Complex Korovkin Theory via inequalities, a quantitative approach

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Abstract

Let K be a compact convex subset of $\mathbb{C}$ and $C(K,\mathbb{C})$ be the space of continuous functions from K into $\mathbb{C}$. We consider bounded linear operators from $C(K,\mathbb{C})$ into itself. We assume that these are bounded by companion real positive linear operators. We study quantitatively the rate of convergence of the approximation and high order approximation of these complex operators to the unit operators. Our results are inequalities of Korovkin type involving the complex modulus of continuity of the engaged function or its derivatives and basic test functions.

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1 Introduction

The study of the convergence of positive linear operators became more intensive and attractive when P. Korovkin (1953) proved his famous theorem (see [6], p. 14).

Korovkin's First Theorem. Let $[a,b]$ be a compact interval in $\mathbb{R}$ and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators L_n mapping $C([a,b])$ into itself. Assume that $(L_n f)$ converges uniformly to f for the three test functions $f=1,x,x^2$. Then $(L_n f)$ converges uniformly to f on $[a,b]$ for all functions of $f\in C([a,b])$.

So a lot of authors since then have worked on the theoretical aspects of the above convergence. But R. A. Mamedov (1959) (see [7]) was the first to put Korovkin's theorem in a quantitative scheme.

Mamedov's Theorem. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators in the space $C([a,b])$, for which $L_n 1=1$, $L_n(t,x)=x+\alpha_n(x)$,

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$L_n\left(t^2,x\right)=x^2+\beta_n(x)$. Then it holds

$$\left\|L_n(f,x)-f(x)\right\|_\infty\leq 3\omega_1\left(f,\sqrt{d_n}\right), \tag{1}$$

where ω_1 is the first modulus of continuity and $d_n=\left\|\beta_n(x)-2x\alpha_n(x)\right\|_\infty$.

An improvement of the last result was the following.

Shisha and Mond's Theorem. (1968, see [9]). Let $[a,b]\subset\mathbb{R}$ be a compact interval. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators acting on $C([a,b])$. For $n=1,2,\ldots$, suppose $L_n(1)$ is bounded. Let $f\in C([a,b])$. Then for $n=1,2,\ldots$, it holds

$$\left\|L_nf-f\right\|_\infty\leq\|f\|_\infty\cdot\left\|L_n1-1\right\|_\infty+\left\|L_n(1)+1\right\|_\infty\cdot\omega_1(f,\mu_n), \tag{2}$$

where

$$\mu_n:=\left\|\left(L_n\left((t-x)^2\right)\right)(x)\right\|_\infty^{\frac{1}{2}}. \tag{3}$$

Shisha-Mond inequality generated and inspired a lot of research done by many authors worldwide on the rate of convergence of a sequence of positive linear operators to the unit operator, always producing similar inequalities however in many different directions.

The author (see [1]) in his 1993 research monograph, produces in many directions best upper bounds for $\left|(L_nf)(x_0)-f(x_0)\right|$, $x_0\in Q\subseteq\mathbb{R}^n$, $n\geq 1$, compact and convex, which lead for the first time to sharp/attained inequalities of Shisha-Mond type. The method of proving is probabilistic from the theory of moments. His pointwise approach is closely related to the study of the weak convergence with rates of a sequence of finite positive measures to the unit measure at a specific point.

The author in [3], pp. 383-412 continued this work in an abstract setting: Let X be a normed vector space, Y be a Banach lattice; $M\subset X$ is a compact and convex subset. Consider the space of continuous functions from M into Y, denoted by $C(M,Y)$; also consider the space of bounded functions $B(M,Y)$. He studied the rate of the uniform convergence of lattice homomorphisms $T:C(M,Y)\to C(M,Y)$ or $T:C(M,Y)\to B(M,Y)$ to the unit operator I. See also [2].

Also the author in [4], pp. 175-188 continued the last abstract work for bounded linear operators that are bounded by companion real positive linear operators. Here the invoved functions are from $[a,b]\subset\mathbb{R}$ into $(X,\|\cdot\|)$ a Banach space.

All the above have inspired and motivated the work of this chapter. Our results are of Shisha-Mond type, i.e., of Korovkin type.

Namely here let K be a convex and compact subset of $\mathbb{C}$ and L be a linear operator from $C(K,\mathbb{C})$ into itself, and let $\widetilde{L}$ be a positive linear operator from $C(K,\mathbb{R})$ into itself, such that $|L(f)|\leq\lambda\widetilde{L}(|f|)$, $\forall\ f\in C(K,\mathbb{C})$, where $\lambda>0$.

Clearly then L is a bounded linear operator. Here we create a complete quantitative Korovkin type theory over the last described setting.

2 Preparation and Motivation

We need

Theorem 1 *([5]) Let $K \subseteq (\mathbb{C}, |\cdot|)$ and f a function from K into $\mathbb{C}$. Consider the first complex modulus of continuity*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in K \\ |x-y|<\delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{4}$$

We have:

(1)' If K is open convex or compact convex, then $\omega_1(f,\delta) < \infty$, $\forall\ \delta > 0$, where $f \in UC(K,\mathbb{C})$ (uniformly continuous functions).

(2)' If K is open convex or compact convex, then $\omega_1(f,\delta)$ is continuous on $\mathbb{R}_+$ in δ, for $f \in UC(K,\mathbb{C})$.

(3)' If K is convex, then

$$\omega_1(f, t_1 + t_2) \leq \omega_1(f, t_1) + \omega_1(f, t_2), \quad t_1, t_2 > 0, \tag{5}$$

that is the subadditivity property is true. Also it holds

$$\omega_1(f, n\delta) \leq n\omega_1(f,\delta) \tag{6}$$

and

$$\omega_1(f, \lambda\delta) \leq \lceil \lambda \rceil \omega_1(f,\delta) \leq (\lambda+1)\,\omega_1(f,\delta), \tag{7}$$

where $n \in \mathbb{N}$, $\lambda > 0$, $\delta > 0$, $\lceil\cdot\rceil$ is the ceiling of the number.

(4)' Clearly in general $\omega_1(f,\delta) \geq 0$ and is increasing in $\delta > 0$ and $\omega_1(f,0) = 0$.

(5)' If K is open or compact, then $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, iff $f \in UC(K,\mathbb{C})$.

(6)' It holds

$$\omega_1(f+g,\delta) \leq \omega_1(f,\delta) + \omega_1(g,\delta), \tag{8}$$

for $\delta > 0$, any $f, g : K \to \mathbb{C}$, $K \subset \mathbb{C}$ is arbitrary.

Next we give examples that motivate our main assumptions in this chapter.

Example 2 *Let $K \subset \mathbb{C}$ be a compact and convex set, $l : C(K,\mathbb{C}) \to \mathbb{C}$ a linear functional and $\widetilde{l} : C(K,\mathbb{R}) \to \mathbb{R}$ a positive linear functional. If $f \in C(K,\mathbb{C})$, then $|f| \in C(K,\mathbb{R})$. We want to see that*

$$|l(f)| \leq \lambda \widetilde{l}(|f|), \quad \text{where } \lambda > 0, \tag{9}$$

is possible.

Also, we want to see that

$$l(cg) = c\widetilde{l}(g), \quad \forall\ g \in C(K,\mathbb{R}), \ \forall\ c \in \mathbb{C}, \tag{10}$$

is also possible.

So here is a concrete example of $l, \widetilde{l}$.

Take $K = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{C}$ a rectangle. Here $z = x + iy \in \mathbb{C}$, and $f(z) = f_1(x, y) + i f_2(x, y)$. We have that $f \in C(K, \mathbb{C})$ iff $f_1, f_2 \in C(K, \mathbb{R})$. Define the following linear functional

$$l(f) := \int_{a_1}^{b_1} \int_{a_2}^{b_2} f_1(x, y)\, dxdy + i \int_{a_1}^{b_1} \int_{a_2}^{b_2} f_2(x, y)\, dxdy, \ \ \forall\ f \in C(K, \mathbb{C}). \tag{11}$$

This is a linear functional from $C(K, \mathbb{C}) \to \mathbb{C}$.

Let now $g \in C(K, \mathbb{R})$, then

$$l(g) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} g(x, y)\, dxdy \in \mathbb{R}, \tag{12}$$

so that $\widetilde{l} := l|_{C(K,\mathbb{R})}$ is a positive linear functional from $C(K, \mathbb{R})$ into $\mathbb{R}$.

