

# A TRACE INEQUALITY FOR A YOUNG-TYPE INEQUALITY

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ABSTRACT. In this paper we will give a local trace inequality starting from a Young-type inequality for three positive variables and some applications.

## 1. Introduction

The classical inequality of Young is

$$a^\nu b^{1-\nu} < \nu a + (1-\nu)b,$$

where  $a$  and  $b$  are distinct positive real numbers and  $0 < \nu < 1$ , see [23]. It is also an inequality between arithmetic and geometric mean.

Many generalizations and refinements of Young's inequality are stated in [1], [2], [13], [12], [15], [5], [4], [6], [9] and references therein.

**Theorem 1.** ([1]) *Let  $\lambda$ ,  $\nu$  and  $\tau$  be real numbers with  $\lambda \geq 1$  and  $0 < \nu < \tau < 1$ . Then*

$$\left(\frac{\nu}{\tau}\right)^\lambda < \frac{A_\nu(a, b)^\lambda - G_\nu(a, b)^\lambda}{A_\tau(a, b)^\lambda - G_\tau(a, b)^\lambda} < \left(\frac{1-\nu}{1-\tau}\right)^\lambda,$$

for all positive and distinct real numbers  $a$  and  $b$ . Moreover, both bounds are sharp.

We suppose that  $a, b, c > 0$  are three distinct numbers and  $p_1, p_2, p_3 > 0$ ,  $p'_1, p'_2, p'_3 > 0$  with  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$  and  $\frac{1}{p'_1} + \frac{1}{p'_2} + \frac{1}{p'_3} = 1$ . We take into account the three variables function

$$f(a, b, c) = \frac{1}{p_1} a^{p_1} + \frac{1}{p_2} b^{p_2} + \frac{1}{p_3} c^{p_3} - abc - \frac{p'_1}{p_1} \left( \frac{1}{p'_1} a^{p'_1} + \frac{1}{p'_2} b^{p'_2} + \frac{1}{p'_3} c^{p'_3} - a^{\frac{p'_1}{p_1}} b^{\frac{p'_2}{p_2}} c^{\frac{p'_3}{p_3}} \right)$$

which have the stationary points  $A(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c)$  with  $c > 0$ ,  $c \neq 1$ .

**Theorem 2.** ([3]) *The local extreme points of the above function are  $A(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c)$ . If the following conditions are satisfied*

$$\frac{p_1}{p'_1} - 1 \geq \max\left\{ \frac{1}{p_2} \left| \frac{p_1}{p'_1} - \frac{p_2}{p_2} \right|, \frac{1}{p_3} \left| \frac{p_1}{p'_1} - \frac{p_3}{p_3} \right| \right\},$$

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*Date:* March 27, 2019.

*2000 Mathematics Subject Classification.* 26D20.

*Key words and phrases.* Young-type inequalities, Arithmetic mean-Geometric mean inequality.

$$1 \geq \frac{1}{p_2} \frac{p_1 - \frac{p_2^2}{(p_2')^2}}{\frac{p_1}{p_1'} - \frac{p_2}{p_2'}} + \frac{1}{p_3} \frac{p_1 - \frac{p_3^2}{(p_3')^2}}{\frac{p_1}{p_1'} - \frac{p_3}{p_3'}} + \frac{p_1}{p_1'} \frac{1}{p_2 p_3} \frac{\left(\frac{p_2}{p_2'} - \frac{p_3}{p_3'}\right)^2}{\left(\frac{p_1}{p_1'} - \frac{p_2}{p_2'}\right) \left(\frac{p_1}{p_1'} - \frac{p_3}{p_3'}\right)}.$$

then these points are local minimum points for the function  $f$ .

**Proposition 1.** ([3]) For any  $p_1, p_2, p_3 > 0, p_1', p_2', p_3' > 0$  with  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$  and  $\frac{1}{p_1'} + \frac{1}{p_2'} + \frac{1}{p_3'} = 1$  which satisfy the conditions

$$\frac{p_1}{p_1'} - 1 \geq \max\left\{\frac{1}{p_2} \left|\frac{p_1}{p_1'} - \frac{p_2}{p_2'}\right|, \frac{1}{p_3} \left|\frac{p_1}{p_1'} - \frac{p_3}{p_3'}\right|\right\},$$

$$1 \geq \frac{1}{p_2} \frac{p_1 - \frac{p_2^2}{(p_2')^2}}{\frac{p_1}{p_1'} - \frac{p_2}{p_2'}} + \frac{1}{p_3} \frac{p_1 - \frac{p_3^2}{(p_3')^2}}{\frac{p_1}{p_1'} - \frac{p_3}{p_3'}} + \frac{p_1}{p_1'} \frac{1}{p_2 p_3} \frac{\left(\frac{p_2}{p_2'} - \frac{p_3}{p_3'}\right)^2}{\left(\frac{p_1}{p_1'} - \frac{p_2}{p_2'}\right) \left(\frac{p_1}{p_1'} - \frac{p_3}{p_3'}\right)},$$

