$f\left(\frac{a+b}{2}\right) \le \int_0^1 f[(1-t)a+tb]\,dt \le \frac{f(a)+f(b)}{2}, \tag{HH}$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $g(a,b) : [0,1] \to \mathbb{R}$

$$g(a,b)\left(\frac{1}{2}\right) \le \int_0^1 g(a,b)(t)\,dt \le \frac{g(a,b)(0)+g(a,b)(1)}{2}.$$

For other related results see the monograph on line [4].

We have the following result [2] related to the first Hermite-Hadamard inequality in (HH):

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Theorem 1. *Let X be a linear space, a, $b \in X$, $a \neq b$ and $f : [a,b] \subset X \to \mathbb{R}$ be a convex function on the segment $[a,b]$. Then for any $s \in (0,1)$ one has the inequality*

$$\begin{aligned}(1.1)\quad & \frac{1}{2}\left[(1-s)^2 (\nabla_+ f[(1-s)a+sb])(b-a) - s^2 (\nabla_- f[(1-s)a+sb])(b-a)\right] \\ & \leq \int_0^1 f[(1-t)a+tb]\,dt - f[(1-s)a+sb] \\ & \leq \frac{1}{2}\left[(1-s)^2 (\nabla_- f(b))(b-a) - s^2 (\nabla_+ f(a))(b-a)\right].\end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $s = 0$ or $s = 1$.

If $f : [a,b] \to \mathbb{R}$ is as in Theorem 1 and Gâteaux differentiable in $c := (1-\lambda)a + \lambda b$, $\lambda \in (0,1)$ along the direction $b-a$, then we have the inequality:

$$(1.2)\qquad \left(\frac{1}{2} - \lambda\right)(\nabla f(c))(b-a) \leq \int_0^1 f[(1-t)a+tb]\,dt - f(c).$$

If f is as in Theorem 1, then

$$\begin{aligned}(1.3)\qquad 0 & \leq \frac{1}{8}\left[\nabla_+ f\left(\frac{a+b}{2}\right)(b-a) - \nabla_- f\left(\frac{a+b}{2}\right)(b-a)\right] \\ & \leq \int_0^1 f[(1-t)a+tb]\,dt - f\left(\frac{a+b}{2}\right) \\ & \leq \frac{1}{8}\left[(\nabla_- f(b))(b-a) - (\nabla_+ f(a))(b-a)\right].\end{aligned}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

Also we have the following result [3] related to the second Hermite-Hadamard inequality in (HH):

Theorem 2. *Let X be a linear space, a, $b \in X$, $a \neq b$ and $f : [a,b] \subset X \to \mathbb{R}$ be a convex function on the segment $[a,b]$. Then for any $s \in (0,1)$ one has the inequality*

$$\begin{aligned}(1.4)\quad & \frac{1}{2}\left[(1-s)^2 (\nabla_+ f[(1-s)a+sb])(b-a) - s^2 (\nabla_- f[(1-s)a+sb])(b-a)\right] \\ & \leq (1-s)f(a) + sf(b) - \int_0^1 f[(1-t)a+tb]\,dt \\ & \leq \frac{1}{2}\left[(1-s)^2 (\nabla_- f(b))(b-a) - s^2 (\nabla_+ f(a))(b-a)\right].\end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $s = 0$ or $s = 1$.

If $f : [a,b] \to \mathbb{R}$ is as in Theorem 2 and Gâteaux differentiable in $c := (1-\lambda)a + \lambda b$, $\lambda \in (0,1)$ along the direction $b-a$, then we have the inequality:

$$(1.5)\quad \left(\frac{1}{2} - \lambda\right)(\nabla f(c))(b-a) \leq (1-\lambda)f(a) + \lambda f(b) - \int_0^1 f[(1-t)a+tb]\,dt.$$

If f is as in Theorem 2, then

$$0 \leq \frac{1}{8}\left[\nabla_{+} f\left(\frac{a+b}{2}\right)(b-a)-\nabla_{-} f\left(\frac{a+b}{2}\right)(b-a)\right] \tag{1.6}$$
$$\leq \frac{f(a)+f(b)}{2}-\int_{0}^{1} f[(1-t) a+t b] d t$$
$$\leq \frac{1}{8}\left[\left(\nabla_{-} f(b)\right)(b-a)-\left(\nabla_{+} f(a)\right)(b-a)\right].$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

Motivated by the above facts, in this paper we provide some new integral inequalities related to the Hermite-Hadamard result for convex functions defined on convex subsets in a linear space. Applications for norms and multivariate real functions are also given.