Let $c \in K$, then $c = a + ib$, hence $cg = (a + ib)\, g = ag + ibg$, thus

$$\begin{aligned} l(cg) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} (ag(x, y))\, dxdy + i \int_{a_1}^{b_1} \int_{a_2}^{b_2} (bg(x, y))\, dxdy = \\ a \int_{a_1}^{b_1} \int_{a_2}^{b_2} g(x, y)\, dxdy + ib \int_{a_1}^{b_1} \int_{a_2}^{b_2} g(x, y)\, dxdy = \\ (a + ib) \int_{a_1}^{b_1} \int_{a_2}^{b_2} g(x, y)\, dxdy = c \int_{a_1}^{b_1} \int_{a_2}^{b_2} g(x, y)\, dxdy = c\widetilde{l}(g). \end{aligned} \tag{13}$$

Thus

$$l(cg) = c\widetilde{l}(g) \tag{14}$$

is true, where

$$\widetilde{l}(g) := \int_{a_1}^{b_1} \int_{a_2}^{b_2} g(x, y)\, dxdy, \ \ \forall\ g \in C(K, \mathbb{R}).$$

Next, we notice that

$$\begin{aligned} |l(f)| \leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} |f_1(x, y)|\, dxdy + \int_{a_1}^{b_1} \int_{a_2}^{b_2} |f_2(x, y)|\, dxdy = \\ \int_{a_1}^{b_1} \int_{a_2}^{b_2} (|f_1(x, y)| + |f_2(x, y)|)\, dxdy \leq \\ \sqrt{2} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \sqrt{(f_1(x, y))^2 + (f_2(x, y))^2} dxdy = \\ \sqrt{2} \int_{a_1}^{b_1} \int_{a_2}^{b_2} |f(z)|\, dxdy = \sqrt{2}\widetilde{l}(|f|). \end{aligned} \tag{15}$$

That is

$$|l(f)| \leq \sqrt{2}\widetilde{l}(|f|), \ \ \forall\ f \in C(K, \mathbb{C}) \tag{16}$$

is valid.

Relations (14) and (16) motivate our major assumptions of our theory here.

We continue with a more general example.

Example 3 *Let K be a compact and convex subset of $\mathbb{C}$, and $f \in C(K, \mathbb{C})$, which is $f(z) = u(x, y) + iv(x, y) = u + iv$, where $z = x + iy$, $z \in K$; $x, y \in \mathbb{R}$.*

All linearities here are over the field of $\mathbb{R}$.

Consider $\widetilde{L} : C(K, \mathbb{R}) \to C(K, \mathbb{R})$ a positive linear operator. And consider $L : C(K, \mathbb{C}) \to C(K, \mathbb{C})$ the linear operator such that:

$$L(f)(z) := \widetilde{L}(u)(x, y) + i\widetilde{L}(v)(x, y), \tag{17}$$

indeed L is a linear operator.

Notice from $|u| \le |u| \Leftrightarrow -|u| \le u \le |u| \Leftrightarrow -\widetilde{L}(|u|) \le \widetilde{L}(u) \le \widetilde{L}(|u|) \Leftrightarrow \left|\widetilde{L}(u)\right| \le \widetilde{L}(|u|)$.

Thus

$$\begin{gathered}|L(f)(z)| \le \left|\widetilde{L}(u)(x, y)\right| + \left|\widetilde{L}(v)(x, y)\right| \le \\ \widetilde{L}(|u|)(x, y) + \widetilde{L}(|v|)(x, y) = \widetilde{L}(|u| + |v|)(x, y) \le \\ \sqrt{2}\widetilde{L}\left(\sqrt{u^2 + v^2}\right)(x, y) = \sqrt{2}\widetilde{L}(|f(z)|) = \sqrt{2}\widetilde{L}(|f|)(z). \end{gathered} \tag{18}$$

We have proved that

$$(1) \qquad |L(f)|(z) \le \sqrt{2}\widetilde{L}(|f|)(z), \ \ \forall\ z \in K. \tag{19}$$

Next, let $g \in C(K, \mathbb{R})$, and $c \in \mathbb{C}$, i.e. $c = a + bi$; $a, b \in \mathbb{R}$. Then $cg = ag + ibg$.

Clearly $L(cg) = \widetilde{L}(ag) + i\widetilde{L}(bg) = a\widetilde{L}(g) + ib\widetilde{L}(g) = c\widetilde{L}(g)$.

That is true

$$(2) \qquad L(cg) = c\widetilde{L}(g), \ \ \forall\ c \in \mathbb{C} \text{ and } \forall\ g \in C(K, \mathbb{R}). \tag{20}$$

Properties (1) and (2), see (19), (20),justify our theory here. Notice that $f \in C(K, \mathbb{C})$, iff $u, v \in C(K, \mathbb{R})$.

Application 4 *Take $K := [0, 1]^2$, $z \in K$ ($z = x + iy$), $x, y \in [0, 1]$. Let $g \in C\left([0, 1]^2, \mathbb{R}\right)$, then the two-dimensional Bernstein polynomials are*

$$\begin{gathered} B_{n_1, n_2}(g)(x, y) := \\ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} g\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right) \binom{n_1}{k_1} \binom{n_2}{k_2} x^{k_1} (1 - x)^{n_1 - k_1} y^{k_2} (1 - y)^{n_2 - k_2}, \end{gathered} \tag{21}$$

and they converge uniformly to g, for $n_1, n_2 \to \infty$.

Thus, for $f \in C\left([0, 1]^2, \mathbb{C}\right)$, we define

$$B^{\mathbb{C}}_{n_1, n_2}(f)(z) := B_{n_1, n_2}(u)(x, y) + iB_{n_1, n_2}(v)(x, y), \tag{22}$$

the complex Bernstein operators.

Indeed it is

$$B^{\mathbb{C}}_{n_1,n_2}(f)(z) =$$

$$\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} u\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\binom{n_1}{k_1}\binom{n_2}{k_2} x^{k_1}(1-x)^{n_1-k_1} y^{k_2}(1-y)^{n_2-k_2} +$$

$$i\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} v\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\binom{n_1}{k_1}\binom{n_2}{k_2} x^{k_1}(1-x)^{n_1-k_1} y^{k_2}(1-y)^{n_2-k_2} =$$

$$\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\left[u\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)+iv\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\right]\binom{n_1}{k_1}\binom{n_2}{k_2} x^{k_1}(1-x)^{n_1-k_1} y^{k_2}(1-y)^{n_2-k_2}, \tag{23}$$

a complex linear operator.

Notice that

$$\left|B^{\mathbb{C}}_{n_1,n_2}(f)(z)-f(z)\right| =$$

$$\left|\left(B_{n_1,n_2}(u)(x,y)-u(x,y)\right)+i\left(B_{n_1,n_2}(v)(x,y)-v(x,y)\right)\right| =$$

$$\sqrt{\left(B_{n_1,n_2}(u)(x,y)-u(x,y)\right)^2+\left(B_{n_1,n_2}(v)(x,y)-v(x,y)\right)^2} =: (*). \tag{24}$$

We have that $|B_{n_1,n_2}(u)(x,y)-u(x,y)|<\varepsilon_1$, $\forall\ x,y\in[0,1]^2$, $\forall\ n_1,n_2\geq N_1$, *and* $|B_{n_1,n_2}(v)(x,y)-v(x,y)|<\varepsilon_2$, $\forall\ x,y\in[0,1]^2$, $\forall\ n_1,n_2\geq N_2$; $N_1,N_2\in\mathbb{N}$, *where* $\varepsilon_1,\varepsilon_2>0$.

Thus, it holds

$$(*)\leq\sqrt{\varepsilon_1^2+\varepsilon_2^2}=:\varepsilon, \tag{25}$$

$\forall\ x,y\in[0,1]^2$, $\forall\ n_1,n_2\geq\max(N_1,N_2)=:N^*$, $\varepsilon>0$.

Hence

$$\left|B^{\mathbb{C}}_{n_1,n_2}(f)(z)-f(z)\right|\leq\varepsilon,\ \forall\ z\in[0,1]^2,\ \forall\ n_1,n_2\geq N^*\in\mathbb{N},\ \text{where } \varepsilon>0.$$

Therefore $B^{\mathbb{C}}_{n_1,n_2}(f)\to f$, *uniformly convergent, as* $n_1,n_2\to\infty$.

3 Main Results

Let K be a compact and convex subset of $\mathbb{C}$. Consider $L: C(K,\mathbb{C})\to C(K,\mathbb{C})$ a linear operator and $\widetilde{L}: C(K,\mathbb{R})\to C(K,\mathbb{R})$ a positive linear operator (i.e. for $f_1.f_2\in C(K,\mathbb{R})$ with $f_1\geq f_2$ we get $\widetilde{L}(f_1)\geq\widetilde{L}(f_2)$) both over the field of $\mathbb{R}$.