and for any  $d > 0$ , there is  $r_d > 0$  so that for any  $c \in (d - r_d, d + r_d)$ ,  $b \in (d^{\frac{p_3}{p_2}} - r_d, d^{\frac{p_3}{p_2}} + r_d)$  and  $a \in (d^{\frac{p_3}{p_1}} - r_d, d^{\frac{p_3}{p_1}} + r_d)$  it is true the inequality:

$$\frac{1}{p_1} a^{p_1} + \frac{1}{p_2} b^{p_2} + \frac{1}{p_3} c^{p_3} - abc \geq \frac{p_1'}{p_1} \left( \frac{1}{p_1'} a^{p_1} + \frac{1}{p_2'} b^{p_2} + \frac{1}{p_3'} c^{p_3} - a^{\frac{p_1'}{p_1}} b^{\frac{p_2'}{p_2}} c^{\frac{p_3'}{p_3}} \right).$$

Now we will use this inequality in order to establish several trace inequalities.

It is necessary to recall some basic things about the functional calculus with continuous functions on spectrum. As in [10], we recall that for selfadjoint operators  $A, B \in B(\mathcal{H})$  we write  $A \leq B$  (or  $B \geq A$ ) if  $\langle Ax, x \rangle \leq \langle Bx, x \rangle$  for every vector  $x \in \mathcal{H}$ . We will consider for beginning  $A$  as being a selfadjoint linear operator on a complex Hilbert space  $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ . The *Gelfand map* establishes a  $*$ -isometrically isomorphism  $\Phi$  between the set  $C(Sp(A))$  of all *continuous functions* defined on the *spectrum* of  $A$ , denoted  $Sp(A)$ , and the  $C^*$ -algebra  $C^*(A)$  generated by  $A$  and the identity operator  $1_{\mathcal{H}}$  on  $\mathcal{H}$  as follows: For any  $f, g \in C(Sp(A))$  and for any  $\alpha, \beta \in \mathbf{C}$  we have

- (i)  $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$ ;
  - (ii)  $\Phi(fg) = \Phi(f)\Phi(g)$  and  $\Phi(f) = \Phi(f^*)$ ;
  - (iii)  $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$ ;
  - (iv)  $\Phi(f_0) = 1_{\mathcal{H}}$  and  $\Phi(f_1) = A$ , where  $f_0(t) = 1$  and  $f_1(t) = t$  for  $t \in Sp(A)$ .
- Using this notation, as in [10] for example, we define

$$f(A) := \Phi(f) \quad \text{for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator  $A$ . It is known that if  $A$  is a selfadjoint operator and  $f$  is a real valued continuous function on  $Sp(A)$ , then  $f(t) \geq 0$  for any  $t \in Sp(A)$  implies that  $f(A) \geq 0$ , i.e.  $f(A)$  is a *positive operator* on  $\mathcal{H}$ . In addition, if  $f$  and  $g$  are real valued functions on  $Sp(A)$  then the following property holds:

- (1)  $f(t) \geq g(t)$  for any  $t \in Sp(A)$  implies that  $f(A) \geq g(A)$

in the operator order of  $B(\mathcal{H})$ .

We need below of some basic properties of trace of an operator. The main properties of the trace can be found in [6] and the references therein, but we mention

here some of them. We consider  $\{e_i\}_{i \in I}$  an orthonormal basis of a separable Hilbert space  $\mathcal{H}$ . We will say that the operator  $A \in \mathcal{B}(\mathcal{H})$  is *trace class* provided

$$\|A\|_1 = \sum_{i \in I} \langle A|e_i, e_i \rangle < \infty.$$

It is known that the definition of  $\|A\|_1$  does not depend on the choice of the orthogonal basis  $\{e_i\}_{i \in I}$  and that the set of trace class operators in  $\mathcal{B}(\mathcal{H})$  is denoted by  $\mathcal{B}_1(\mathcal{H})$ .

We will enumerate below several well-known properties of trace:

- (a)  $\|A\|_1 = \|A^*\|_1$ , for any  $A \in \mathcal{B}_1(\mathcal{H})$ ;
- (b)  $\mathcal{B}_1(\mathcal{H})$  is an operator ideal in  $\mathcal{B}(\mathcal{H})$ , that is

$$\mathcal{B}(\mathcal{H})\mathcal{B}_1(\mathcal{H})\mathcal{B}(\mathcal{H}) \subseteq \mathcal{B}_1(\mathcal{H});$$

- (c)  $(\mathcal{B}_1(\mathcal{H}), \|\cdot\|_1)$  is a Banach space.

We consider, as in [6], the *trace* of a trace class operator  $A \in \mathcal{B}_1(\mathcal{H})$  to be

$$tr(A) = \sum_{i \in I} \langle Ae_i, e_i \rangle,$$

where  $\{e_i\}_{i \in I}$  an orthonormal basis of  $\mathcal{H}$ . We can see that this definition coincides with the usual definition of the trace if  $\mathcal{H}$  is finite dimensional, and that previous series also converges absolutely and it is independent from the choice of basis.