2. The Results

Let $f : C \subset X \to \mathbb{R}$ be a convex function on C. We define the function $F_f : C \times C \to \mathbb{R}$ by

$$F_{f}(x, y):=\int_{0}^{1} f((1-t) x+t y) d t. \tag{2.1}$$

Theorem 3. *If $f : C \subset X \to \mathbb{R}$ is a convex function on C, then the function F_f is convex on $C \times C$.*

Proof. Let (x, y), $(u, v) \in C \times C$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned}
F_{f}(\alpha(x, y)+\beta(u, v)) &= F_{f}(\alpha x+\beta u, \alpha y+\beta v) \\
&= \int_{0}^{1} f((1-t)(\alpha x+\beta u)+t(\alpha y+\beta v)) d t \\
&= \int_{0}^{1} f(\alpha[(1-t) x+t y]+\beta[(1-t) u+t v]) d t \\
&\leq \int_{0}^{1}[\alpha f((1-t) x+t y)+\beta f((1-t) u+t v)] d t \\
&= \alpha \int_{0}^{1} f((1-t) x+t y) d t+\beta \int_{0}^{1} f((1-t) u+t v) d t \\
&= \alpha F_{f}(x, y)+\beta F_{f}(u, v),
\end{aligned}$$

which proves the joint convexity of the function F_f. □

We also have the double integral Hermite-Hadamard type inequalities:

Theorem 4. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on C. Then for all (x, y), $(u, v) \in C \times C$ we have*

$$f\left(\frac{x+u+y+v}{4}\right) \leq \int_0^1 f\left((1-t)\frac{x+u}{2} + t\frac{y+v}{2}\right) dt \tag{2.2}$$
$$\leq \int_0^1 \int_0^1 f((1-t)(1-\tau)x + (1-t)\tau u + t(1-\tau)y + t\tau v)\, dt d\tau$$
$$\leq \int_0^1 \frac{f((1-t)x+ty) + f((1-t)u+tv)}{2} dt$$
$$\leq \frac{1}{4}[f(x) + f(u) + f(y) + f(v)].$$

Proof. We write the Hermite-Hadamard inequality for the convex function $F_f : C \times C \to \mathbb{R}$ to get

$$F_f\left(\frac{(x,y)+(u,v)}{2}\right) \leq \int_0^1 F_f[(1-\tau)(x,y) + \tau(u,v)]\, d\tau \tag{2.3}$$
$$\leq \frac{F_f(x,y) + F_f(u,v)}{2}.$$

Since

$$F_f\left(\frac{(x,y)+(u,v)}{2}\right) = F_f\left(\frac{x+u}{2}, \frac{y+v}{2}\right) = \int_0^1 f\left((1-t)\frac{x+u}{2} + t\frac{y+v}{2}\right) dt,$$

$$\int_0^1 F_f[(1-\tau)(x,y) + \tau(u,v)]\, d\tau$$
$$= \int_0^1 F_f[(1-\tau)x + \tau u, (1-\tau)y + \tau v]\, d\tau$$
$$= \int_0^1 \left(\int_0^1 f((1-t)[(1-\tau)x + \tau u] + t[(1-\tau)y + \tau v])\, dt\right) d\tau$$
$$= \int_0^1 \int_0^1 f((1-t)(1-\tau)x + (1-t)\tau u + t(1-\tau)y + t\tau v)\, dt d\tau$$

and

$$\frac{1}{2}[F_f(x,y) + F_f(u,v)] = \int_0^1 \frac{f((1-t)x+ty) + f((1-t)u+tv)}{2} dt,$$

hence by (2.3) we get the second and third inequalities in (2.2).

