BOUNDS FOR THE HH $f$-DIVERGENCE MEASURES IN TERMS OF $\chi^2$-DIVERGENCE

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ABSTRACT. In this paper we establish some inequalities for the Hermite-Hadamard (HH) $f$-divergence measures in terms of $\chi^2$-divergence. An application for Kullback-Leibler divergence is also provided.

1. INTRODUCTION

Let the set $X$ and the $\sigma$-finite measure $\mu$ be given and consider the set of all probability densities on $\mu$ to be defined on $\Omega := \{p : X \to \mathbb{R}, p (x) \geq 0, \int_X p (x) \mu (x) = 1\}$

The $f$-divergence is defined as follows \cite{2}, \cite{3}

$$D_f (p, q) := \int_X p (x) f \left[ \frac{q (x)}{p(x)} \right] d\mu (x), \quad p, q \in \Omega,$$

where the function $f$ is convex on $(0, \infty)$. It is assumed that $f (u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. For instance, the following celebrated divergences are particular cases of $f$-divergence

$$D_{KL} (p, q) := \int_X p (x) \log \left[ \frac{p (x)}{q (x)} \right] d\mu (x), \quad p, q \in \Omega,$$

(Kullback-Leibler divergence \cite{9})

$$D_v (p, q) := \int_X |p (x) - q (x)| d\mu (x), \quad p, q \in \Omega;$$

(variation distance)

$$D_H (p, q) := \int_X \sqrt{p (x) - q (x)} d\mu (x), \quad p, q \in \Omega;$$

(Hellinger distance \cite{7})

$$D_{\chi}^2 (p, q) := \int_X p (x) \left[ \left( \frac{q (x)}{p (x)} \right)^2 - 1 \right] d\mu (x), \quad p, q \in \Omega;$$

($\chi^2$-divergence)

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\begin{equation}
D_J (p, q) := \int_X [p(x) - q(x)] \ln \frac{p(x)}{q(x)} \, d\mu(x), \quad p, q \in \Omega;
\end{equation}
(Jeffreys distance \cite{8})

\begin{equation}
D_{\Delta} (p, q) := \int_X \frac{[p(x) - q(x)]^2}{p(x) + q(x)} \, d\mu(x), \quad p, q \in \Omega.
\end{equation}
(triangular discrimination \cite{12})

In \cite{10}, Lin and Wong (see also \cite{11}) introduced the following divergence

\begin{equation}
D_{LW} (p, q) := \int_X p(x) \log \left[ \frac{p(x)}{\frac{1}{2} p(x) + \frac{1}{2} q(x)} \right] \, d\mu(x), \quad p, q \in \Omega.
\end{equation}

This can be represented as follows, using the Kullback-Leibler divergence:

\[ D_{LW} (p, q) = D_{KL} \left( p, \frac{p + q}{2} \right). \]

Lin and Wong have established the following inequalities

\begin{equation}
D_{LW} (p, q) \leq \frac{1}{2} D_{KL} (p, q); \tag{1.9}
\end{equation}

\begin{equation}
D_{LW} (p, q) + D_{LW} (q, p) \leq D_v (p, q) \leq 2; \tag{1.10}
\end{equation}

\begin{equation}
D_{LW} (p, q) \leq 1. \tag{1.11}
\end{equation}

In \cite{11}, Shioya and Da-te improved (1.9)-(1.11) by showing that

\[ D_{LW} (p, q) \leq \frac{1}{2} D_v (p, q) \leq 1. \]

In the same paper \cite{11}, the authors introduced the generalised Lin-Wong \( f \)-divergence \( D_f \left( p, \frac{1}{2} p + \frac{1}{2} q \right) \) and the Hermite-Hadamard (HH) \( f \)-divergence

\begin{equation}
D_{HH}^f (p, q) := \int_X p(x) \int_t^{g(x)} \frac{f(t)}{g(x)} \frac{dt}{p(x)} \, d\mu(x), \quad p, q \in \Omega
\end{equation}

and, by use of the Hermite-Hadamard inequality for convex functions, proved the following basic inequality

\begin{equation}
D_f \left( p, \frac{p + q}{2} \right) \leq D_{HH}^f (p, q) \leq \frac{1}{2} D_f (p, q), \tag{1.13}
\end{equation}

provided that \( f \) is convex and normalised, i.e., \( f (1) = 0 \).

