SOME INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. In this paper we prove among others that

$$\begin{aligned}\|f(y)z - zf(x)\| &\leq \frac{1}{2\pi}\|yz - zx\| \int_\gamma |f(\xi)| \|R_y(\xi)\| \|R_x(\xi)\| |d\xi| \\ &\leq \frac{1}{2\pi}\|yz - zx\| \int_\gamma \frac{|f(\xi)| |d\xi|}{(|\xi| - \|y\|)(|\xi| - \|x\|)},\end{aligned}$$

where $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the domain D, $x, y, z \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D$, $R_y(\cdot), R_x(\cdot)$ are the resolvent functions for the elements y and x, and γ is a closed rectifiable path in D and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Applications for the exponential function on the Banach algebra $\mathcal{B}$ are also given.

1. Introduction

Let $\mathcal{B}$ be an algebra over $\mathbb{C}$. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*. We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a, b \in \operatorname{Inv}(\mathcal{B})$ then $ab \in \operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \operatorname{Inv}(\mathcal{B})$;
(ii) $\{b \in \mathcal{B}: \|1 - b\| < 1\} \subset \operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}\mathcal{B}$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}\mathcal{B} \ni a \longmapsto a^{-1} \in \operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by λ. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} : \lambda - a \in \operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \operatorname{Inv}(\mathcal{B})$,

$$R_a(\lambda) := (\lambda - a)^{-1}.$$

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For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma).$$

We also have that

$$\sigma(a) \subset \{\lambda \in \mathbb{C} : \ |\lambda| \leq \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functional* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) We have

$$\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Let f be an analytic functions on the open disk $D(0, R)$ given by the *power series*

$$f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j \ \ (|\lambda| < R).$$

If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a + b) = \exp(a) \exp(b).$$

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [15] and [18].

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_{\delta} f(\xi)(\xi - a)^{-1} d\xi, \tag{1.1}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [6, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.2}$$

holds.

Let $\mathfrak{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\mathfrak{Hol}(a)$ is an algebra where if f, $g \in \mathfrak{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f+g$ have domain $D(f) \cap D(g)$. $\mathfrak{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [6, p. 201-203]:

Theorem 1. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) *The map $f \mapsto f(a)$ of $\mathfrak{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.*
(b) *If $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ has radius of convergence $r > \nu(a)$, then $f \in \mathfrak{Hol}(a)$ and $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.*
(c) *If $f(z) \equiv 1$, then $f(a) = 1$.*
(d) *If $f(z) = z$ for all z, $f(a) = a$.*
(e) *If f, $f_1, ..., f_n ...$ are analytic on G, $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\|f_n(a) - f(a)\| \to 0$ as $n \to \infty$.*
(f) *The Riesz Functional Calculus is unique and if a, b are commuting elements in $\mathcal{B}$ and $f \in \mathfrak{Hol}(a)$, then $f(a)b = bf(a)$.*

For some recent norm inequalities for functions on Banach algebras, see [3]-[5] and [7]-[14].

2. Some Inequalities for Generalised Commutator

We start with the following identity for the resolvent that is of interest in itself as well:

Lemma 1. *Let $\lambda \in \mathbb{C}$ and a, b, $c \in \mathcal{B}$ such that $\lambda \in \rho(a) \cap \rho(b)$, then*

$$R_a(\lambda)c - cR_b(\lambda) = R_a(\lambda)(ac - cb)R_b(\lambda). \tag{2.1}$$

In particular, we have the second resolvent identity

$$R_a(\lambda) - R_b(\lambda) = R_a(\lambda)(a-b)R_b(\lambda) \tag{2.2}$$

and the commutator identity for the resolvent

$$R_a(\lambda)c - cR_a(\lambda) = R_a(\lambda)(ac - ca)R_a(\lambda). \tag{2.3}$$

Proof. We have the following simple identity

$$x^{-1}(cy - xc)y^{-1} = x^{-1}cyy^{-1} - x^{-1}xcy^{-1} = x^{-1}c - cy^{-1}, \tag{2.4}$$

that holds for any invertible x, $y \in B$ and $c \in B$.