The first and last inequality in (2.2) are obvious by Hermite-Hadamard inequality for the single integral. □

Remark 1. *By taking $u = y$ and $v = x$ in (2.2), we have*

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 \int_0^1 f([(1-t)(1-\tau) + t\tau]x + [(1-t)\tau + t(1-\tau)]y)\, dt d\tau$$
$$\leq \int_0^1 \frac{f((1-t)x+ty) + f((1-t)y+tx)}{2} dt$$

namely, the following refinement of the first Hermite-Hadamard inequality:

$$
\begin{aligned}
f\left(\frac{x+y}{2}\right) &\leq \int_0^1 \int_0^1 f\left(\left(1-\tau-t+2t\tau\right)x+\left(\tau+t-2t\tau\right)y\right)dtd\tau \\
&\leq \int_0^1 f\left(\left(1-t\right)x+ty\right)dt.
\end{aligned}
\tag{2.4}
$$

We have the following reverse inequalities:

Theorem 5. *With the assumptions of Theorem 4 we have*

$$
\begin{aligned}
0 &\leq \frac{1}{8}\left[\int_0^1 \left(\nabla_+ f\left(\left(1-t\right)\frac{x+u}{2}+t\frac{y+v}{2}\right)\right)\left(\left(1-t\right)\left(u-x\right)+t\left(v-y\right)\right)dt\right. \\
&\quad \left.-\int_0^1 \left(\nabla_- f\left(\left(1-t\right)\frac{x+u}{2}+t\frac{y+v}{2}\right)\right)\left(\left(1-t\right)\left(u-x\right)+t\left(v-y\right)\right)dt\right] \\
&\leq \int_0^1 \int_0^1 f\left(\left(1-t\right)\left(1-\tau\right)x+\left(1-t\right)\tau u+t\left(1-\tau\right)y+t\tau v\right)dtd\tau \\
&\quad -\int_0^1 f\left(\left(1-t\right)\frac{x+u}{2}+t\frac{y+v}{2}\right)dt \\
&\leq \frac{1}{8}\left[\int_0^1 \left(\nabla_- f\left(\left(1-t\right)u+tv\right)\right)\left(\left(1-t\right)\left(u-x\right)+t\left(v-y\right)\right)dt\right. \\
&\quad \left.-\int_0^1 \left(\nabla_+ f\left(\left(1-t\right)x+ty\right)\right)\left(\left(1-t\right)\left(u-x\right)+t\left(v-y\right)\right)dt\right]
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
0 &\leq \frac{1}{8}\left[\int_0^1 \left(\nabla_+ f\left(\left(1-t\right)\frac{x+u}{2}+t\frac{y+v}{2}\right)\right)\left(\left(1-t\right)\left(u-x\right)+t\left(v-y\right)\right)dt\right. \\
&\quad \left.-\int_0^1 \left(\nabla_- f\left(\left(1-t\right)\frac{x+u}{2}+t\frac{y+v}{2}\right)\right)\left(\left(1-t\right)\left(u-x\right)+t\left(v-y\right)\right)dt\right] \\
&\leq \int_0^1 \frac{f\left(\left(1-t\right)x+ty\right)+f\left(\left(1-t\right)u+tv\right)}{2}dt \\
&\quad -\int_0^1 \int_0^1 f\left(\left(1-t\right)\left(1-\tau\right)x+\left(1-t\right)\tau u+t\left(1-\tau\right)y+t\tau v\right)dtd\tau \\
&\leq \frac{1}{8}\left[\int_0^1 \left(\nabla_- f\left(\left(1-t\right)u+tv\right)\right)\left(\left(1-t\right)\left(u-x\right)+t\left(v-y\right)\right)dt\right. \\
&\quad \left.-\int_0^1 \left(\nabla_+ f\left(\left(1-t\right)x+ty\right)\right)\left(\left(1-t\right)\left(u-x\right)+t\left(v-y\right)\right)dt\right].
\end{aligned}
\tag{2.6}
$$