In 2002, Barnett, Cerone & Dragomir \cite{1} improved the inequality (1.13) as follows:
Theorem 1. Assume that the function $f : (0, \infty) \to \mathbb{R}$ is convex and normalised, i.e. $f(1) = 0$. Let $p, q \in \Omega$ then we have the inequality,

\begin{align*}
0 \leq D_f \left( p, \frac{p+q}{2} \right) \\
\leq \lambda D_f \left( p, p + \frac{\lambda}{2} (q-p) \right) + (1-\lambda) D_f \left( p, \frac{p+q}{2} + \frac{\lambda}{2} (q-p) \right) \\
\leq D_{HH}^f (p,q) \leq \frac{1}{2} \left[ D_f (p, (1-\lambda) p + \lambda q) + (1-\lambda) D_f (p,q) \right] \\
\leq \frac{1}{2} D_f(p,q),
\end{align*}

for all $\lambda \in [0,1]$.

In particular,

\begin{align*}
0 \leq D_f \left( p, \frac{p+q}{2} \right) &\leq \frac{1}{2} \left[ D_f \left( p, \frac{3p+q}{4} \right) + D_f \left( p, \frac{p+3q}{4} \right) \right] \\
\leq D_{HH}^f (p,q) &\leq \frac{1}{2} \left[ D_f \left( p, \frac{p+q}{2} \right) + \frac{1}{2} D_f (p,q) \right] \\
&\leq \frac{1}{2} D_f(p,q).
\end{align*}

In 2005, [5], the author obtained the following estimate for a differentiable convex and normalised function $f : (0, \infty) \to \mathbb{R}$

\begin{equation}
0 \leq D_{HH}^f (p,q) \leq \frac{1}{8} D_f (p,q)
\end{equation}

for $p, q \in \Omega$, where

\begin{equation}
f^1(t) := (t-1) f'(t), \ t \in (0, \infty).
\end{equation}

In the paper [6] we also obtained the dual inequality

\begin{equation}
0 \leq \frac{1}{2} D_f (p,q) - D_{HH}^f (p,q) \leq \frac{1}{8} D_f (p,q)
\end{equation}

for $p, q \in \Omega$.

Motivated by the above results, we establish in this paper other inequalities for the HH $f$-divergence.

2. General Results

We start with the following useful representation for the HH $f$-divergence:

Lemma 1. Assume that the function $f : (0, \infty) \to \mathbb{R}$ is convex and normalised, then we have the representation

\begin{align*}
D_{HH}^f (p,q) &= \int_X p(x) \left( \int_0^1 f \left( \frac{sq(x) + (1-s)p(x)}{p(x)} \right) ds \right) d\mu(x) \\
&= \int_0^1 D_f (p, sq + (1-s)p) ds
\end{align*}

for $p, q \in \Omega$. 
Proof. Using the change of variable
\[ t = \frac{sq(x) + (1 - s)p(x)}{p(x)}, \quad s \in [0, 1] \]
we have
\[ \int_0^{1/p(x)} f(t) \, dt = \int_0^1 f\left( \frac{sq(x) + (1 - s)p(x)}{p(x)} \right) ds \]
for \( x \in X \) for which \( p(x), q(x), q(x) - p(x) \neq 0 \).
Therefore
\[
D_{f, HH}^f (p, q) := \int_X p(x) \int_0^{q(x)/p(x)} f(t) \, dt \frac{q(x)}{p(x)} - 1 \, d\mu(x)
= \int_X p(x) \left( \int_0^1 f \left( \frac{sq(x) + (1 - s)p(x)}{p(x)} \right) ds \right) d\mu(x)
= \int_0^1 \left( \int_X p(x) f \left( \frac{sq(x) + (1 - s)p(x)}{p(x)} \right) d\mu(x) \right) ds,
\]
where for the last equality we used Fubini’s theorem.
Since
\[
\int_X p(x) f \left( \frac{sq(x) + (1 - s)p(x)}{p(x)} \right) d\mu(x) = D_f (p, sq + (1 - s)p)
\]
hence
\[
\int_0^1 \left( \int_X p(x) f \left( \frac{sq(x) + (1 - s)p(x)}{p(x)} \right) d\mu(x) \right) ds
= \int_0^1 D_f (p, sq + (1 - s)p) \, ds
\]
and the equalities in are proved. \( \square \)