We assume that

$$|L(f)|\leq\lambda\widetilde{L}(|f|),\ \ \forall\ f\in C(K,\mathbb{C}),\ \text{where } \lambda>0, \tag{27}$$

(i.e. $|L(f)(z)|\leq\lambda\widetilde{L}(|f|)(z)$, $\forall\ z\in K$).

We call $\widetilde{L}$ the companion operator of L.

Let $z_0 \in K$. Clearly, then $L(\cdot)(z_0)$ is a linear functional from $C(K,\mathbb{C})$ into $\mathbb{C}$, and $\widetilde{L}(\cdot)(z_0)$ is a positive linear functional from $C(K,\mathbb{R})$ into $\mathbb{R}$. Notice $L(f)(z) \in \mathbb{C}$ and $\widetilde{L}(|f|)(z) \in \mathbb{R}$, $\forall\ f \in C(K,\mathbb{C})$ (thus $|f| \in C(K,\mathbb{R})$). Here $L(f) \in C(K,\mathbb{C})$, and $\widetilde{L}(|f|) \in C(K,\mathbb{R})$, $\forall\ f \in C(K,\mathbb{C})$.

Notice that $C(K,\mathbb{C}) = UC(K,\mathbb{C})$, also $C(K,\mathbb{R}) = UC(K,\mathbb{R})$ (uniformly continuous functions).

By [3], p. 388, we have that $\widetilde{L}(|\cdot - z_0|^r)(z_0)$, $r>0$, is a continuous function in $z_0 \in K$.

We have the following approximation result with rates of Korovkin type.

Theorem 5 *Here K is a convex and compact subset of $\mathbb{C}$ and L_n is a sequence of linear operators from $C(K,\mathbb{C})$ into itself, $n \in \mathbb{N}$. There is a sequence of companion positive linear operators $\widetilde{L}_n$ from $C(K,\mathbb{R})$ into itself, such that*

$$|L_n(f)| \leq \lambda \widetilde{L}_n(|f|), \quad \lambda > 0, \forall\ f \in C(K,\mathbb{C}), \ \forall\ n \in \mathbb{N} \tag{28}$$

(i.e. $|L_n(f)(z_0)| \leq \lambda \left(\widetilde{L}_n(|f|)\right)(z_0)$, $\forall\ z_0 \in K$).

Additionally, we assume that

$$L_n(cg) = c\widetilde{L}_n(g), \quad \forall\ g \in C(K,\mathbb{R}), \ \forall\ c \in \mathbb{C} \tag{29}$$

(i.e. $(L_n(cg))(z_0) = c\left(\widetilde{L}_n(g)\right)(z_0)$, $\forall\ z_0 \in K$).

Then, for any $f \in C(K,\mathbb{C})$, we have

$$|(L_n(f))(z_0) - f(z_0)| \leq |f(z_0)| \left|\widetilde{L}_n(1(\cdot))(z_0) - 1\right| +$$

$$\lambda \left(\widetilde{L}_n(1(\cdot))(z_0) + 1\right) \omega_1\left(f, \widetilde{L}_n(|\cdot - z_0|)(z_0)\right), \tag{30}$$

$\forall\ z_0 \in K$, $\forall\ n \in \mathbb{N}$.

If $\widetilde{L}_n(1(\cdot))(z_0) = 1$, $\forall\ z_0 \in K$, then

$$|(L_n(f))(z_0) - f(z_0)| \leq 2\lambda\omega_1\left(f, \widetilde{L}_n(|\cdot - z_0|)(z_0)\right), \tag{31}$$

$\forall\ z_0 \in K$, $\forall\ n \in \mathbb{N}$.

If $\widetilde{L}_n(1(\cdot))(z_0) \to 1$, and $\widetilde{L}_n(|\cdot - z_0|)(z_0) \to 0$, as $n \to \infty$, then $L_n(f)(z_0) \to f(z_0)$, $\forall\ f \in C(K,\mathbb{C})$. Here $\widetilde{L}_n(1(\cdot))(z_0)$ is bounded.

Proof. We notice that

$$|(L_n(f))(z_0) - f(z_0)| =$$

$$|(L_n(f))(z_0) - L_n(f(z_0)(\cdot))(z_0) + L_n(f(z_0)(\cdot))(z_0) - f(z_0)| \overset{(29)}{=}$$

$$\left|(L_n(f))(z_0) - L_n(f(z_0)(\cdot))(z_0) + f(z_0)\widetilde{L}_n(1(\cdot))(z_0) - f(z_0)\right| \leq$$

$$|(L_n(f))(z_0) - L_n(f(z_0)(\cdot))(z_0)| + |f(z_0)| \left|\widetilde{L}_n(1(\cdot))(z_0) - 1\right| = \tag{32}$$

$$|L_n(f(\cdot)-f(z_0))(z_0)|+|f(z_0)|\left|\widetilde{L}_n(1(\cdot))(z_0)-1\right| \overset{(28)}{\leq}$$

$$|f(z_0)|\left|\widetilde{L}_n(1(\cdot))(z_0)-1\right|+\lambda\left(\widetilde{L}_n(|f(\cdot)-f(z_0)|)\right)(z_0)\leq$$

$$|f(z_0)|\left|\widetilde{L}_n(1(\cdot))(z_0)-1\right|+\lambda\left(\widetilde{L}_n\left(\omega_1\left(f,\frac{\delta|\cdot-z_0|}{\delta}\right)\right)\right)(z_0)\leq$$

$$|f(z_0)|\left|\widetilde{L}_n(1(\cdot))(z_0)-1\right|+\lambda\left(\widetilde{L}_n\left(\omega_1(f,\delta)\left(1(\cdot)+\frac{1}{\delta}|\cdot-z_0|\right)\right)\right)(z_0)=$$

$$|f(z_0)|\left|\widetilde{L}_n(1(\cdot))(z_0)-1\right|+\lambda\omega_1(f,\delta)\left[\widetilde{L}_n(1(\cdot))(z_0)+\frac{1}{\delta}\widetilde{L}_n(|\cdot-z_0|)(z_0)\right]= \tag{33}$$

$$|f(z_0)|\left|\widetilde{L}_n(1(\cdot))(z_0)-1\right|+\lambda\omega_1\left(f,\widetilde{L}_n(|\cdot-z_0|)(z_0)\right)\left[\widetilde{L}_n(1(\cdot))(z_0)+1\right], \tag{34}$$

by choosing

$$\delta:=\widetilde{L}_n(|\cdot-z_0|)(z_0), \tag{35}$$

if $\widetilde{L}_n(|\cdot-z_0|)(z_0)>0$.

Next we consider the case of

$$\widetilde{L}_n(|\cdot-z_0|)(z_0)=0. \tag{36}$$

By Riesz representation theorem ([8], p. 304) there exists a positive finite measure μ_{z_0} such that

$$\widetilde{L}_n(g)(z_0)=\int_K g(t)\,d\mu_{z_0}(t),\quad \forall\, g\in C(K,\mathbb{R}). \tag{37}$$

That is

$$\int_K |t-z_0|\,d\mu_{z_0}(t)=0,$$

which implies $|t-z_0|=0$, a.e., hence $t-z_0=0$, a.e., and $t=z_0$, a.e. on K. Consequently $\mu_{z_0}(\{t\in K: t\neq z_0\})=0$.

That is $\mu_{z_0}=\delta_{z_0}M$ (where $0<M:=\mu_{z_0}(K)=\widetilde{L}_n(1(\cdot))(z_0)$). Hence, in that case $\widetilde{L}_n(g)(z_0)=g(z_0)M$. Consequently, it holds $\omega_1\left(f,\widetilde{L}_n(|\cdot-z_0|)(z_0)\right)=0$, and the right hand side of (30) equals $|f(z_0)|\,|M-1|$.

Also, it is $\widetilde{L}_n(|f(\cdot)-f(z_0)(\cdot)|)(z_0)=|f(z_0)-f(z_0)|\,M=0$.

And by (28) we obtain

$$|(L_n(f(\cdot)-f(z_0)(\cdot)))(z_0)|=0,$$

that is

$$|L_n(f)(z_0)-L_n(f(z_0)(\cdot))(z_0)|=0.$$

The last says that

$$L_n(f)(z_0)=L_n(f(z_0)(\cdot))(z_0)\overset{(29)}{=}f(z_0)\,\widetilde{L}_n(1(\cdot))(z_0)=Mf(z_0).$$

Consequently the left hand side of (30) becomes

$$|L_n(f)(z_0) - f(z_0)| = |Mf(z_0) - f(z_0)| = |f(z_0)||M-1|.$$

So that (30) becomes an equality, and both sides equal $|f(z_0)||M-1|$ in the extreme case of $\widetilde{L}_n(|\cdot - z_0|)(z_0) = 0$. Thus inequality (30) is proved completely in both cases. ■

A similar result follows:

Theorem 6 *Here all as in Theorem 5. Then, for any $f \in C(K, \mathbb{C})$, we have*

$$|(L_n(f))(z_0) - f(z_0)| \leq |f(z_0)| \left| \widetilde{L}_n(1(\cdot))(z_0) - 1 \right| + \tag{38}$$

$$\lambda \left(\widetilde{L}_n(1(\cdot))(z_0) + 1 \right) \omega_1 \left(f, \left(\widetilde{L}_n \left(|\cdot - z_0|^2 \right)(z_0) \right)^{\frac{1}{2}} \right),$$

$\forall\ z_0 \in K$, $\forall\ n \in \mathbb{N}$.