Proof. Now, observe that, by utilising the properties of the integral and limits under the sign of integral, we have succesively that

$$
\begin{aligned}
(2.7)\qquad (\triangledown_{+}F_{f}(x,y))(u,v) &:= \lim_{h\to 0+}\left[\frac{F_{f}((x,y)+h(u,v))-F_{f}(x,y)}{h}\right]\\
&= \lim_{h\to 0+}\left[\frac{F_{f}(x+hu,y+hv)-F_{f}(x,y)}{h}\right]\\
&= \lim_{h\to 0+}\frac{1}{h}\left[\int_{0}^{1} f((1-t)(x+hu)+t(y+hv))\,dt-\int_{0}^{1} f((1-t)x+ty)\,dt\right]\\
&= \lim_{h\to 0+}\frac{1}{h}\left[\int_{0}^{1} f((1-t)x+ty+h((1-t)u+tv))\,dt-\int_{0}^{1} f((1-t)x+ty)\,dt\right]\\
&= \int_{0}^{1}\lim_{h\to 0+}\frac{1}{h}\left[f((1-t)x+ty+h((1-t)u+tv))-f((1-t)x+ty)\right]dt\\
&= \int_{0}^{1}(\triangledown_{+}f((1-t)x+ty))((1-t)u+tv)\,dt
\end{aligned}
$$

and, similarly

$$
(2.8)\qquad (\triangledown_{-}F_{f}(x,y))(u,v)=\int_{0}^{1}(\triangledown_{-}f((1-t)x+ty))((1-t)u+tv)\,dt,
$$

for all $(x,y)\,,\ (u,v)\in C\times C$.

If we use the inequality (1.3) for the function F_{f}, then we have

$$
\begin{aligned}
(2.9)\qquad 0 &\le \frac{1}{8}\left[\triangledown_{+}F_{f}\left(\frac{x+u}{2},\frac{y+v}{2}\right)(u-x,v-y)\right.\\
&\qquad\left. -\triangledown_{-}F_{f}\left(\frac{x+u}{2},\frac{y+v}{2}\right)(u-x,v-y)\right]\\
&\le \int_{0}^{1}F_{f}[(1-t)(x,y)+t(u,v)]\,dt-F_{f}\left(\frac{x+u}{2},\frac{y+v}{2}\right)\\
&\le \frac{1}{8}\left[(\triangledown_{-}F_{f}(u,v))(u-x,v-y)-(\triangledown_{+}F_{f}(x,y))(u-x,v-y)\right].
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left(\triangledown_{\pm}F_{f}\left(\frac{x+u}{2},\frac{y+v}{2}\right)\right)(u-x,v-y)\\
&\qquad=\int_{0}^{1}\left(\triangledown_{\pm}f\left((1-t)\frac{x+u}{2}+t\frac{y+v}{2}\right)\right)((1-t)(u-x)+t(v-y))\,dt,
\end{aligned}
$$

$$
\begin{aligned}
&(\triangledown_{-}F_{f}(u,v))(u-x,v-y)\\
&\qquad=\int_{0}^{1}(\triangledown_{-}f((1-t)u+tv))((1-t)(u-x)+t(v-y))\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
&(\triangledown_{+}F_{f}(x,y))(u-x,v-y)\\
&\qquad=\int_{0}^{1}(\triangledown_{+}f((1-t)x+ty))((1-t)(u-x)+t(v-y))\,dt,
\end{aligned}
$$

hence by (2.9) we get (2.5).