For \( s \in [0, 1] \) and the convex function \( f : (0, \infty) \to \mathbb{R} \) we define the \( s \)-weighted perspective \( \mathcal{P}_{f, s} : (0, \infty) \times (0, \infty) \to \mathbb{R} \) by
\[
(2.2) \quad \mathcal{P}_{f, s} (u, v) := uf \left( \frac{sv + (1 - s)u}{u} \right).
\]
We have the following lemma that is of interest in itself as well:

**Lemma 2.** Assume that the function \( f : (0, \infty) \to \mathbb{R} \) is convex, then for all \( s \in [0, 1] \) the \( s \)-weighted perspective \( \mathcal{P}_{f, s} \) is also convex as a function of two variables.
Proof. Let \((u, v), (w, z) \in (0, \infty) \times (0, \infty)\) and \(\alpha, \beta \geq 0\) with \(\alpha + \beta = 1\). Then

\[
\mathcal{P}_{f,s} (\alpha (u, v) + \beta (w, z)) \\
= \mathcal{P}_{f,s} (\alpha u + \beta w, \alpha v + \beta z) \\
= (\alpha u + \beta w) f \left( \frac{s (\alpha v + \beta z) + (1 - s) (\alpha u + \beta w)}{\alpha u + \beta w} \right) \\
= (\alpha u + \beta w) f \left( \frac{\alpha (sv + (1 - s) u) + \beta (sz + (1 - s) w)}{\alpha u + \beta w} \right) \\
= (\alpha u + \beta w) f \left( \frac{\alpha u sv + (1 - s) u}{u} + \beta w sz + (1 - s) w}{\alpha u + \beta w} \right) \\
\leq (\alpha u + \beta w) \\
\times \left[ \frac{\alpha u}{\alpha u + \beta w} f \left( \frac{sv + (1 - s) u}{u} \right) + \frac{\beta w}{\alpha u + \beta w} f \left( \frac{sz + (1 - s) w}{w} \right) \right] \\
= \alpha uf \left( \frac{sv + (1 - s) u}{u} \right) + \beta wf \left( \frac{sz + (1 - s) w}{w} \right) \\
= \alpha \mathcal{P}_{f,s} (u, v) + \beta \mathcal{P}_{f,s} (w, z) ,
\]

which proves the joint convexity of the perspective \(\mathcal{P}_{f,s}\). \(\square\)

Remark 1. If we use the perspective concept, then by (2.1) we also have

\[
(2.3) \quad D^f_{HH} (p, q) = \int_0^1 \left( \int_X \mathcal{P}_{f,s} (p(x), q(x)) d\mu(x) \right) ds .
\]

The following joint convexity of the HH \(f\)-divergence holds:

Theorem 2. Assume that the function \(f : (0, \infty) \rightarrow \mathbb{R}\) is convex and normalised, then
\(D^f_{HH}\) is convex as a mapping of two variables on \(\Omega \times \Omega\).

Proof. Let \((p_1, q_1), (p_2, q_2) \in \Omega\) and and \(\alpha, \beta \geq 0\) with \(\alpha + \beta = 1\). Then by the representation (2.3) and Lemma 2 we have

\[
D^f_{HH} (\alpha (p_1, q_1) + \beta (p_2, q_2)) \\
= D^f_{HH} (\alpha p_1 + \beta p_2, \alpha q_1 + \beta q_2) \\
= \int_0^1 \left( \int_X \mathcal{P}_{f,s} (\alpha p_1 (x) + \beta p_2 (x), \alpha q_1 (x) + \beta q_2 (x)) d\mu(x) \right) ds \\
= \int_0^1 \left( \int_X \mathcal{P}_{f,s} (\alpha (p_1 (x), q_1 (x)) + \beta (p_2 (x), q_2 (x))) d\mu(x) \right) ds \\
\geq \int_0^1 \left( \int_X [\alpha \mathcal{P}_{f,s} (p_1 (x), q_1 (x)) + \beta \mathcal{P}_{f,s} (p_2 (x), q_2 (x))] d\mu(x) \right) ds \\
= \alpha \int_0^1 \left( \int_X \mathcal{P}_{f,s} (p_1 (x), q_1 (x)) d\mu(x) \right) ds \\
+ \beta \int_0^1 \left( \int_X \mathcal{P}_{f,s} (p_2 (x), q_2 (x)) d\mu(x) \right) ds \\
= \alpha D^f_{HH} (p_1, q_1) + \beta D^f_{HH} (p_2, q_2) ,
\]

which proves the desired convexity. \(\square\)
3. Bounds in Terms of \( \chi^2 \)-Divergence

The above definitions \( D_f (p, q) \) and \( D_{HH}^f (p, q) \) can be extended to continuous functions \( f \) defined on \((0, \infty)\), however, in this general case, the positivity properties of the divergences under consideration do not hold in general.