In addition, we can mention that if  $A \in \mathcal{B}_1(\mathcal{H})$  then  $A^* \in \mathcal{B}_1(\mathcal{H})$  and  $tr(A^*) = \overline{tr(A)}$ . If  $A \in \mathcal{B}_1(\mathcal{H})$  and  $T \in \mathcal{B}(\mathcal{H})$  then  $AT, TA \in \mathcal{B}_1(\mathcal{H})$  and  $tr(AT) = tr(TA)$  and  $|tr(AT)| \leq \|A\|_1 \|T\|$ . The application  $tr(\cdot)$  is a bounded linear functional on  $\mathcal{B}_1(\mathcal{H})$  with  $\|tr\| = 1$ .

In 2007, L. Liu in [14] showed that  $tr[(AB)^k] \leq (tr(A))^k (tr(B))^k$ , where  $k$  is any positive integer and  $A, B$  any positive operators in  $\mathcal{B}_1(\mathcal{H})$ . Many trace inequalities for matrices and operators can be found for example in [18], [20], [21], [6],[8], [15], [22], [16], [17] and also, references therein.

We also recall that if  $A$  and  $B$  are positive invertible operators on a complex Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  then the following notation,  $A \sharp_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}$  is used for the *weighted geometric mean*, where  $\nu \in (0, 1)$ . We will use here the same notation for the case when  $\nu > 0$ .

In the following, we will give a local trace inequality starting from a Young-type inequality for three positive variables and then some consequences will be also presented.

## 2. Main results

Let  $\mathcal{B}(\mathcal{H})$  be the  $C^*$ -algebra of all bounded linear operators on a complex Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  and  $A, B, C \in \mathcal{B}(\mathcal{H})$  be three positive operators.

The following result is obtained as an application of Proposition 1 for trace of an operator.

**Theorem 3.** Let  $p_1, p_2, p_3 > 0, p'_1, p'_2, p'_3 > 0$  with  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$  and  $\frac{1}{p'_1} + \frac{1}{p'_2} + \frac{1}{p'_3} = 1$  which satisfy the conditions

$$\begin{aligned} \frac{p_1}{p'_1} - 1 &\geq \max\left\{\frac{1}{p_2} \left| \frac{p_1}{p'_1} - \frac{p_2}{p'_2} \right|, \frac{1}{p_3} \left| \frac{p_1}{p'_1} - \frac{p_3}{p'_3} \right|\right\}, \\ 1 &\geq \frac{1}{p_2} \frac{\frac{p_1}{p'_1} - \frac{p_2^2}{(p'_2)^2}}{\frac{p_1}{p'_1} - \frac{p_2}{p'_2}} + \frac{1}{p_3} \frac{\frac{p_1}{p'_1} - \frac{p_3^2}{(p'_3)^2}}{\frac{p_1}{p'_1} - \frac{p_3}{p'_3}} + \frac{p_1}{p'_1} \frac{1}{p_2 p_3} \frac{\left(\frac{p_2}{p'_2} - \frac{p_3}{p'_3}\right)^2}{\left(\frac{p_1}{p'_1} - \frac{p_2}{p'_2}\right) \left(\frac{p_1}{p'_1} - \frac{p_3}{p'_3}\right)}, \end{aligned}$$

and let  $A, B, C$  be three positive operators on  $\mathcal{H}$  and  $P, Q, R \in \mathcal{B}_1(\mathcal{H})$  with  $P, Q, R > 0$ .

For any  $d > 0$  there is  $r_d > 0$  so that if  $(d - r_d)I \leq C \leq (d + r_d)I, (d^{\frac{p_3}{p_2}} - r_d)I \leq B \leq (d^{\frac{p_3}{p_2}} + r_d)I,$  and  $(d^{\frac{p_3}{p_1}} - r_d)I \leq A \leq (d^{\frac{p_3}{p_1}} + r_d)I$  then the following inequality takes place:

$$\begin{aligned} &\frac{1}{p_1} \text{tr}(PA^{p_1}) \text{tr}(Q) \text{tr}(R) + \frac{1}{p_2} \text{tr}(P) \text{tr}(QB^{p_2}) \text{tr}(R) + \frac{1}{p_3} \text{tr}(P) \text{tr}(Q) \text{tr}(RC^{p_3}) - \\ & - \text{tr}(PA) \text{tr}(QB) \text{tr}(RC) \geq \frac{p'_1}{p_1} \left[ \frac{1}{p'_1} \text{tr}(PA^{p_1}) \text{tr}(Q) \text{tr}(R) + \frac{1}{p'_2} \text{tr}(P) \text{tr}(QB^{p_2}) \text{tr}(R) + \right. \\ & \left. + \frac{1}{p'_3} \text{tr}(P) \text{tr}(Q) \text{tr}(RC^{p_3}) - \text{tr}(PA^{\frac{p_1}{p'_1}}) \text{tr}(QB^{\frac{p_2}{p'_2}}) \text{tr}(RC^{\frac{p_3}{p'_3}}) \right]. \end{aligned}$$