The inequality (2.6) follows in a similar way from (1.6). □

Remark 2. *If we take in Theorem 5 $u = y$ and $v = x$, then we get*

$$\begin{aligned}(2.10)\quad 0 &\leq \frac{1}{4}\left[\int_0^1 \left(\nabla_+ f\left(\frac{x+y}{2}\right)\right)\left(\left(\frac{1}{2}-t\right)(y-x)\right)dt\right.\\ &\qquad\left.-\int_0^1 \left(\nabla_- f\left(\frac{x+y}{2}\right)\right)\left(\left(\frac{1}{2}-t\right)(y-x)\right)dt\right]\\ &\leq \int_0^1\int_0^1 f\left(\left[(1-t)(1-\tau)+t\tau\right]x+\left[(1-t)\tau+t(1-\tau)\right]y\right)dtd\tau\\ &\qquad - f\left(\frac{x+y}{2}\right)\\ &\leq \frac{1}{4}\left[\int_0^1 \left(\nabla_- f\left((1-t)y+tx\right)\right)\left(\left(\frac{1}{2}-t\right)(y-x)\right)dt\right.\\ &\qquad\left.-\int_0^1 \left(\nabla_+ f\left((1-t)x+ty\right)\right)\left(\left(\frac{1}{2}-t\right)(y-x)\right)dt\right].\end{aligned}$$

Observe that by the change of variable $t = 1 - s$, $s \in [0, 1]$ we have

$$\begin{aligned}&\int_0^1 \left(\nabla_- f\left((1-t)y+tx\right)\right)\left(\left(\frac{1}{2}-t\right)(y-x)\right)dt\\ &\qquad = \int_0^1 \left(\nabla_- f\left((1-t)x+ty\right)\right)\left(\left(t-\frac{1}{2}\right)(y-x)\right)dt.\end{aligned}$$

Also, since

$$\left(\nabla_- f(x)\right)(-y) = -\left(\nabla_+ f(x)\right)(y),$$

then

$$\begin{aligned}&\int_0^1 \left(\nabla_+ f\left((1-t)x+ty\right)\right)\left(\left(\frac{1}{2}-t\right)(y-x)\right)dt\\ &\qquad = -\int_0^1 \left(\nabla_- f\left((1-t)x+ty\right)\right)\left(\left(t-\frac{1}{2}\right)(y-x)\right)dt\end{aligned}$$

and by (2.10) we get

$$\begin{aligned}(2.11)\quad 0 &\leq \frac{1}{4}\left[\int_0^1 \left(\nabla_+ f\left(\frac{x+y}{2}\right)\right)\left(\left(\frac{1}{2}-t\right)(y-x)\right)dt\right.\\ &\qquad\left.-\int_0^1 \left(\nabla_- f\left(\frac{x+y}{2}\right)\right)\left(\left(\frac{1}{2}-t\right)(y-x)\right)dt\right]\\ &\leq \int_0^1\int_0^1 f\left(\left[(1-t)(1-\tau)+t\tau\right]x+\left[(1-t)\tau+t(1-\tau)\right]y\right)dtd\tau\\ &\qquad - f\left(\frac{x+y}{2}\right)\\ &\leq \frac{1}{2}\int_0^1 \left(\nabla_- f\left((1-t)x+ty\right)\right)\left(\left(t-\frac{1}{2}\right)(y-x)\right)dt\end{aligned}$$

for x, $y \in C$.

Similarly, we have

$$0\le\frac{1}{4}\left[\int_0^1\left(\nabla_+f\left(\frac{x+y}{2}\right)\right)\left(\left(\frac{1}{2}-t\right)(y-x)\right)dt\right.$$
$$\left.-\int_0^1\left(\nabla_-f\left(\frac{x+y}{2}\right)\right)\left(\left(\frac{1}{2}-t\right)(y-x)\right)dt\right]$$
$$\le\int_0^1 f((1-t)x+ty)\,dt$$
$$-\int_0^1\int_0^1 f([(1-t)(1-\tau)+t\tau]x+[(1-t)\tau+t(1-\tau)]y)\,dtd\tau$$
$$\le\frac{1}{2}\int_0^1(\nabla_-f((1-t)x+ty))\left(\left(t-\frac{1}{2}\right)(y-x)\right)dt. \tag{2.12}$$