We have:

**Theorem 3.** Assume that the function \( f : (0, \infty) \to \mathbb{R} \) is twice differentiable and normalised. Let \( 0 < r \leq 1 \leq R < \infty \) and \( p, q \in \Omega \) are such that

\[
(3.1) \quad r \leq \frac{q(x)}{p(x)} \leq R \text{ for } \mu\text{-almost every } x \in X.
\]

(i) If there exists a real number \( m \) such that

\[
(3.2) \quad m \leq f''(t) \text{ for all } t \in [r, R],
\]

then we have the inequality

\[
(3.3) \quad 0 \leq D_f \left( p, \frac{p + q}{2} \right) - \frac{1}{8} mD_{\chi^2} (p, q) \leq D_{HH}^f (p, q) - \frac{1}{6} mD_{\chi^2} (p, q).
\]

(ii) If there exists the real number \( M \) such that

\[
(3.4) \quad f''(t) \leq M \text{ for all } t \in [r, R],
\]

then we have the inequality

\[
(3.5) \quad 0 \leq \frac{1}{8} MD_{\chi^2} (p, q) - D_f \left( p, \frac{p + q}{2} \right) \leq \frac{1}{6} MD_{\chi^2} (p, q) - D_{HH}^f (p, q).
\]

**Proof.** (i) Consider the auxiliary function \( g_m : [r, R] \to \mathbb{R}, \ g_m(t) := f(t) - \frac{1}{2} m \ell^2(t) - 1 \), where \( \ell(t) = t \) is the identity function. This function is convex and normalized on \([r, R]\), since \( g_m \) is twice differentiable and

\[
(3.2) \quad g_m''(t) := f''(t) - m \geq 0 \text{ for all } t \in [r, R].
\]

We have for \( p, q \in \Omega \) that

\[
D_{HH}^{g_m} (p, q)
= D_{HH}^f (p, q) - \frac{1}{2} mD_{HH}^{\ell^2 - 1} (p, q)
= D_{HH}^f (p, q) - \frac{1}{2} m \int_X p(x) \left( \int_0^1 \left[ \left( \frac{sq(x) + (1 - s)p(x)}{p(x)} \right)^2 - 1 \right] ds \right) d\mu(x)
= D_{HH}^f (p, q) - \frac{1}{2} m \int_X p(x) \left( \int_0^1 \left( \frac{sq(x) + (1 - s)p(x)}{p(x)} \right)^2 ds \right) d\mu(x)
+ \frac{1}{2} m \int_X p(x) d\mu(x)
= D_{HH}^f (p, q) - \frac{1}{2} m \int_X p(x) \left( \int_0^1 \left( \frac{sq(x) + (1 - s)p(x)}{p(x)} \right)^2 ds \right) d\mu(x) + \frac{1}{2} m.
\]
Observe that
\[
\int_0^1 \left( \frac{sq(x) + (1-s)p(x)}{p(x)} \right)^2 ds = \int_0^1 \left[ s^2 \left( \frac{q(x)}{p(x)} \right)^2 + 2s(1-s) \frac{q(x)}{p(x)} + (1-s)^2 \right] ds
\]
\[
= \frac{1}{3} \left( \frac{q(x)}{p(x)} \right)^2 + \frac{1}{3} \frac{q(x)}{p(x)} + \frac{1}{3} = \frac{1}{3} \left[ \left( \frac{q(x)}{p(x)} \right)^2 + \frac{q(x)}{p(x)} + 1 \right]
\]