If $\widetilde{L}_n(1(\cdot))(z_0) = 1$, $\forall\ z_0 \in K$, then

$$|(L_n(f))(z_0) - f(z_0)| \leq 2\lambda\omega_1 \left(f, \widetilde{L}_n \left(|\cdot - z_0|^2 \right)(z_0) \right)^{\frac{1}{2}}, \tag{39}$$

$\forall\ z_0 \in K$, $\forall\ n \in \mathbb{N}$.

Remark 7 *(to Theorem 6) If $\widetilde{L}_n(1(\cdot))(z_0) \to 1$, and $\widetilde{L}_n \left(|\cdot - z_0|^2 \right)(z_0) \to 0$, as $n \to \infty$, we get that $L_n(f)(z_0) \to f(z_0)$, $\forall\ f \in C(K, \mathbb{C})$. Here $\widetilde{L}_n(1(\cdot))(z_0)$ is bounded.*

For $t, z_0 \in K$ with $t = t_1 + it_2$ and $z_0 = z_{01} + iz_{02}$ we have

$$\widetilde{L}_n \left(|t - z_0|^2 \right)(z_0) = \widetilde{L}_n \left((t_1 - z_{01})^2 + (t_2 - z_{02})^2 \right)(z_0) = \tag{40}$$

$$\widetilde{L}_n \left((t_1 - z_{01})^2 \right)(z_0) + \widetilde{L}_n \left((t_2 - z_{02})^2 \right)(z_0).$$

so if $\widetilde{L}_n \left((t_1 - z_{01})^2 \right)(z_0)$ and $\widetilde{L}_n \left((t_2 - z_{02})^2 \right)(z_0)$ converge to zero, as $n \to \infty$, we get that $\widetilde{L}_n \left(|t - z_0|^2 \right)(z_0) \to 0$.

We also notice that

$$\widetilde{L}_n \left(|t - z_0|^2 \right)(z_0) = \left(\widetilde{L}_n \left(t_1^2 \right)(z_0) - z_{01}^2 \right) + \left(\widetilde{L}_n \left(t_2^2 \right)(z_2) - z_{02}^2 \right) + \tag{41}$$

$$|z|^2 \left(\widetilde{L}_n(1(\cdot))(z_0) - 1 \right) - 2z_{01} \left(\widetilde{L}_n(t_1)(z_0) - z_{01} \right) - 2z_{02} \left(\widetilde{L}_n(t_2)(z_0) - z_{02} \right).$$

Thus, if $\widetilde{L}_n(1(\cdot))(z_0) \to 1$, $\widetilde{L}_n(t_1)(z_0) \to z_{01}$, $\widetilde{L}_n(t_2)(z_0) \to z_{02}$, $\widetilde{L}_n \left(t_1^2 \right)(z_0) \to z_{01}^2$ and $\widetilde{L}_n \left(t_2^2 \right)(z_0) \to z_{02}^2$, as $n \to \infty$, then we get that $L_n(f)(z_0) \to f(z_0)$, $\forall\ f \in C(K, \mathbb{C})$.

Proof. of Theorem 6.

Let $t, z_0 \in K$ and $\delta > 0$. If $|t - z_0| > \delta$, then

$$|f(t) - f(z_0)| \leq \omega_1(f, |t - z_0|) = \omega_1\left(f, |t - z_0| \delta^{-1} \delta\right) \leq \tag{42}$$

$$\left(1 + \frac{|t - z_0|}{\delta}\right) \omega_1(f, \delta) \leq \left(1 + \frac{|t - z_0|^2}{\delta^2}\right) \omega_1(f, \delta).$$

The estimate

$$|f(t) - f(z_0)| \leq \left(1 + \frac{|t - z_0|^2}{\delta^2}\right) \omega_1(f, \delta) \tag{43}$$

also holds trivially when $|t - z_0| \leq \delta$.

So (43) is true always, $\forall\, t \in K$, for any $z_0 \in K$.

As in the proof of Theorem we have

$$|(L_n(f))(z_0) - f(z_0)| \leq ... \leq |f(z_0)| \left|\widetilde{L}_n(1(\cdot))(z_0) - 1\right|$$

$$+\lambda \left(\widetilde{L}_n(|f(\cdot) - f(z_0)|)\right)(z_0) \overset{(43)}{\leq} |f(z_0)| \left|\widetilde{L}_n(1(\cdot))(z_0) - 1\right| +$$

$$\lambda \left(\widetilde{L}_n\left(\left(1(\cdot) + \frac{|\cdot - z_0|^2}{\delta^2}\right) \omega_1(f, \delta)\right)\right)(z_0) =$$

$$|f(z_0)| \left|\widetilde{L}_n(1(\cdot))(z_0) - 1\right| + \lambda \omega_1(f, \delta) \left[\widetilde{L}_n(1(\cdot))(z_0) + \frac{1}{\delta^2} \widetilde{L}_n\left(|\cdot - z_0|^2\right)(z_0)\right] =$$

$$|f(z_0)| \left|\widetilde{L}_n(1(\cdot))(z_0) - 1\right| + \lambda \omega_1\left(f, \left(\widetilde{L}_n\left(|\cdot - z_0|^2\right)(z_0)\right)^{\frac{1}{2}}\right) \left[\widetilde{L}_n(1(\cdot))(z_0) + 1\right], \tag{44}$$

by choosing

$$\delta := \left(\widetilde{L}_n\left(|\cdot - z_0|^2\right)(z_0)\right)^{\frac{1}{2}}, \tag{45}$$

if $\widetilde{L}_n\left(|\cdot - z_0|^2\right)(z_0) > 0$.

Next we consider the case of

$$\widetilde{L}_n\left(|\cdot - z_0|^2\right)(z_0) = 0.$$

By Riesz representation theorem there exists a positive finite measure μ_{z_0} such that

$$\widetilde{L}_n(g)(z_0) = \int_K g(t)\, d\mu_{z_0}(t), \quad \forall\, g \in C(K, \mathbb{R}). \tag{46}$$

That is

$$\int_K |t - z_0|^2\, d\mu_{z_0}(t) = 0,$$

which implies $|t - z_0|^2 = 0$, a.e., hence $|t - z_0| = 0$, a.e., thus $t - z_0 = 0$, a.e., and $t = z_0$, a.e. on K. Consequently $\mu_{z_0}(\{t \in K : t \neq z_0\}) = 0$.

That is $\mu_{z_0} = \delta_{z_0} M$ (where $0 < M := \mu_{z_0}(K) = \widetilde{L}_n(1(\cdot))(z_0)$). Hence, in that case $\widetilde{L}_n(g)(z_0) = g(z_0) M$.

Consequently, it holds $\omega_1 \left(f, \left(\widetilde{L}_n \left(|\cdot - z_0|^2 \right)(z_0) \right)^{\frac{1}{2}} \right) = 0$, and the right hand side of (38) equals $|f(z_0)| |M-1|$. Also, it is $\widetilde{L}_n(|f(\cdot) - f(z_0)(\cdot)|)(z_0) = |f(z_0) - f(z_0)| M = 0$.

And by (28) we obtain

$$|(L_n(f(\cdot) - f(z_0)(\cdot)))(z_0)| = 0,$$

that is

$$|L_n(f)(z_0) - L_n(f(z_0)(\cdot))(z_0)| = 0.$$

The last says that

$$L_n(f)(z_0) = L_n(f(z_0)(\cdot))(z_0) \overset{(29)}{=} f(z_0) \widetilde{L}_n(1(\cdot))(z_0) = M f(z_0).$$

Consequently the left hand side of (38) becomes

$$|L_n(f)(z_0) - f(z_0)| = |Mf(z_0) - f(z_0)| = |f(z_0)| |M-1|.$$

So that (38) becomes an equality, and both sides equal $|f(z_0)| |M-1|$ in the extreme case of $\widetilde{L}_n \left(|\cdot - z_0|^2 \right)(z_0) = 0$. Thus inequality (38) is proved completely in both cases. ■

We give

Corollary 8 *All as in Theorem 5, $z_0 \in K$. Then*

$$\|L_n(f) - f\|_\infty \le \|f\|_\infty \left\| \widetilde{L}_n(1(\cdot)) - 1 \right\|_\infty + \tag{47}$$

$$\lambda \left\| \widetilde{L}_n(1(\cdot)) + 1 \right\|_\infty \omega_1 \left(f, \left\| \widetilde{L}_n(|\cdot - z_0|)(z_0) \right\|_{\infty, z_0} \right),$$

$\forall\, n \in \mathbb{N}$.