3. Eamples for Norms

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s:=(\nabla_+f_0(y))(x)=\lim_{t\to0+}\left[\frac{\|y+tx\|^2-\|y\|^2}{2t}\right]$;
(v) $\langle x,y\rangle_i:=(\nabla_-f_0(y))(x)=\lim_{s\to0-}\left[\frac{\|y+sx\|^2-\|y\|^2}{2s}\right]$;

for any x, $y\in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [1] or [5]), assuming that p, $q\in\{s,i\}$ and $p\neq q$:

(a) $\langle x,x\rangle_p=\|x\|^2$ for all $x\in X$;
(aa) $\langle\alpha x,\beta y\rangle_p=\alpha\beta\langle x,y\rangle_p$ if $\alpha,\beta\ge0$ and x, $y\in X$;
(aaa) $\left|\langle x,y\rangle_p\right|\le\|x\|\,\|y\|$ for all x, $y\in X$;
(av) $\langle\alpha x+y,x\rangle_p=\alpha\langle x,x\rangle_p+\langle y,x\rangle_p$ if x, $y\in X$ and $\alpha\in\mathbb{R}$;
(v) $\langle -x,y\rangle_p=-\langle x,y\rangle_q$ for all x, $y\in X$;
(va) $\langle x+y,z\rangle_p\le\|x\|\,\|z\|+\langle y,z\rangle_p$ for all x, y, $z\in X$;
(vaa) The mapping $\langle\cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p=s$ (or $p=i$);
(vaaa) The normed linear space $(X,\|\cdot\|)$ is smooth at the point $x_0\in X\setminus\{0\}$ if and only if $\langle y,x_0\rangle_s=\langle y,x_0\rangle_i$ for all $y\in X$; in general $\langle y,x\rangle_i\le\langle y,x\rangle_s$ for all x, $y\in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y,x\rangle_i=\langle y,x\rangle=\langle y,x\rangle_s$ for all x, $y\in X$.

Applying inequality (HH) for the convex function $f_0(x)=\frac{1}{2}\|x\|^2$, one may deduce the inequality

$$\left\|\frac{x+y}{2}\right\|^2\le\int_0^1\|(1-t)x+ty\|^2\,dt\le\frac{\|x\|^2+\|y\|^2}{2} \tag{3.1}$$

for any x, $y \in X$. The same (HH) inequality applied for $f_1(x) = \|x\|$, will give the following refinement of the triangle inequality:

$$\left\|\frac{x+y}{2}\right\| \le \int_0^1 \|(1-t)x + ty\| \, dt \le \frac{\|x\| + \|y\|}{2}, \quad x, y \in X. \tag{3.2}$$

If we write the inequality (2.4) for the convex function $f(z) = \|z\|^r$ with $r \ge 1$ we have

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|^r &\le \int_0^1 \int_0^1 \|((1-\tau-t+2t\tau)x + (\tau+t-2t\tau)y)\|^r \, dt d\tau \\ &\le \int_0^1 \|(1-t)x + ty\|^r \, dt. \end{aligned} \tag{3.3}$$

From (2.11) we get for the convex function $f_0(x) = \frac{1}{2}\|x\|^2$ that

$$\begin{aligned} 0 &\le \frac{1}{2}\int_0^1 \left[\left\langle \left(\frac{1}{2} - t\right)(y-x), \frac{x+y}{2}\right\rangle_s - \left\langle \left(\frac{1}{2} - t\right)(y-x), \frac{x+y}{2}\right\rangle_i\right] dt \\ &\le \int_0^1 \int_0^1 \|[(1-t)(1-\tau) + t\tau]x + [(1-t)\tau + t(1-\tau)]y\|^2 \, dt d\tau \\ &\qquad - \left\|\frac{x+y}{2}\right\|^2 \\ &\le \int_0^1 \left\langle \left(t - \frac{1}{2}\right)(y-x), (1-t)x + ty\right\rangle_i dt, \end{aligned} \tag{3.4}$$

while from (2.12) we get

$$\begin{aligned} 0 &\le \frac{1}{2}\int_0^1 \left[\left\langle \left(\frac{1}{2} - t\right)(y-x), \frac{x+y}{2}\right\rangle_s - \left\langle \left(\frac{1}{2} - t\right)(y-x), \frac{x+y}{2}\right\rangle_i\right] dt \\ &\le \int_0^1 \|(1-t)x + ty\|^2 \, dt \\ &\qquad - \int_0^1 \int_0^1 \|[(1-t)(1-\tau) + t\tau]x + [(1-t)\tau + t(1-\tau)]y\|^2 \, dt d\tau \\ &\le \int_0^1 \left\langle \left(t - \frac{1}{2}\right)(y-x), (1-t)x + ty\right\rangle_i dt. \end{aligned} \tag{3.5}$$