and

\[
\int_X p(x) \left( \int_0^1 \left( \frac{sq(x) + (1-s)p(x)}{p(x)} \right)^2 ds \right) d\mu(x)
\]
\[
= \frac{1}{3} \int_X p(x) \left( \left( \frac{q(x)}{p(x)} \right)^2 + \frac{q(x)}{p(x)} + 1 \right) d\mu(x)
\]
\[
= \frac{1}{3} \left[ \int_X p(x) \left( \frac{q(x)}{p(x)} \right)^2 d\mu(x) + \int_X p(x) \frac{q(x)}{p(x)} d\mu(x) + \int_X p(x) d\mu(x) \right]
\]
\[
= \frac{1}{3} \left[ \int_X q(x)^2 d\mu(x) + \int_X q(x) d\mu(x) + \int_X p(x) d\mu(x) \right]
\]
\[
= \frac{1}{3} \left[ \int_X \frac{q(x)}{p(x)} d\mu(x) + 1 + 1 \right] = \frac{1}{3} \left[ D_{\chi^2}(p, q) + 3 \right] = \frac{1}{3} D_{\chi^2}(p, q) + 1.
\]

Therefore
\[
D_{HH}^g (p, q) = D_{HH}^f (p, q) - \frac{1}{2} m \left[ \frac{1}{3} D_{\chi^2}(p, q) + 1 \right] + \frac{1}{2} m
\]
\[
= D_{HH}^f (p, q) - \frac{1}{6} m D_{\chi^2}(p, q).
\]

We also have
\[
D_{gm} \left( p, \frac{p+q}{2} \right) = D_f \left( p, \frac{p+q}{2} \right) - \frac{1}{2} m D_{\chi^2-1} \left( p, \frac{p+q}{2} \right)
\]
\[
= D_f \left( p, \frac{p+q}{2} \right) - \frac{1}{2} m D_{\chi^2} \left( p, \frac{p+q}{2} \right).
\]

Now,
\[
D_{\chi^2} \left( p, \frac{p+q}{2} \right) = \int_X p(x) \left[ \left( \frac{p(x)+q(x)}{2p(x)} \right)^2 - 1 \right] d\mu(x)
\]
\[
= \int_X p(x) \left[ \left( \frac{p(x) + q(x)}{2p(x)} \right)^2 - 1 \right] d\mu(x)
\]
\[
\frac{1}{4} \int_X p(x) \left( \frac{q(x)}{p(x)} \right)^2 + 2 \frac{q(x)}{p(x)} + 1 \right] \, d\mu(x)
\]
\[
= \frac{1}{4} \left[ \int_X \frac{q^2(x)}{p(x)} \, d\mu(x) + 2 \int_X \frac{q(x)}{p(x)} \, d\mu(x) + \int_X p(x) \, d\mu(x) \right] - 1
\]
\[
= \frac{1}{4} D_{\chi^2} (p, q) + 1 - 1 = \frac{1}{4} D_{\chi^2} (p, q),
\]
therefore
\[
D_{g_m} \left( p, \frac{p+q}{2} \right) = D_f \left( p, \frac{p+q}{2} \right) - \frac{1}{8} mD_{\ell^2} - 1 \left( p, \frac{p+q}{2} \right)
\]
\[
= D_f \left( p, \frac{p+q}{2} \right) - \frac{1}{8} mD_{\chi^2} (p, q).
\]
If we use the first inequality in (1.13) for \( g_m \), we have
\[
0 \leq D_{g_m} \left( p, \frac{p+q}{2} \right) \leq D_{g_m}^H (p, q),
\]
which by above calculations gives
\[
0 \leq D_f \left( p, \frac{p+q}{2} \right) - \frac{1}{8} mD_{\chi^2} (p, q) \leq D_f^H (p, q) - \frac{1}{6} mD_{\chi^2} (p, q).
\]
This proves (3.3).

(ii) Consider the auxiliary function \( g_M : [r, R] \to \mathbb{R}, g_M (t) := \frac{1}{2} M (\ell^2 (t) - 1) - f (t), \) where \( \ell (t) = t \) is the identity function. This function is convex and normalized on \([r, R]\), since \( g_M \) is twice differentiable and
\[
g''_M (t) = M - f'' (t) \geq 0 \text{ for all } t \in [r, R].
\]
Now, by using a similar argument to the one for the auxiliary function \( g_m \), we deduce the desired result (3.5).