*Proof.* From Proposition 1, we see that for any  $d > 0$  there is  $r_d > 0$  so that for any  $c \in (d - r_d, d + r_d), b \in (d^{\frac{p_3}{p_2}} - r_d, d^{\frac{p_3}{p_2}} + r_d)$  and  $a \in (d^{\frac{p_3}{p_1}} - r_d, d^{\frac{p_3}{p_1}} + r_d)$  it is true the inequality:

$$\frac{1}{p_1} a^{p_1} + \frac{1}{p_2} b^{p_2} + \frac{1}{p_3} c^{p_3} - abc \geq \frac{p'_1}{p_1} \left( \frac{1}{p'_1} a^{p_1} + \frac{1}{p'_2} b^{p_2} + \frac{1}{p'_3} c^{p_3} - a^{\frac{p_1}{p'_1}} b^{\frac{p_2}{p'_2}} c^{\frac{p_3}{p'_3}} \right).$$

Then we will use the same method as in [6]. Using the functional calculus with continuous functions on spectrum for the operator  $A$ , we get

$$\begin{aligned} &\frac{1}{p_1} \langle A^{p_1} x, x \rangle + \frac{1}{p_2} b^{p_2} \langle x, x \rangle + \frac{1}{p_3} c^{p_3} \langle x, x \rangle - bc \langle Ax, x \rangle \geq \\ &\geq \frac{p'_1}{p_1} \left( \frac{1}{p'_1} \langle A^{p_1} x, x \rangle + \frac{1}{p'_2} b^{p_2} \langle x, x \rangle + \frac{1}{p'_3} c^{p_3} \langle x, x \rangle - b^{\frac{p_2}{p'_2}} c^{\frac{p_3}{p'_3}} \langle A^{\frac{p_1}{p'_1}} x, x \rangle \right), \end{aligned}$$

for any  $x \in \mathcal{H}, b \in (d^{\frac{p_3}{p_2}} - r_d, d^{\frac{p_3}{p_2}} + r_d), c \in (d - r_d, d + r_d)$  and  $p_1, p_2, p_3, p'_1, p'_2, p'_3$  as in our hypothesis.

Using in last inequality the functional calculus with continous functions on spectrum for the operator  $B$  we have:

$$\begin{aligned} &\frac{1}{p_1} \langle A^{p_1} x, x \rangle \langle y, y \rangle + \frac{1}{p_2} \langle x, x \rangle \langle B^{p_2} y, y \rangle + \frac{1}{p_3} c^{p_3} \langle x, x \rangle \langle y, y \rangle - \\ & - c \langle Ax, x \rangle \langle By, y \rangle \geq \\ &\geq \frac{p'_1}{p_1} \left( \frac{1}{p'_1} \langle A^{p_1} x, x \rangle \langle y, y \rangle + \frac{1}{p'_2} \langle x, x \rangle \langle B^{p_2} y, y \rangle + \frac{1}{p'_3} c^{p_3} \langle x, x \rangle \langle y, y \rangle - \right. \\ & \left. - c^{\frac{p_3}{p'_3}} \langle A^{\frac{p_1}{p'_1}} x, x \rangle \langle B^{\frac{p_2}{p'_2}} y, y \rangle \right), \end{aligned}$$

for any  $x, y \in \mathcal{H}$ ,  $c \in (d-r_d, d+r_d)$  and  $p_1, p_2, p_3, p'_1, p'_2, p'_3$  as in our hypothesis.

Now, by functional calculus with continuous functions on spectrum for the operator  $C$ , we will find out from last inequality that,

$$\begin{aligned} & \frac{1}{p_1} \langle A^{p_1} x, x \rangle \langle y, y \rangle \langle z, z \rangle + \frac{1}{p_2} \langle x, x \rangle \langle z, z \rangle \langle B^{p_2} y, y \rangle + \\ & + \frac{1}{p_3} \langle C^{p_3} z, z \rangle \langle x, x \rangle \langle y, y \rangle - \langle Cz, z \rangle \langle Ax, x \rangle \langle By, y \rangle \geq \\ & \geq \frac{p'_1}{p_1} \left( \frac{1}{p_1} \langle A^{p_1} x, x \rangle \langle y, y \rangle \langle z, z \rangle + \frac{1}{p_2} \langle x, x \rangle \langle z, z \rangle \langle B^{p_2} y, y \rangle + \right. \\ & \left. + \frac{1}{p_3} \langle C^{p_3} z, z \rangle \langle x, x \rangle \langle y, y \rangle - \langle C^{\frac{p_3}{p_1}} z, z \rangle \langle A^{\frac{p_1}{p_2}} x, x \rangle \langle B^{\frac{p_2}{p_3}} y, y \rangle \right), \end{aligned}$$

for any  $x, y, z \in \mathcal{H}$  and  $p_1, p_2, p_3, p'_1, p'_2, p'_3$  as in our hypothesis.