By the Schwarz inequality we have

$$\begin{aligned} &\int_0^1 \left\langle \left(t - \frac{1}{2}\right)(y-x), (1-t)x + ty\right\rangle_i dt \\ &\le \int_0^1 \left\|\left(t - \frac{1}{2}\right)(y-x)\right\| \|(1-t)x + ty\| \, dt \\ &= \|y-x\| \int_0^1 \left|t - \frac{1}{2}\right| \|(1-t)x + ty\| \, dt \\ &\le \|y-x\| \int_0^1 \left|t - \frac{1}{2}\right| [(1-t)\|x\| + t\|y\|] \, dt. \end{aligned}$$

Consider

$$I := \int_0^1 \left|t - \frac{1}{2}\right| [(1-t)\|x\| + t\|y\|] \, dt.$$

By the change of variable $t = 1 - s$, $s \in [0,1]$, we also have

$$I = \int_0^1 \left| \frac{1}{2} - s \right| [s \|x\| + (1-s) \|y\|] \, dt = \int_0^1 \left| t - \frac{1}{2} \right| [t \|x\| + (1-t) \|y\|] \, dt.$$

If we add these two equalities, we get

$$2I = (\|x\| + \|y\|) \int_0^1 \left| t - \frac{1}{2} \right| dt = \frac{1}{4} (\|x\| + \|y\|),$$

which gives

$$I = \frac{1}{8} (\|x\| + \|y\|).$$

This implies that

$$\begin{aligned} &\int_0^1 \left\langle \left(t - \frac{1}{2} \right) (y - x), (1-t) x + ty \right\rangle_i dt \\ &\leq \|y - x\| \int_0^1 \left| t - \frac{1}{2} \right| \|(1-t) x + ty\| \, dt \leq \frac{1}{8} (\|x\| + \|y\|) \|y - x\|. \end{aligned}$$

Therefore, for any x, $y \in X$ we have the simple upper bounds:

$$\begin{aligned} 0 &\leq \int_0^1 \int_0^1 \|[(1-t)(1-\tau) + t\tau] x + [(1-t)\tau + t(1-\tau)] y\|^2 \, dt d\tau \\ &\qquad - \left\| \frac{x+y}{2} \right\|^2 \\ &\leq \|y - x\| \int_0^1 \left| t - \frac{1}{2} \right| \|(1-t) x + ty\| \, dt \leq \frac{1}{8} (\|x\| + \|y\|) \|y - x\| \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} 0 &\leq \int_0^1 \|(1-t) x + ty\|^2 \, dt \\ &\quad - \int_0^1 \int_0^1 \|[(1-t)(1-\tau) + t\tau] x + [(1-t)\tau + t(1-\tau)] y\|^2 \, dt d\tau \\ &\leq \|y - x\| \int_0^1 \left| t - \frac{1}{2} \right| \|(1-t) x + ty\| \, dt \leq \frac{1}{8} (\|x\| + \|y\|) \|y - x\|. \end{aligned} \tag{3.7}$$

4. Aplications for Functions of Real Variables

Now, let $\Omega \subset \mathbb{R}^n$ be an open and convex set in $\mathbb{R}^n$. If $F : \Omega \to \mathbb{R}$ is a differentiable convex function on Ω, then, obviously, for any $\bar{c} \in \Omega$ we have

$$\nabla F(\bar{c})(\bar{y}) = \sum_{i=1}^n \frac{\partial F(\bar{c})}{\partial x_i} y_i, \qquad \bar{y} \in \mathbb{R}^n,$$

where $\frac{\partial F}{\partial x_i}$ are the partial derivatives of F with respect to the variable x_i $(i = 1, \ldots, n)$.