**Corollary 1.** With the assumptions of Theorem 3 and if
\[
0 < m \leq f'' (t) \leq M < \infty \text{ for all } t \in [r, R],
\]
then we have
\[
\frac{1}{8} mD_{\chi^2} (p, q) \leq D_f \left( p, \frac{p+q}{2} \right) \leq \frac{1}{8} M D_{\chi^2} (p, q),
\]
\[
\frac{1}{6} mD_{\chi^2} (p, q) \leq D_f^H (p, q) \leq \frac{1}{6} M D_{\chi^2} (p, q)
\]
and
\[
\frac{1}{24} mD_{\chi^2} (p, q) \leq D_{H^H} (p, q) - D_f \left( p, \frac{p+q}{2} \right) \leq \frac{1}{24} M D_{\chi^2} (p, q).
\]
We also have:
Theorem 4. Assume that the function \( f : (0, \infty) \to \mathbb{R} \) is twice differentiable and normalised. Let \( 0 < r \leq 1 \leq R < \infty \) and \( p, q \in \Omega \) are such that the condition (3.1) is valid.

(i) If there exists a real number \( m \) such that the assumption (3.2) holds, then we have the inequality

\[
0 \leq D_{HH}^f(p, q) - \frac{1}{6} m D_{\chi^2}(p, q) \leq \frac{1}{2} D_f(p, q) - \frac{1}{4} m D_{\chi^2}(p, q) .
\]

(ii) If there exists the real number \( M \) such that the assumption (3.4) holds, then we have the inequality

\[
0 \leq \frac{1}{6} M D_{\chi^2}(p, q) - D_{HH}^f(p, q) \leq \frac{1}{2} M D_{\chi^2}(p, q) - D_f(p, q) .
\]

Proof. (i) Consider the auxiliary function \( g_m : [r, R] \to \mathbb{R} \), \( g_m(t) := f(t) - \frac{1}{2} m \left( \ell^2(t) - 1 \right) \), where \( \ell(t) = t \) is the identity function. This function is convex and normalized on \([r, R]\).

We have

\[
D_{HH}^{g_m}(p, q) = D_{HH}^f(p, q) - \frac{1}{6} m D_{\chi^2}(p, q)
\]

and

\[
D_{g_m}(p, q) := \int_X p(x) g_m \left[ \frac{q(x)}{p(x)} \right] d\mu(x)
= \int_X p(x) \left[ f \left( \frac{q(x)}{p(x)} \right) - \frac{1}{2} m \left( \ell^2 \left( \frac{q(x)}{p(x)} \right) - 1 \right) \right] d\mu(x)
= D_f(p, q) - \frac{1}{2} m D_{\chi^2}(p, q) .
\]

If we use the second inequality in (1.13) we have

\[
0 \leq D_{HH}^{g_m}(p, q) \leq \frac{1}{2} D_{g_m}(p, q) ,
\]

namely

\[
0 \leq D_{HH}^{f}(p, q) - \frac{1}{6} m D_{\chi^2}(p, q) \leq \frac{1}{2} \left[ D_f(p, q) - \frac{1}{2} m D_{\chi^2}(p, q) \right] \]

\[
= \frac{1}{2} D_f(p, q) - \frac{1}{4} m D_{\chi^2}(p, q) ,
\]

which proves (3.10).

(ii) Follows in a similar way for the auxiliary function \( g_M : [r, R] \to \mathbb{R} \), \( g_M(t) := \frac{1}{2} M \left( \ell^2(t) - 1 \right) - f(t) . \)

Corollary 2. With the assumptions of Theorem 3 and if the condition (3.6) holds, then we have

\[
\frac{1}{2} m D_{\chi^2}(p, q) \leq D_f(p, q) \leq \frac{1}{2} M D_{\chi^2}(p, q) \quad \text{(see also [4])}
\]

and

\[
\frac{1}{12} m D_{\chi^2}(p, q) \leq \frac{1}{2} D_f(p, q) - D_{HH}^f(p, q) \leq \frac{1}{12} M D_{\chi^2}(p, q) .
\]
Further, we observe that by using the definitions of the auxiliary mappings \(g_m(t)\) and \(g_M(t)\) we have

\[
g_m^+ (t) = (t - 1) \left( f(t) - \frac{1}{2} m (t^2 - 1) \right) ' = f^+ (t) - mt (t - 1)
\]

and

\[
g_M^+ (t) = Mt (t - 1) - f^+ (t).
\]