We put now  $x = P^{\frac{1}{2}} e$ ,  $y = Q^{\frac{1}{2}} f$  and  $z = R^{\frac{1}{2}} g$  where  $e, f, g \in \mathcal{H}$  and then we rewrite below our last inequality

$$\begin{aligned} & \frac{1}{p_1} \langle A^{p_1} P^{\frac{1}{2}} e, P^{\frac{1}{2}} e \rangle \langle Q^{\frac{1}{2}} f, Q^{\frac{1}{2}} f \rangle \langle R^{\frac{1}{2}} g, R^{\frac{1}{2}} g \rangle + \\ & + \frac{1}{p_2} \langle P^{\frac{1}{2}} e, P^{\frac{1}{2}} e \rangle \langle R^{\frac{1}{2}} g, R^{\frac{1}{2}} g \rangle \langle B^{p_2} Q^{\frac{1}{2}} f, Q^{\frac{1}{2}} f \rangle + \\ & + \frac{1}{p_3} \langle C^{p_3} R^{\frac{1}{2}} g, R^{\frac{1}{2}} g \rangle \langle P^{\frac{1}{2}} e, P^{\frac{1}{2}} e \rangle \langle Q^{\frac{1}{2}} f, Q^{\frac{1}{2}} f \rangle - \\ & - \langle CR^{\frac{1}{2}} g, R^{\frac{1}{2}} g \rangle \langle AP^{\frac{1}{2}} e, P^{\frac{1}{2}} e \rangle \langle BQ^{\frac{1}{2}} f, Q^{\frac{1}{2}} f \rangle \geq \\ & \geq \frac{p'_1}{p_1} \left( \frac{1}{p_1} \langle A^{p_1} P^{\frac{1}{2}} e, P^{\frac{1}{2}} e \rangle \langle Q^{\frac{1}{2}} f, Q^{\frac{1}{2}} f \rangle \langle R^{\frac{1}{2}} g, R^{\frac{1}{2}} g \rangle + \right. \\ & + \frac{1}{p_2} \langle P^{\frac{1}{2}} e, P^{\frac{1}{2}} e \rangle \langle R^{\frac{1}{2}} g, R^{\frac{1}{2}} g \rangle \langle B^{p_2} Q^{\frac{1}{2}} f, Q^{\frac{1}{2}} f \rangle + \\ & + \frac{1}{p_3} \langle C^{p_3} R^{\frac{1}{2}} g, R^{\frac{1}{2}} g \rangle \langle P^{\frac{1}{2}} e, P^{\frac{1}{2}} e \rangle \langle Q^{\frac{1}{2}} f, Q^{\frac{1}{2}} f \rangle - \\ & \left. - \langle C^{\frac{p_3}{p_1}} R^{\frac{1}{2}} g, R^{\frac{1}{2}} g \rangle \langle A^{\frac{p_1}{p_2}} P^{\frac{1}{2}} e, P^{\frac{1}{2}} e \rangle \langle B^{\frac{p_2}{p_3}} Q^{\frac{1}{2}} f, Q^{\frac{1}{2}} f \rangle \right), \end{aligned}$$

for any  $e, f, g \in \mathcal{H}$  and  $p_1, p_2, p_3, p'_1, p'_2, p'_3$  as in our hypothesis.

Let  $\{e_i\}_{i \in I}$ ,  $\{f_j\}_{j \in J}$  and  $\{g_k\}_{k \in K}$  be three orthonormal bases of  $\mathcal{H}$ . We take in previous inequality  $e = e_i$ ,  $i \in I$ ,  $f = f_j$ ,  $j \in J$  and  $g = g_k$ ,  $k \in K$  and then summing over  $i \in I$ ,  $j \in J$ , and  $k \in K$  we get the following:

$$\begin{aligned} & \frac{1}{p_1} \sum_{i \in I} \langle P^{\frac{1}{2}} A^{p_1} P^{\frac{1}{2}} e_i, e_i \rangle \sum_{j \in J} \langle Q f_j, f_j \rangle \sum_{k \in K} \langle R g_k, g_k \rangle + \\ & + \frac{1}{p_2} \sum_{i \in I} \langle P e_i, e_i \rangle \sum_{k \in K} \langle R g_k, g_k \rangle \sum_{j \in J} \langle Q^{\frac{1}{2}} B^{p_2} Q^{\frac{1}{2}} f_j, f_j \rangle + \\ & + \frac{1}{p_3} \sum_{k \in K} \langle R^{\frac{1}{2}} C^{p_3} R^{\frac{1}{2}} g_k, g_k \rangle \sum_{i \in I} \langle P e_i, e_i \rangle \sum_{j \in J} \langle Q f_j, f_j \rangle - \\ & - \sum_{k \in K} \langle R^{\frac{1}{2}} C R^{\frac{1}{2}} g_k, g_k \rangle \sum_{i \in I} \langle P^{\frac{1}{2}} A P^{\frac{1}{2}} e_i, e_i \rangle \sum_{j \in J} \langle Q^{\frac{1}{2}} B Q^{\frac{1}{2}} f_j, f_j \rangle \geq \end{aligned}$$