For any $\bar{a} = (a_1, ..., a_n)$, $\bar{b} = (b_1, ..., b_n) \in \Omega$ we have for (2.11) and (2.12) that

$$0 \leq \int_0^1 \int_0^1 f\left([(1-t)(1-\tau) + t\tau]\,\bar{a} + [(1-t)\tau + t(1-\tau)]\,\bar{b}\right) dt d\tau - f\left(\frac{\bar{a}+\bar{b}}{2}\right) \leq \frac{1}{2}\int_0^1 \left(t - \frac{1}{2}\right)\left(\sum_{i=1}^n \frac{\partial F\left((1-t)\,\bar{a} + t\bar{b}\right)}{\partial x_i}(b_i - x_i)\right) dt \tag{4.1}$$

and

$$0 \leq \int_0^1 f\left((1-t)\,\bar{a} + t\bar{b}\right) dt - \int_0^1 \int_0^1 f\left([(1-t)(1-\tau) + t\tau]\,\bar{a} + [(1-t)\tau + t(1-\tau)]\,\bar{b}\right) dt d\tau \leq \frac{1}{2}\int_0^1 \left(t - \frac{1}{2}\right)\left(\sum_{i=1}^n \frac{\partial F\left((1-t)\,\bar{a} + t\bar{b}\right)}{\partial x_i}(b_i - x_i)\right) dt. \tag{4.2}$$

The case of a single variable is as follows. If f is convex on the interval I, then for all $a, b \in \mathring{I}$, the interior of I, we have

$$0 \leq \int_0^1 \int_0^1 f([(1-t)(1-\tau) + t\tau]\,a + [(1-t)\tau + t(1-\tau)]\,b)\, dt d\tau - f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\int_0^1 \left(t - \frac{1}{2}\right) f'((1-t)\,a + tb)\,(b-a)\, dt$$

and

$$0 \leq \int_0^1 f((1-t)\,a + tb)\, dt - \int_0^1 \int_0^1 f([(1-t)(1-\tau) + t\tau]\,a + [(1-t)\tau + t(1-\tau)]\,b)\, dt d\tau \leq \frac{1}{2}\int_0^1 \left(t - \frac{1}{2}\right) f'((1-t)\,a + tb)\,(b-a)\, dt.$$

Since

$$\begin{aligned}
&\int_0^1 \left(t - \frac{1}{2}\right) f'((1-t)\,a + tb)\,(b-a)\, dt \\
&= \int_0^1 \left(t - \frac{1}{2}\right) d\left(f((1-t)\,a + tb)\right) \\
&= \left(t - \frac{1}{2}\right) f((1-t)\,a + tb)\Big|_0^1 - \int_0^1 f((1-t)\,a + tb)\, dt \\
&= \frac{f(a) + f(b)}{2} - \int_0^1 f((1-t)\,a + tb)\, dt,
\end{aligned}$$

hence we have the inequalities

$$0 \le \int_0^1 \int_0^1 f\left(\left[(1-t)(1-\tau)+t\tau\right]a+\left[(1-t)\tau+t(1-\tau)\right]b\right)dtd\tau - f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[\frac{f(a)+f(b)}{2}-\int_0^1 f((1-t)a+tb)\,dt\right] \le \frac{1}{16}\left[f'(b)-f'(a)\right](b-a) \tag{4.3}$$

and

$$0 \le \int_0^1 f((1-t)a+tb)\,dt - \int_0^1 \int_0^1 f\left(\left[(1-t)(1-\tau)+t\tau\right]a+\left[(1-t)\tau+t(1-\tau)\right]b\right)dtd\tau \le \frac{1}{2}\left[\frac{f(a)+f(b)}{2}-\int_0^1 f((1-t)a+tb)\,dt\right] \le \frac{1}{16}\left[f'(b)-f'(a)\right](b-a). \tag{4.4}$$

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