If $\widetilde{L}_n(1(\cdot)) = 1$, then

$$\|L_n(f) - f\|_\infty \le 2\lambda\omega_1 \left(f, \left\| \widetilde{L}_n(|\cdot - z_0|)(z_0) \right\|_{\infty, z_0} \right), \tag{48}$$

$\forall\, n \in \mathbb{N}$.

As $\widetilde{L}_n(1) \overset{u}{\to} 1$, and $\widetilde{L}_n(|\cdot - z_0|)(z_0) \overset{u}{\to} 0$ (u is uniformly), as $n \to \infty$, then $L_n(f) \overset{u}{\to} f$, $\forall\, f \in C(K, \mathbb{C})$. Notice $\widetilde{L}_n(1)$ is bounded, and all suprema in (47) are finite.

Corollary 9 *All as in Theorem 6, $z_0 \in K$. Then*

$$\|L_n(f) - f\|_\infty \le \|f\|_\infty \left\| \widetilde{L}_n(1(\cdot)) - 1 \right\|_\infty + \tag{49}$$

$$\lambda \left\| \widetilde{L}_n (1(\cdot)) + 1 \right\|_\infty \omega_1 \left(f, \left\| \widetilde{L}_n \left(|\cdot - z_0|^2 \right) (z_0) \right\|_{\infty, z_0}^{\frac{1}{2}} \right),$$

$\forall\ n \in \mathbb{N}$.

If $\widetilde{L}_n (1(\cdot)) = 1$, *then*

$$\| L_n (f) - f \|_\infty \leq 2\lambda \omega_1 \left(f, \left\| \widetilde{L}_n \left(|\cdot - z_0|^2 \right) (z_0) \right\|_{\infty, z_0}^{\frac{1}{2}} \right), \tag{50}$$

$\forall\ n \in \mathbb{N}$.

As $\widetilde{L}_n (1) \overset{u}{\to} 1$, *and* $\widetilde{L}_n \left(|\cdot - z_0|^2 \right) (z_0) \overset{u}{\to} 0$, *then* $L_n (f) \overset{u}{\to} f$, *as* $n \to \infty$, $\forall$ $f \in C(K, \mathbb{C})$.

We need

Theorem 10 *([5]) Let* $K \subseteq \mathbb{C}$ *convex,* $x_0 \in K^0$ *(interior of* K*) and* $f : K \to \mathbb{C}$ *such that* $|f(t) - f(x_0)|$ *is convex in* $t \in K$*. Furthermore let* $\delta > 0$ *so that the closed disk* $D(x_0, \delta) \subset K$*. Then*

$$|f(t) - f(x_0)| \leq \frac{\omega_1 (f, \delta)}{\delta} |t - x_0|, \quad \forall\ t \in K. \tag{51}$$

We present a convex Korovkin type result:

Theorem 11 *Here all as in Theorem 5. Let a fixed* $z_0 \in K^0$ *and assume that* $|f(t) - f(z_0)|$ *is convex in* $t \in K$*. Assume the closed disk* $D\left(z_0, \widetilde{L}_n (|\cdot - z_0|)(z_0) \right)$ $\subset K$*. Then*

$$|(L_n (f))(z_0) - f(z_0)| \leq |f(z_0)| \left| \widetilde{L}_n (1(\cdot))(z_0) - 1 \right|$$

$$+ \lambda \omega_1 \left(f, \widetilde{L}_n (|\cdot - z_0|)(z_0) \right), \quad \forall\ n \in \mathbb{N}. \tag{52}$$

As $\widetilde{L}_n (1(\cdot))(z_0) \to 1$, *and* $\widetilde{L}_n (|\cdot - z_0|)(z_0) \to 0$, *then* $(L_n (f))(z_0) \to f(z_0)$, *as* $n \to \infty$.

Proof. As in the proof of Theorem 5 we have

$$|(L_n (f))(z_0) - f(z_0)| \leq$$

$$|f(z_0)| \left| \widetilde{L}_n (1(\cdot))(z_0) - 1 \right| + \lambda \left(\widetilde{L}_n (|f(\cdot) - f(z_0)|) \right)(z_0) \overset{(51)}{\leq}$$

$(\delta > 0 : D(z_0, \delta) \subset K)$

$$|f(z_0)| \left| \widetilde{L}_n (1(\cdot))(z_0) - 1 \right| + \frac{\lambda \omega_1 (f, \delta)}{\delta} \widetilde{L}_n (|\cdot - z_0|)(z_0) =$$

$$|f(z_0)| \left| \widetilde{L}_n (1(\cdot))(z_0) - 1 \right| + \lambda \omega_1 \left(f, \widetilde{L}_n (|\cdot - z_0|)(z_0) \right), \tag{53}$$

by choosing

$$\delta := \widetilde{L}_n\left(|\cdot - z_0|\right)(z_0),$$

if $\widetilde{L}_n\left(|\cdot - z_0|\right)(z_0) > 0$.

The case $\widetilde{L}_n\left(|\cdot - z_0|\right)(z_0) = 0$ is treated similarly as in the proof of Theorem 5. The theorem is proved. ■

We make

Remark 12 *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $y, x \in D$, then we have the following Taylor's expansion with integral remainder*

$$f(y) = \sum_{k=0}^{N-1} \frac{f^{(k)}(x)}{k!}(y-x)^k +$$

$$\frac{1}{(N-1)!}(y-x)^N \int_0^1 f^{(N)}\left[(1-s)x + sy\right](1-s)^{N-1} ds, \tag{54}$$

for $N \in \mathbb{N}$, see [10], p. 8.

Clearly then

$$f(y) = \sum_{k=0}^{N} \frac{f^{(k)}(x)}{k!}(y-x)^k +$$

$$\frac{1}{(N-1)!}(y-x)^N \int_0^1 \left[f^{(N)}\left[(1-s)x + sy\right] - f^{(N)}(x)\right](1-s)^{N-1} ds, \tag{55}$$

for $N \in \mathbb{N}$.

Call the remainder of (55) as

$$R_N(x,y) := \frac{(y-x)^N}{(N-1)!} \int_0^1 \left[f^{(N)}\left[(1-s)x + sy\right] - f^{(N)}(x)\right](1-s)^{N-1} ds. \tag{56}$$

We have that

$$|R_N(x,y)| \le \frac{|y-x|^N}{(N-1)!} \int_0^1 \left|f^{(N)}\left[(1-s)x + sy\right] - f^{(N)}(x)\right|(1-s)^{N-1} ds =: (*), \tag{57}$$

$N \in \mathbb{N}$.

Next assume $f^{(N)} \in UC(D, \mathbb{C})$.

We observe that

$$(*) \le \frac{|y-x|^N}{(N-1)!} \int_0^1 \omega_1\left(f^{(N)}, \frac{\delta s |y-x|}{\delta}\right)(1-s)^{N-1} ds \le$$

$$\frac{|y-x|^N}{(N-1)!}\omega_1\left(f^{(N)}, \delta\right) \int_0^1 \left[1 + \frac{s|y-x|}{\delta}\right](1-s)^{N-1} ds =$$

$$\frac{|y-x|^N}{(N-1)!}\omega_1\left(f^{(N)}, \delta\right)\left[\int_0^1 (1-s)^{N-1} ds + \frac{|y-x|}{\delta}\int_0^1 (1-s)^{N-1}(s-0)^{2-1} ds\right] =$$

$$\frac{|y-x|^N}{(N-1)!}\omega_1\left(f^{(N)},\delta\right)\left[\frac{1}{N}+\frac{|y-x|}{\delta}\frac{1}{N(N+1)}\right]=$$

$$\frac{|y-x|^N}{N!}\omega_1\left(f^{(N)},\delta\right)\left[1+\frac{|y-x|}{\delta(N+1)}\right]. \tag{59}$$

We have proved

$$|R_N(x,y)|\leq\frac{|y-x|^N}{N!}\omega_1\left(f^{(N)},\delta\right)\left[1+\frac{|y-x|}{\delta(N+1)}\right], \tag{60}$$

$N\in\mathbb{N}$, $\delta>0$.