$$\begin{aligned}
&\geq \frac{p_1'}{p_1} \left( \frac{1}{p_1} \sum_{i \in I} \langle P^{\frac{1}{2}} A^{p_1} P^{\frac{1}{2}} e_i, e_i \rangle \sum_{j \in J} \langle Q f_j, f_j \rangle \sum_{k \in K} \langle R g_k, g_k \rangle + \right. \\
&\quad + \frac{1}{p_2} \sum_{i \in I} \langle P e_i, e_i \rangle \sum_{k \in K} \langle R g_k, g_k \rangle \sum_{j \in J} \langle Q^{\frac{1}{2}} B^{p_2} Q^{\frac{1}{2}} f_j, f_j \rangle + \\
&\quad + \frac{1}{p_3} \sum_{k \in K} \langle R^{\frac{1}{2}} C^{p_3} R^{\frac{1}{2}} g_k, g_k \rangle \sum_{i \in I} \langle P e_i, e_i \rangle \sum_{j \in J} \langle Q f_j, f_j \rangle - \\
&\quad \left. - \sum_{k \in K} \langle R^{\frac{1}{2}} C^{\frac{p_3}{p_1}} R^{\frac{1}{2}} g_k, g_k \rangle \sum_{i \in I} \langle P^{\frac{1}{2}} A^{\frac{p_1}{p_1}} P^{\frac{1}{2}} e_i, e_i \rangle \sum_{j \in J} \langle Q^{\frac{1}{2}} B^{\frac{p_2}{p_2}} Q^{\frac{1}{2}} f_j, f_j \rangle \right),
\end{aligned}$$

for any  $p_1, p_2, p_3, p_1', p_2', p_3'$  as in our hypothesis. Now, by the properties of the trace, we have,

$$\begin{aligned}
&\frac{1}{p_1} \text{tr}(P A^{p_1}) \text{tr}(Q) \text{tr}(R) + \frac{1}{p_2} \text{tr}(P) \text{tr}(Q B^{p_2}) \text{tr}(R) + \frac{1}{p_3} \text{tr}(P) \text{tr}(Q) \text{tr}(R C^{p_3}) - \\
&- \text{tr}(P A) \text{tr}(Q B) \text{tr}(R C) \geq \frac{p_1'}{p_1} \left[ \frac{1}{p_1} \text{tr}(P A^{p_1}) \text{tr}(Q) \text{tr}(R) + \frac{1}{p_2} \text{tr}(P) \text{tr}(Q B^{p_2}) \text{tr}(R) + \right. \\
&\quad \left. + \frac{1}{p_3} \text{tr}(P) \text{tr}(Q) \text{tr}(R C^{p_3}) - \text{tr}(P A^{\frac{p_1}{p_1}}) \text{tr}(Q B^{\frac{p_2}{p_2}}) \text{tr}(R C^{\frac{p_3}{p_3}}) \right],
\end{aligned}$$

for any  $p_1, p_2, p_3, p_1', p_2', p_3'$  as in our hypothesis.

■

Next three results are several applications of Theorem 3.

**Corollary 1.** *Let  $p_1, p_2, p_3 > 0, p_1', p_2', p_3' > 0$  with  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$  and  $\frac{1}{p_1'} + \frac{1}{p_2'} + \frac{1}{p_3'} = 1$  which satisfy the conditions*

$$\begin{aligned}
&\frac{p_1'}{p_1} - 1 \geq \max \left\{ \frac{1}{p_2} \left| \frac{p_1}{p_1'} - \frac{p_2}{p_2'} \right|, \frac{1}{p_3} \left| \frac{p_1}{p_1'} - \frac{p_3}{p_3'} \right| \right\}, \\
&1 \geq \frac{1}{p_2} \frac{\frac{p_1}{p_1'} - \frac{p_2^2}{(p_2')^2}}{\frac{p_1}{p_1'} - \frac{p_2}{p_2'}} + \frac{1}{p_3} \frac{\frac{p_1}{p_1'} - \frac{p_3^2}{(p_3')^2}}{\frac{p_1}{p_1'} - \frac{p_3}{p_3'}} + \frac{p_1}{p_1'} \frac{1}{p_2 p_3} \frac{\left( \frac{p_2}{p_2'} - \frac{p_3}{p_3'} \right)^2}{\left( \frac{p_1}{p_1'} - \frac{p_2}{p_2'} \right) \left( \frac{p_1}{p_1'} - \frac{p_3}{p_3'} \right)},
\end{aligned}$$

and let  $A, B, C$  be three positive operators on  $\mathcal{H}$  and  $P \in \mathcal{B}_1(\mathcal{H})$  with  $P > 0$ .