This gives

\[
D g_m^+ (p, q) = D f^+ (p, q) - m \int_X p(x) \frac{q(x)}{p(x)} \left( \frac{q(x)}{p(x)} - 1 \right) d\mu(x)
\]

\[
= D f^+ (p, q) - m D_{\chi^2} (p, q)
\]

and

\[
D g_M^+ (p, q) = M D_{\chi^2} (p, q) - D f^+ (p, q).
\]

**Theorem 5.** Assume that the function \(f : (0, \infty) \to \mathbb{R}\) is twice differentiable and normalised. Let \(0 < r \leq 1 \leq R < \infty\) and \(p, q \in \Omega\) are such that the condition (3.1) is valid.

(i) If there exists a real number \(m\) such that the assumption (3.2) holds, then we have the inequality

\[
0 \leq D_{HH}^f (p, q) - D_f \left( p, \frac{p + q}{2} \right) - \frac{1}{24} m D_{\chi^2} (p, q)
\]

\[
\leq \frac{1}{8} \left[ D f^+ (p, q) - m D_{\chi^2} (p, q) \right]
\]

and

\[
0 \leq \frac{1}{2} D_f (p, q) - D_{HH}^f (p, q) - \frac{1}{12} m D_{\chi^2} (p, q)
\]

\[
\leq \frac{1}{8} \left[ D f^+ (p, q) - m D_{\chi^2} (p, q) \right].
\]

(ii) If there exists the real number \(M\) such that the assumption (3.4) holds, then we have the inequality

\[
0 \leq \frac{1}{24} M D_{\chi^2} (p, q) - D_{HH}^f (p, q) + D_f \left( p, \frac{p + q}{2} \right)
\]

\[
\leq \frac{1}{8} \left[ M D_{\chi^2} (p, q) - D f^+ (p, q) \right]
\]

and

\[
0 \leq \frac{1}{12} M D_{\chi^2} (p, q) - \frac{1}{2} D_f (p, q) + D_{HH}^f (p, q)
\]

\[
\leq \frac{1}{8} \left[ M D_{\chi^2} (p, q) - D f^+ (p, q) \right].
\]

**Proof.** (i) If we use the inequality (1.16) for \(g_m\), then we have

\[
0 \leq D_{HH}^m (p, q) - D g_m \left( p, \frac{p + q}{2} \right) \leq \frac{1}{8} D g_m^+ (p, q),
\]
namely

\[
0 \leq D_{HH}^f(p, q) - \frac{1}{6} mD_{\chi^2}(p, q) - D_f\left(p, \frac{p + q}{2}\right) + \frac{1}{8} mD_{\chi^2}(p, q)
\]

\[
\leq \frac{1}{8} \left[ D_f^1(p, q) - mD_{\chi^2}(p, q) \right],
\]

which is equivalent to (3.16).

If we use (1.18) for \( g_m \), then we have

\[
0 \leq \frac{1}{2} D_{g_m}(p, q) - D_{HH}^g(p, q) \leq \frac{1}{8} D_{g_h}(p, q),
\]

namely

\[
0 \leq \frac{1}{2} \left[ D_f(p, q) - \frac{1}{2} mD_{\chi^2}(p, q) \right] - D_{HH}^f(p, q) + \frac{1}{6} mD_{\chi^2}(p, q)
\]

\[
\leq \frac{1}{8} \left[ D_f^1(p, q) - mD_{\chi^2}(p, q) \right],
\]

(ii) Follows in a similar way for \( g_M \).

Finally, we have:

**Corollary 3.** With the assumptions of Theorem 3 and if the condition (3.6) holds, then we have

\[
(3.20) \quad \frac{1}{12} mD_{\chi^2}(p, q) \leq \frac{1}{8} D_f^1(p, q) - D_{HH}^f(p, q) + D_f\left(p, \frac{p + q}{2}\right)
\]

\[
\leq \frac{1}{12} MD_{\chi^2}(p, q)
\]

and

\[
(3.21) \quad \frac{1}{24} mD_{\chi^2}(p, q) \leq \frac{1}{8} D_f(p, q) + D_{HH}^f(p, q) - \frac{1}{2} D_f(p, q) \leq \frac{1}{24} MD_{\chi^2}(p, q).
\]

4. **A Example**

We consider the convex and normalized function \( f : (0, \infty) \to R, f(t) = -\ln t \).