The last means that

$$\left|f(y)-\sum_{k=0}^{N}\frac{f^{(k)}(x)}{k!}(y-x)^k\right|\leq$$

$$\omega_1\left(f^{(N)},\delta\right)\frac{|y-x|^N}{N!}\left[1+\frac{|y-x|}{\delta(N+1)}\right], \tag{61}$$

$N\in\mathbb{N}$, $\delta>0$, $\forall\ x,y\in D$, *where* $f^{(N)}\in UC(D,\mathbb{C})$.

We make

Remark 13 *Let* $f:K\subseteq\mathbb{C}\to\mathbb{C}$ *be an analytic function on the convex and compact set* K*, and* $z_0\in K$*, where* $\delta>0$.

Then, as in (61), we get

$$\left|f(\cdot)-\sum_{k=0}^{N}\frac{f^{(k)}(z_0)}{k!}(\cdot-z_0)^k\right|\leq$$

$$\omega_1\left(f^{(N)},\delta\right)\frac{|\cdot-z_0|^N}{N!}\left[1+\frac{|\cdot-z_0|}{\delta(N+1)}\right], \tag{62}$$

$\forall\ N\in\mathbb{N}$. *Here* ω_1 *is on* K.

Above we mean that $f:D\subseteq\mathbb{C}\to\mathbb{C}$ is analytic on the convex domain D, where $K\subseteq D$. For convenience we set and use $f=f|_K$.

We have proved

Theorem 14 *Let* $f:K\subseteq\mathbb{C}\to\mathbb{C}$ *be an analytic function on the convex and compact set* K*;* $z_0\in K$*,* $\delta>0$*, and* $f^{(k)}(z_0)=0$*,* $k=1,2,...,N$*. Then*

$$|f(\cdot)-f(z_0)|\leq\frac{\omega_1\left(f^{(N)},\delta\right)}{N!}\left[|\cdot-z_0|^N+\frac{|\cdot-z_0|^{N+1}}{\delta(N+1)}\right], \tag{63}$$

over K*,* $N\in\mathbb{N}$.

We present higher order of approximation:

Theorem 15 *Here K is a convex and compact subset of $\mathbb{C}$ and L_n is a sequence of linear operators from $C(K,\mathbb{C})$ into itself, $n \in \mathbb{N}$. There is a sequence of companion positive linear operators $\widetilde{L}_n$ from $C(K,\mathbb{R})$ into itself, such that*

$$|L_n(f)| \leq \lambda \widetilde{L}_n(|f|), \quad \lambda > 0, \ \forall\, f \in C(K,\mathbb{C}), \ \forall\, n \in \mathbb{N}. \tag{64}$$

Additionally, we assume that

$$L_n(cg) = c\widetilde{L}_n(g), \quad \forall\, g \in C(K,\mathbb{R}), \ \forall\, c \in \mathbb{C}. \tag{65}$$

Here we consider $f : K \to \mathbb{C}$ that are analytic, so that $f^{(k)}(z_0) = 0$, $k = 1,2,...,N$, where $z_0 \in K$.

Then

$$|(L_n(f))(z_0) - f(z_0)| \leq |f(z_0)| \left|\widetilde{L}_n(1(\cdot))(z_0) - 1\right| +$$

$$\frac{\lambda\omega_1\left(f^{(N)}, \left(\left(\widetilde{L}_n\left(|\cdot - z_0|^{N+1}\right)\right)(z_0)\right)^{\frac{1}{(N+1)}}\right)}{N!} \left(\left(\widetilde{L}_n\left(|\cdot - z_0|^{N+1}\right)\right)(z_0)\right)^{\left(\frac{N}{N+1}\right)}$$

$$\left[\left(\widetilde{L}_n(1(\cdot))(z_0)\right)^{\frac{1}{(N+1)}} + \frac{1}{(N+1)}\right], \tag{66}$$

$\forall\, n \in \mathbb{N}$.

If $\widetilde{L}_n(1(\cdot))(z_0) = 1$, then

$$|(L_n(f))(z_0) - f(z_0)| \leq \frac{\lambda(N+2)\,\omega_1\left(f^{(N)}, \left(\left(\widetilde{L}_n\left(|\cdot - z_0|^{N+1}\right)\right)(z_0)\right)^{\frac{1}{(N+1)}}\right)}{(N+1)!}$$

$$\left(\left(\widetilde{L}_n\left(|\cdot - z_0|^{N+1}\right)\right)(z_0)\right)^{\left(\frac{N}{N+1}\right)}, \tag{67}$$

$\forall\, n \in \mathbb{N}$.

If $\widetilde{L}_n(1(\cdot))(z_0) \to 1$ and $\widetilde{L}_n\left(|\cdot - z_0|^{N+1}\right)(z_0) \to 0$, then $(L_n(f))(z_0) \to f(z_0)$, as $n \to \infty$. Here $\widetilde{L}_n(1(\cdot))(z_0)$ is bounded.

Proof. We notice that

$$|(L_n(f))(z_0) - f(z_0)| =$$

$$|(L_n(f))(z_0) - L_n(f(z_0)(\cdot))(z_0) + L_n(f(z_0)(\cdot))(z_0) - f(z_0)| \overset{(65)}{=}$$

$$\left|(L_n(f))(z_0) - L_n(f(z_0)(\cdot))(z_0) + f(z_0)\,\widetilde{L}_n(1(\cdot))(z_0) - f(z_0)\right| \leq$$

$$|(L_n(f))(z_0) - L_n(f(z_0)(\cdot))(z_0)| + |f(z_0)| \left|\widetilde{L}_n(1(\cdot))(z_0) - 1\right| = \tag{68}$$

$$\left|L_n\left(f\left(\cdot\right)-f\left(z_0\right)\right)\left(z_0\right)\right|+\left|f\left(z_0\right)\right|\left|\widetilde{L}_n\left(1\left(\cdot\right)\right)\left(z_0\right)-1\right|\overset{(64)}{\leq}$$

$$\left|f\left(z_0\right)\right|\left|\widetilde{L}_n\left(1\left(\cdot\right)\right)\left(z_0\right)-1\right|+\lambda\left(\widetilde{L}_n\left(\left|f\left(\cdot\right)-f\left(z_0\right)\right|\right)\right)\left(z_0\right)\overset{(63)}{\leq}$$

$$\left|f\left(z_0\right)\right|\left|\widetilde{L}_n\left(1\left(\cdot\right)\right)\left(z_0\right)-1\right|+$$

$$\lambda\frac{\omega_1\left(f^{(N)},\delta\right)}{N!}\left[\widetilde{L}_n\left(\left|\cdot-z_0\right|^N\right)\left(z_0\right)+\frac{1}{\delta\left(N+1\right)}\widetilde{L}_n\left(\left|\cdot-z_0\right|^{N+1}\right)\left(z_0\right)\right]=:\left(*\right).$$

By Hölder's inequality and Riesz representation theorem we obtain

$$\widetilde{L}_n\left(\left|\cdot-z_0\right|^N\right)\left(z_0\right)\leq\left(\left(\widetilde{L}_n\left(\left|\cdot-z_0\right|^{N+1}\right)\right)\left(z_0\right)\right)^{\left(\frac{N}{N+1}\right)}\left(\widetilde{L}_n\left(1\left(\cdot\right)\right)\left(z_0\right)\right)^{\frac{1}{N+1}}. \tag{70}$$

Therefore

$$\left(*\right)\leq\left|f\left(z_0\right)\right|\left|\widetilde{L}_n\left(1\left(\cdot\right)\right)\left(z_0\right)-1\right|+$$

$$\lambda\frac{\omega_1\left(f^{(N)},\delta\right)}{N!}\left[\left(\widetilde{L}_n\left(\left|\cdot-z_0\right|^{N+1}\right)\left(z_0\right)\right)^{\left(\frac{N}{N+1}\right)}\left(\widetilde{L}_n\left(1\left(\cdot\right)\right)\left(z_0\right)\right)^{\frac{1}{N+1}}\right.$$

$$\left.+\frac{1}{\delta\left(N+1\right)}\widetilde{L}_n\left(\left|\cdot-z_0\right|^{N+1}\right)\left(z_0\right)\right]=:\left(\xi\right). \tag{71}$$

We choose

$$\delta:=\left(\left(\widetilde{L}_n\left(\left|\cdot-z_0\right|^{N+1}\right)\right)\left(z_0\right)\right)^{\frac{1}{(N+1)}}, \tag{72}$$

in case of $\widetilde{L}_n\left(\left|\cdot-z_0\right|^{N+1}\right)\left(z_0\right)>0$.