For any  $d > 0$  there is  $r_d > 0$  so that if  $(d - r_d)I \leq C \leq (d + r_d)I, (d^{\frac{p_3}{p_1}} - r_d)I \leq B \leq (d^{\frac{p_3}{p_2}} + r_d)I,$  and  $(d^{\frac{p_3}{p_1}} - r_d)I \leq A \leq (d^{\frac{p_3}{p_1}} + r_d)I$  then we have:

$$\begin{aligned}
&\frac{1}{p_1} \text{tr}(P A^{p_1}) + \frac{1}{p_2} \text{tr}(P B^{p_2}) + \frac{1}{p_3} \text{tr}(P C^{p_3}) - \frac{\text{tr}(P A) \text{tr}(P B) \text{tr}(P C)}{(\text{tr}(P))^2} \geq \\
&\geq \frac{p_1'}{p_1} \left[ \frac{1}{p_1} \text{tr}(P A^{p_1}) + \frac{1}{p_2} \text{tr}(P B^{p_2}) + \frac{1}{p_3} \text{tr}(P C^{p_3}) - \frac{\text{tr}(P A^{\frac{p_1}{p_1}}) \text{tr}(P B^{\frac{p_2}{p_2}}) \text{tr}(P C^{\frac{p_3}{p_3}})}{(\text{tr}(P))^2} \right].
\end{aligned}$$

*Proof.* We take in Theorem 3,  $P = Q = R$ .

■

**Corollary 2.** Let  $p_1, p_2, p_3 > 0, p'_1, p'_2, p'_3 > 0$  with  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$  and  $\frac{1}{p'_1} + \frac{1}{p'_2} + \frac{1}{p'_3} = 1$  which satisfy the conditions

$$\frac{p_1}{p'_1} - 1 \geq \max\left\{\frac{1}{p_2} \left| \frac{p_1}{p'_1} - \frac{p_2}{p'_2} \right|, \frac{1}{p_3} \left| \frac{p_1}{p'_1} - \frac{p_3}{p'_3} \right|\right\},$$

$$1 \geq \frac{1}{p_2} \frac{\frac{p_1}{p'_1} - \frac{p_2^2}{(p'_2)^2}}{\frac{p_1}{p'_1} - \frac{p_2}{p'_2}} + \frac{1}{p_3} \frac{\frac{p_1}{p'_1} - \frac{p_3^2}{(p'_3)^2}}{\frac{p_1}{p'_1} - \frac{p_3}{p'_3}} + \frac{p_1}{p'_1} \frac{1}{p_2 p_3} \frac{\left(\frac{p_2}{p'_2} - \frac{p_3}{p'_3}\right)^2}{\left(\frac{p_1}{p'_1} - \frac{p_2}{p'_2}\right) \left(\frac{p_1}{p'_1} - \frac{p_3}{p'_3}\right)},$$

and let  $P, Q, R, S, V, W$  be invertible positive operators on  $\mathcal{H}$  and  $P, Q, R, S, V, W \in \mathcal{B}_1(\mathcal{H})$ .

For any  $d > 0$  there is  $r_d > 0$  so that if  $(d-r_d)R \leq W \leq (d+r_d)R$ ,  $(d^{\frac{p_3}{p_2}} - r_d)Q \leq V \leq (d^{\frac{p_3}{p_2}} + r_d)Q$ , and  $(d^{\frac{p_3}{p_1}} - r_d)P \leq S \leq (d^{\frac{p_3}{p_1}} + r_d)P$  then the following inequality takes place:

$$\begin{aligned} & \frac{1}{p_1} \text{tr}(P \#_{p_1} S) \text{tr}(Q) \text{tr}(R) + \frac{1}{p_2} \text{tr}(P) \text{tr}(Q \#_{p_2} V) \text{tr}(R) + \frac{1}{p_3} \text{tr}(P) \text{tr}(Q) \text{tr}(R \#_{p_3} W) - \\ & - \text{tr}(S) \text{tr}(V) \text{tr}(W) \geq \frac{p'_1}{p_1} \left[ \frac{1}{p'_1} \text{tr}(P \#_{p'_1} S) \text{tr}(Q) \text{tr}(R) + \frac{1}{p'_2} \text{tr}(Q \#_{p'_2} V) \text{tr}(P) \text{tr}(R) + \right. \\ & \left. + \frac{1}{p'_3} \text{tr}(P) \text{tr}(Q) \text{tr}(R \#_{p'_3} W) - \text{tr}(P \#_{p'_1} S) \text{tr}(Q \#_{p'_2} V) \text{tr}(R \#_{p'_3} W) \right]. \end{aligned}$$