We have

\[
D_f(p, q) := D_{KL}(p, q)
\]

and

\[
D_f\left(p, \frac{p + q}{2}\right) = D_{LW}(p, q)
\]

for all \( p, q \in \Omega \).

We define the *identric mean* of two positive numbers \( a, b > 0 \)

\[
I(a, b) := \begin{cases} 
\frac{1}{b-a} \left( \frac{b^a}{a^b} \right)^{1/(b-a)} & \text{if } b \neq a, \\
\frac{b}{b-a} & \text{if } b = a.
\end{cases}
\]

We observe that

\[
\frac{1}{b-a} \int_a^b \ln t \, dt = \frac{b \ln b - b - b \ln b + a}{b - a} = \ln I(a, b).
\]
Therefore

\[ D_{HH}^f (p, q) = - \int_X p(x) \int_1^{q(x)} \ln t dt \frac{d \mu(x)}{q(x) - 1} = - \int_X p(x) \ln \left[ I \left( \frac{q(x)}{p(x)}, 1 \right) \right] d \mu(x) \]

\[ = \int_X p(x) \ln \left[ I \left( \frac{q(x)}{p(x)}, 1 \right) \right]^{-1} d \mu(x) =: D_{HH}^{KL} (p, q), \]

where we call \( D_{HH}^{KL} (p, q) \) the Kullback-Leibler HH divergence.

If \( 0 < r < 1 < R < \infty \) then for \( f(t) = - \ln t \),

\[ \inf_{t \in [r, R]} f''(t) = \inf_{t \in [r, R]} \frac{1}{t^2} = \frac{1}{R^2}, \quad \sup_{t \in [r, R]} f''(t) = \sup_{t \in [r, R]} \frac{1}{t^2} = \frac{1}{r^2}. \]

If \( p, q \in \Omega \) satisfy the condition (3.1), then by using (3.7)-(3.9) for \( m = \frac{1}{r^2} \) and \( M = \frac{1}{R^2} \) we get

\[ \frac{1}{8r^2} D_{\chi^2} (p, q) \leq D_{KL} (p, q) \leq \frac{1}{8r^2} D_{\chi^2} (p, q), \]

\[ \frac{1}{6r^2} D_{\chi^2} (p, q) \leq D_{HH}^{KL} (p, q) \leq \frac{1}{6r^2} D_{\chi^2} (p, q) \]

and

\[ \frac{1}{24r^2} D_{\chi^2} (p, q) \leq D_{HH}^{KL} (p, q) - D_{KL} (p, q) \leq \frac{1}{24r^2} D_{\chi^2} (p, q). \]

By (3.13), we also have

\[ \frac{1}{12r^2} D_{\chi^2} (p, q) \leq \frac{1}{2} D_{KL} (p, q) - D_{HH}^{KL} (p, q) \leq \frac{1}{12r^2} D_{\chi^2} (p, q). \]

Now, if \( f(t) = - \ln t \), then

\[ f'(t) := - \left( \frac{t - 1}{t} \right) = \frac{1}{t} - 1 \]

and

\[ D_{f'} (p, q) = \int_X p(x) \left( \frac{p(x)}{q(x)} - 1 \right) d \mu(x) = \int_X \left( \frac{p^2(x)}{q(x)} - p(x) \right) d \mu(x) \]

\[ = \int_X \left( \frac{p^2(x)}{q(x)} - 1 \right) d \mu(x) = D_{\chi^2} (q, p) \]

for all \( p, q \in \Omega \).

Finally, by the (3.20) and (3.21) we also have

\[ \frac{1}{12r^2} D_{\chi^2} (p, q) \leq \frac{1}{8} D_{\chi^2} (q, p) - D_{HH}^{KL} (p, q) + D_{KL} (p, q) \]

\[ \leq \frac{1}{12r^2} D_{\chi^2} (p, q) \]

and

\[ \frac{1}{24r^2} D_{\chi^2} (p, q) \leq \frac{1}{8} D_{\chi^2} (q, p) + D_{HH}^{KL} (p, q) - \frac{1}{2} D_{KL} (p, q) \]

\[ \leq \frac{1}{24r^2} D_{\chi^2} (p, q), \]

provided \( p, q \in \Omega \) satisfy the condition (3.1).
References


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