Then it holds

$$\left(\xi\right)=\left|f\left(z_0\right)\right|\left|\widetilde{L}_n\left(1\left(\cdot\right)\right)\left(z_0\right)-1\right|+$$

$$\frac{\lambda\omega_1\left(f^{(N)},\left(\left(\widetilde{L}_n\left(\left|\cdot-z_0\right|^{N+1}\right)\right)\left(z_0\right)\right)^{\frac{1}{(N+1)}}\right)}{N!}$$

$$\left[\delta^N\left(\widetilde{L}_n\left(1\left(\cdot\right)\right)\left(z_0\right)\right)^{\frac{1}{N+1}}+\frac{\delta^N}{\left(N+1\right)}\right]=$$

$$\left|f\left(z_0\right)\right|\left|\widetilde{L}_n\left(1\left(\cdot\right)\right)\left(z_0\right)-1\right|+$$

$$\frac{\lambda\omega_1\left(f^{(N)},\left(\left(\widetilde{L}_n\left(\left|\cdot-z_0\right|^{N+1}\right)\right)\left(z_0\right)\right)^{\frac{1}{(N+1)}}\right)}{N!}\left(\left(\widetilde{L}_n\left(\left|\cdot-z_0\right|^{N+1}\right)\right)\left(z_0\right)\right)^{\left(\frac{N}{N+1}\right)}$$

$$\left[\left(\widetilde{L}_n\left(1\left(\cdot\right)\right)\left(z_0\right)\right)^{\frac{1}{(N+1)}}+\frac{1}{\left(N+1\right)}\right]. \tag{73}$$

Next we treat the case of

$$\widetilde{L}_n\left(\left|\cdot-z_0\right|^{N+1}\right)\left(z_0\right)=0.$$

By Riesz representation theorem there exists a positive finite measure μ_{z_0} such that

$$\widetilde{L}_n (g) (z_0) = \int_K g (t) \, d\mu_{z_0} (t) , \quad \forall \, g \in C (K, \mathbb{R}) . \tag{74}$$

That is

$$\int_K |t - z_0|^{N+1} \, d\mu_{z_0} (t) = 0,$$

which implies $|t - z_0|^{N+1} = 0$, a.e., hence $|t - z_0| = 0$, a.e., thus $t - z_0 = 0$, a.e., and $t = z_0$, a.e. on K. Consequently $\mu_{z_0} (\{t \in K : t \neq z_0\}) = 0$.

That is $\mu_{z_0} = \delta_{z_0} M$ (where $0 < M := \mu_{z_0} (K) = \widetilde{L}_n (1 (\cdot)) (z_0)$). Hence, in that case $\widetilde{L}_n (g) (z_0) = g (z_0) M$.

Consequently, it holds $\omega_1 \left(f^{(N)}, \left(\widetilde{L}_n \left(|\cdot - z_0|^{N+1} \right) (z_0) \right)^{\frac{1}{(N+1)}} \right) = 0$, and the right hand side of (66) equals $|f (z_0)| \, |M - 1|$. Also, it is $\widetilde{L}_n (|f (\cdot) - f (z_0) (\cdot)|) (z_0) = |f (z_0) - f (z_0)| \, M = 0$.

And by (64) we obtain

$$|(L_n (f (\cdot) - f (z_0) (\cdot))) (z_0)| = 0,$$

that is

$$|L_n (f) (z_0) - L_n (f (z_0) (\cdot)) (z_0)| = 0.$$

The last says that

$$L_n (f) (z_0) = L_n (f (z_0) (\cdot)) (z_0) \overset{(65)}{=} f (z_0) \, \widetilde{L}_n (1 (\cdot)) (z_0) = M f (z_0) .$$

Consequently the left hand side of (66) becomes

$$|L_n (f) (z_0) - f (z_0)| = |M f (z_0) - f (z_0)| = |f (z_0)| \, |M - 1| .$$

So that (66) becomes an equality, and both sides equal $|f (z_0)| \, |M - 1|$ in the extreme case of $\widetilde{L}_n \left(|\cdot - z_0|^{N+1} \right) (z_0) = 0$. Thus inequality (66) is proved completely in both cases. ∎

We give

Corollary 16 *All as in Theorem 15. Here $N = 1$, i.e. $f' (z_0) = 0$. Then*

$$|(L_n (f)) (z_0) - f (z_0)| \leq |f (z_0)| \left| \widetilde{L}_n (1 (\cdot)) (z_0) - 1 \right| +$$

$$\lambda \omega_1 \left(f', \left(\left(\widetilde{L}_n \left(|\cdot - z_0|^2 \right) \right) (z_0) \right)^{\frac{1}{2}} \right) \left(\left(\widetilde{L}_n \left(|\cdot - z_0|^2 \right) \right) (z_0) \right)^{\frac{1}{2}}$$

$$\left[\left(\widetilde{L}_n (1 (\cdot)) (z_0) \right)^{\frac{1}{2}} + \frac{1}{2} \right] , \quad \forall \, n \in \mathbb{N}. \tag{75}$$

If $\widetilde{L}_n(1(\cdot))(z_0) = 1$, *then*

$$|(L_n(f))(z_0) - f(z_0)| \leq \frac{3\lambda\omega_1\left(f', \left(\left(\widetilde{L}_n\left(|\cdot - z_0|^2\right)\right)(z_0)\right)^{\frac{1}{2}}\right)}{2}$$

$$\left(\left(\widetilde{L}_n\left(|\cdot - z_0|^2\right)\right)(z_0)\right)^{\frac{1}{2}}, \quad \forall\, n \in \mathbb{N}. \tag{76}$$

If $\widetilde{L}_n(1(\cdot))(z_0) \to 1$ *and* $\widetilde{L}_n\left(|\cdot - z_0|^2\right)(z_0) \to 0$, *as* $n \to \infty$, *we get that* $(L_n(f))(z_0) \to f(z_0)$.

We make

Remark 17 *Let* $f : D \subseteq \mathbb{C} \to \mathbb{C}$ *be an analytic function on the convex domain* D *and* K *be a compact and convex subset of* D *and* $t, z_0 \in K$, *with* $z_0 \in K^0$ *(interior of* K*), then we have the following modified Taylor's expansion with integral remainder:*

$$f(t) = \sum_{k=0}^{N} \frac{f^{(k)}(z_0)}{k!}(t - z_0)^k +$$

$$\frac{(t - z_0)^N}{(N-1)!}\int_0^1 \left[f^{(N)}[(1-s)z_0 + st] - f^{(N)}(z_0)\right](1-s)^{N-1}\,ds, \tag{77}$$

for $N \in \mathbb{N}$.

Assuming $f^{(k)}(z_0) = 0$, $k = 1, \ldots, N$, *we get*

$$f(t) - f(z_0) = \frac{(t - z_0)^N}{(N-1)!}\int_0^1 \left[f^{(N)}[(1-s)z_0 + st] - f^{(N)}(z_0)\right](1-s)^{N-1}\,ds, \tag{78}$$

$N \in \mathbb{N}$.

We have that

$$|f(t) - f(z_0)| \leq$$

$$\frac{|t - z_0|^N}{(N-1)!}\int_0^1 \left|f^{(N)}[(1-s)z_0 + st] - f^{(N)}(z_0)\right|(1-s)^{N-1}\,ds =: (*). \tag{79}$$

We assume that $\left|f^{(N)}(t) - f^{(N)}(z_0)\right|$ *is convex in* $t \in K$. *Let* $\delta > 0$ *such that the closed disk* $D(z_0, \delta) \subset K$. *Then, by Theorem 10, we obtain that*

$$\left|f^{(N)}(t) - f^{(N)}(z_0)\right| \leq \frac{\omega_1\left(f^{(N)}, \delta\right)}{\delta}|t - z_0|, \quad \forall\, t \in K. \tag{80}$$

Notice that by convexity of K, $(1-s)z_0 + st \in K$, $0 \leq s \leq 1$. *Therefore*

$$(*) \overset{(80)}{\leq} \frac{|t - z_0|^N}{(N-1)!}\frac{\omega_1\left(f^{(N)}, \delta\right)}{\delta}\int_0^1 s\,|t - z_0|(1-s)^{N-1}\,ds =$$

$$\frac{|t-z_0|^{N+1}}{(N-1)!}\frac{\omega_1\left(f^{(N)},\delta\right)}{\delta}\int_0^1 (1-s)^{N-1}(s-0)^{2-1}\,ds =$$

$$\frac{|t-z_0|^{N+1}}{(N-1)!}\frac{\omega_1\left(f^{(N)},\delta\right)}{\delta}\frac{1}{N(N+1)} = \frac{|t-z_0|^{N+1}}{(N+1)!}\frac{\omega_1\left(f^{(N)},\delta\right)}{\delta}. \tag{81}$$

We have proved that

$$|f(t)-f(z_0)| \leq \frac{\omega_1\left(f^{(N)},\delta\right)}{\delta(N+1)!}|t-z_0|^{N+1}, \tag{82}$$

$\forall\, t \in K,\ N \in \mathbb{N}.$

We have proved

Theorem 18 *Let K be a compact and convex subset of the convex domain $D \subseteq \mathbb{C}$, $z_0 \in K^0$. Here $f: K \to \mathbb{C}$ is analytic such that $f^{(k)}(z_0) = 0$, $k = 1,2,...,N \in \mathbb{N}$. We assume that $\left|f^{(N)}(\cdot) - f^{(N)}(z_0)\right|$ is convex over K. Let $\delta > 0$ such that the closed disk $D(z_0,\delta) \subset K$. Then*

$$|f(\cdot)-f(z_0)| \leq \frac{\omega_1\left(f^{(N)},\delta\right)}{\delta(N+1)!}|\cdot - z_0|^{N+1}, \tag{83}$$

over K.