*Proof.* Taking into account our hypothesis, we see that  $0 < (d-r_d)I \leq R^{-\frac{1}{2}}WR^{-\frac{1}{2}} \leq (d+r_d)I$ ,  $0 < (d^{\frac{p_3}{p_2}} - r_d)I \leq Q^{-\frac{1}{2}}VQ^{-\frac{1}{2}} \leq (d^{\frac{p_3}{p_2}} + r_d)I$ , and  $(d^{\frac{p_3}{p_1}} - r_d)I \leq P^{-\frac{1}{2}}SP^{-\frac{1}{2}} \leq (d^{\frac{p_3}{p_1}} + r_d)I$  and then we use Theorem 3 for  $A = P^{-\frac{1}{2}}SP^{-\frac{1}{2}}$ ,  $B = Q^{-\frac{1}{2}}VQ^{-\frac{1}{2}}$  and  $C = R^{-\frac{1}{2}}WR^{-\frac{1}{2}}$ .

We obtain then,

$$\begin{aligned} & \frac{1}{p_1} \text{tr}(P(P^{-\frac{1}{2}}SP^{-\frac{1}{2}})^{p_1}) \text{tr}(Q) \text{tr}(R) + \frac{1}{p_2} \text{tr}(P) \text{tr}(Q(Q^{-\frac{1}{2}}VQ^{-\frac{1}{2}})^{p_2}) \text{tr}(R) + \\ & + \frac{1}{p_3} \text{tr}(P) \text{tr}(Q) \text{tr}(R(R^{-\frac{1}{2}}WR^{-\frac{1}{2}})^{p_3}) - \text{tr}(P^{\frac{1}{2}}SP^{-\frac{1}{2}}) \text{tr}(Q^{\frac{1}{2}}VQ^{-\frac{1}{2}}) \text{tr}(R^{\frac{1}{2}}WR^{-\frac{1}{2}}) \geq \\ & \geq \frac{p'_1}{p_1} \left[ \frac{1}{p'_1} \text{tr}(P(P^{-\frac{1}{2}}SP^{-\frac{1}{2}})^{p_1}) \text{tr}(Q) \text{tr}(R) + \frac{1}{p'_2} \text{tr}(P) \text{tr}(Q(Q^{-\frac{1}{2}}VQ^{-\frac{1}{2}})^{p_2}) \text{tr}(R) + \right. \\ & \quad \left. + \frac{1}{p'_3} \text{tr}(P) \text{tr}(Q) \text{tr}(R(R^{-\frac{1}{2}}WR^{-\frac{1}{2}})^{p_3}) - \right. \\ & \quad \left. - \text{tr}(P(P^{-\frac{1}{2}}SP^{-\frac{1}{2}})^{\frac{p_1}{p'_1}}) \text{tr}(Q(Q^{-\frac{1}{2}}VQ^{-\frac{1}{2}})^{\frac{p_2}{p'_2}}) \text{tr}(R(R^{-\frac{1}{2}}WR^{-\frac{1}{2}})^{\frac{p_3}{p'_3}}) \right]. \end{aligned}$$

and using the properties of trace we get the desired inequality.

■

**Corollary 3.** Let  $p_1, p_2, p_3, p'_1, p'_2, p'_3$  be as in Corollary 2 and we consider  $P, S, V, W$  be invertible positive operators on  $\mathcal{H}$  and  $P, S, V, W \in \mathcal{B}_1(\mathcal{H})$ .

In these conditions, for any  $d > 0$  there is  $r_d > 0$  so that if  $(d-r_d)P \leq W \leq (d+r_d)P$ ,  $(d^{\frac{p_3}{p_2}} - r_d)P \leq V \leq (d^{\frac{p_3}{p_2}} + r_d)P$ , and  $(d^{\frac{p_3}{p_1}} - r_d)P \leq S \leq (d^{\frac{p_3}{p_1}} + r_d)P$  then the following inequality takes place:

$$\begin{aligned} & \frac{1}{p_1} \operatorname{tr}(P_{\#p_1}^\# S) + \frac{1}{p_2} \operatorname{tr}(P_{\#p_2}^\# V) + \frac{1}{p_3} \operatorname{tr}(P_{\#p_3}^\# W) - \frac{\operatorname{tr}(S)\operatorname{tr}(V)\operatorname{tr}(W)}{(\operatorname{tr}(P))^2} \geq \\ & \geq \frac{p_1'}{p_1} \left[ \frac{1}{p_1} \operatorname{tr}(P_{\#p_1}^\# S) + \frac{1}{p_2} \operatorname{tr}(P_{\#p_2}^\# V) + \frac{1}{p_3} \operatorname{tr}(P_{\#p_3}^\# W) - \right. \\ & \quad \left. - \frac{1}{(\operatorname{tr}(P))^2} \operatorname{tr}(P_{\#p_1}^\# S) \operatorname{tr}(P_{\#p_2}^\# V) \operatorname{tr}(P_{\#p_3}^\# W) \right]. \end{aligned}$$

*Proof.* We will use Corollary 2 where we take  $P = Q = R$ .

■

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