The convex analog of Theorem 15 follows:

Theorem 19 *All as in Theorem 15. Additionally we assume that: $z_0 \in K^0$, $\left|f^{(N)}(\cdot) - f^{(N)}(z_0)\right|$ is convex over K, and that the closed disk $D\left(z_0, \widetilde{L}_n\left(|\cdot - z_0|^{N+1}\right)(z_0)\right) \subset K$. Then*

$$|(L_n(f))(z_0) - f(z_0)| \leq |f(z_0)|\left|\widetilde{L}_n(1(\cdot))(z_0) - 1\right| +$$

$$\frac{\lambda\omega_1\left(f^{(N)}, \widetilde{L}_n\left(|\cdot - z_0|^{N+1}\right)(z_0)\right)}{(N+1)!}, \tag{84}$$

$\forall\, n \in \mathbb{N}.$

If $\widetilde{L}_n(1(\cdot))(z_0) \to 1$ and $\widetilde{L}_n\left(|\cdot - z_0|^{N+1}\right)(z_0) \to 0$, as $n \to \infty$, then $L_n(f)(z_0) \to f(z_0)$.

Proof. As in the proof of Theorem 15 we have

$$|(L_n(f))(z_0) - f(z_0)| \leq ... \leq$$

$$|f(z_0)|\left|\widetilde{L}_n(1(\cdot))(z_0) - 1\right| + \lambda\left(\widetilde{L}_n(|f(\cdot) - f(z_0)|)\right)(z_0) \overset{(83)}{\leq}$$

$$|f(z_0)|\left|\widetilde{L}_n(1(\cdot))(z_0) - 1\right| +$$

$$\lambda \frac{\omega_1 \left(f^{(N)}, \delta\right)}{\delta (N+1)!} \widetilde{L}_n \left(|\cdot - z_0|^{N+1}\right)(z_0) = \tag{85}$$

$$|f(z_0)| \left|\widetilde{L}_n (1(\cdot))(z_0) - 1\right| + \frac{\lambda}{(N+1)!} \omega_1 \left(f^{(N)}, \widetilde{L}_n \left(|\cdot - z_0|^{N+1}\right)(z_0)\right),$$

by choosing

$$\delta := \widetilde{L}_n \left(|\cdot - z_0|^{N+1}\right)(z_0),$$

if $\widetilde{L}_n \left(|\cdot - z_0|^{N+1}\right)(z_0) > 0$.

If $\widetilde{L}_n \left(|\cdot - z_0|^{N+1}\right)(z_0) = 0$ then this case is treated similarly to the proof of Theorem 15. ■

We give

Corollary 20 *(to Theorem 19, case of $N = 1$) All as in Theorem 19. Assume $|f'(\cdot) - f'(z_0)|$ is convex over K, and the closed disk $D\left(z_0, \widetilde{L}_n \left(|\cdot - z_0|^2\right)(z_0)\right) \subset K$. Then*

$$|(L_n(f))(z_0) - f(z_0)| \leq |f(z_0)| \left|\widetilde{L}_n (1(\cdot))(z_0) - 1\right| +$$

$$\frac{\lambda \omega_1 \left(f', \widetilde{L}_n \left(|\cdot - z_0|^2\right)(z_0)\right)}{2}, \tag{86}$$

$\forall\ n \in \mathbb{N}$.

If $\widetilde{L}_n (1(\cdot))(z_0) \to 1$ and $\widetilde{L}_n \left(|\cdot - z_0|^2\right)(z_0) \to 0$, as $n \to \infty$, then $L_n(f)(z_0) \to f(z_0)$.

4 Illistration

Here we go according to Example 3 and Application 4. We will study the quantitative uniform convergence of complex Bernstein operators $B^{\mathbb{C}}_{n_1,n_2}(f)$ to $f \in C\left([0,1]^2, \mathbb{C}\right)$. Indeed we have

$$\left|B^{\mathbb{C}}_{n_1,n_2}(f)\right|(z) \leq \sqrt{2} B_{n_1,n_2}(|f|)(z), \ \ \forall\ z \in [0,1]^2 \ \text{ and } \forall\ f \in C\left([0,1]^2, \mathbb{C}\right), \tag{87}$$

and

$$B^{\mathbb{C}}_{n_1,n_2}(cg) = c B_{n_1,n_2}(g), \ \ \forall\ c \in \mathbb{C} \text{ and } \forall\ g \in C\left([0,1]^2, \mathbb{R}\right). \tag{88}$$

Clearly $B^{\mathbb{C}}_{n_1,n_2}$ maps $C\left([0,1]^2, \mathbb{C}\right)$ into itself and B_{n_1,n_2} maps $C\left([0,1]^2 R\right)$ into itself. Notice that $B_{n_1,n_2}(1(\cdot))(x,y) = 1$.

Hence by Theorem 6 (39) we get:

$$\left|B^{\mathbb{C}}_{n_1,n_2}(f)(z_0) - f(z_0)\right| \leq 2\sqrt{2}\omega_1 \left(f, \sqrt{B_{n_1,n_2}\left(|\cdot - z_0|^2\right)(z_0)}\right), \tag{89}$$

$\forall\ z_0 \in [0,1]^2$, $\forall\ n_1, n_2 \in \mathbb{N}$.

Here $z_0 = z_{01} + iz_{02}$, $z_{01}, z_{02} \in [0,1]$, and $t = t_1 + it_2$, where $t_1, t_2 \in [0,1]$. We notice that

$$B_{n_1,n_2}\left(|t-z_0|^2\right)(z_0) = B_{n_1,n_2}\left((t_1-z_{01})^2 + (t_2-z_{02})^2\right)(z_{01}, z_{02}) =$$

$$B_{n_1,n_2}\left((t_1-z_{01})^2\right)(z_{01}, z_{02}) + B_{n_1,n_2}\left((t_2-z_{02})^2\right)(z_{01}, z_{02}) = \tag{90}$$

$$\left(B_{n_1}\left((t_1-z_{01})^2\right)\right)(z_{01}) + \left(B_{n_2}\left((t_2-z_{02})^2\right)\right)(z_{02}) =$$

$$\frac{z_{01}(1-z_{01})}{n_1} + \frac{z_{02}(1-z_{02})}{n_2}, \tag{91}$$

where B_{n_1}, B_{n_2} are the basic univariate Bernstein operators over $[0,1]$.

That is

$$B_{n_1,n_2}\left(|t-z_0|^2\right)(z_0) = \frac{z_{01}(1-z_{01})}{n_1} + \frac{z_{02}(1-z_{02})}{n_2}, \tag{92}$$

$\forall\ z_0 \in [0,1]^2$.

Therefore we find

$$B_{n_1,n_2}\left(|t-z_0|^2\right)(z_0) \leq \frac{1}{4}\left(\frac{1}{n_1} + \frac{1}{n_2}\right), \tag{93}$$

$\forall\ z_0 \in [0,1]^2$.

That is

$$\sqrt{B_{n_1,n_2}\left(|\cdot - z_0|^2\right)(z_0)} \leq \frac{1}{2}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, \tag{94}$$

$\forall\ z_0 \in [0,1]^2$.

By (89), finally, we obtain

$$\left\|B^{\mathbb{C}}_{n_1,n_2}(f) - f\right\|_\infty \leq 2\sqrt{2}\omega_1\left(f, \frac{1}{2}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right), \tag{95}$$

$\forall\ n_1, n_2 \in \mathbb{N}$.

Consequently, as $n_1, n_2 \to \infty$, we get that $B^{\mathbb{C}}_{n_1,n_2}(f) \overset{u}{\to} f$, uniformly, $\forall$ $f \in C\left([0,1]^2, \mathbb{C}\right)$.

Many other examples as above could be given but we choose to omit this task.

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