If $\lambda \in \rho(a) \cap \rho(b)$, then by taking $x = \lambda - a$ and $y = \lambda - b$ we get

$$\begin{aligned} R_a(\lambda)c - cR_b(\lambda) &= (\lambda - a)^{-1}c - c(\lambda - b)^{-1} \\ &= (\lambda - a)^{-1}(c(\lambda - b) - (\lambda - a)c)(\lambda - b)^{-1} \\ &= (\lambda - a)^{-1}(\lambda c - cb - \lambda c + ac)(\lambda - b)^{-1} \\ &= (\lambda - a)^{-1}(ac - cb)(\lambda - b)^{-1} \\ &= R_a(\lambda)(ac - cb)R_b(\lambda), \end{aligned}$$

namely the identity (2.1). □

Our first main result is as follows:

Theorem 2. *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D, x, y, $z \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D$ and γ be a closed rectifiable path in D such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then we have*

$$\begin{aligned} \|f(y)z - zf(x)\| &\leq \frac{1}{2\pi}\|yz - zx\| \int_\gamma |f(\xi)| \|R_y(\xi)\| \|R_x(\xi)\| |d\xi| \\ &\leq \frac{1}{2\pi}\|yz - zx\| \int_\gamma \frac{|f(\xi)| |d\xi|}{(|\xi| - \|y\|)(|\xi| - \|x\|)}. \end{aligned} \tag{2.5}$$

In particular,

$$\begin{aligned} \|f(x)z - zf(x)\| &\leq \frac{1}{2\pi}\|xz - zx\| \int_\gamma |f(\xi)| \|R_x(\xi)\|^2 |d\xi| \\ &\leq \frac{1}{2\pi}\|xz - zx\| \int_\gamma \frac{|f(\xi)| |d\xi|}{(|\xi| - \|x\|)^2} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \|f(y) - f(x)\| &\leq \frac{1}{2\pi}\|y - x\| \int_\gamma |f(\xi)| \|R_y(\xi)\| \|R_x(\xi)\| |d\xi| \\ &\leq \frac{1}{2\pi}\|y - x\| \int_\gamma \frac{|f(\xi)| |d\xi|}{(|\xi| - \|y\|)(|\xi| - \|x\|)}. \end{aligned} \tag{2.7}$$

Proof. Using the Riesz functional calculus we have

$$\begin{aligned} f(y)z - zf(x) &= \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - y)^{-1} z d\xi - \int_\gamma f(\xi) z (\xi - x)^{-1} d\xi \\ &= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi - y)^{-1} z - z(\xi - x)^{-1}\right] d\xi \\ &= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[R_y(\xi) z - zR_x(\xi)\right] d\xi. \end{aligned}$$

By taking the norm in this equality and using the properties of Bochner's integral [17] we get

$$\|f(y)z - zf(x)\| \leq \frac{1}{2\pi}\int_\gamma |f(\xi)| \|R_y(\xi) z - zR_x(\xi)\| |d\xi|. \tag{2.8}$$

By taking the norm in the equality (2.1) and using the properties of the norm, we get

$$\begin{aligned} \|R_y(\xi) z - zR_x(\xi)\| &= \|R_y(\xi)(yz - zx) R_x(\xi)\| \\ &\leq \|R_y(\xi)\| \|yz - zx\| \|R_x(\xi)\| \end{aligned}$$

and by (2.8) we get

$$\|f(y)z - zf(x)\| \leq \frac{1}{2\pi}\|yz - zx\| \int_\gamma |f(\xi)| \|R_y(\xi)\| \|R_x(\xi)\| |d\xi|, \tag{2.9}$$

which proves the first inequality in (2.5).

For $\xi \in \gamma$ we get

$$\begin{aligned}\|R_y(\xi)\| &= \frac{1}{|\xi|}\left\|\left(1-\frac{y}{\xi}\right)^{-1}\right\| = \frac{1}{|\xi|}\left\|\sum_{n=0}^{\infty}\left(\frac{y}{\xi}\right)^n\right\| \\ &\leq \frac{1}{|\xi|}\sum_{n=0}^{\infty}\left\|\frac{y}{\xi}\right\|^n = \frac{1}{|\xi|}\frac{1}{1-\left\|\frac{y}{\xi}\right\|} = \frac{1}{|\xi|-\|y\|}\end{aligned}$$

and, similarly

$$\|R_x(\xi)\| \leq \frac{1}{|\xi|-\|x\|}.$$

By making use of (2.9), we deduce the second inequality in (2.5). □

Remark 1. *The inequality between the first and last term in (2.6) has been obtained in* [10] *while the inequality between the first and last term in (2.7), in* [9].

Corollary 1. *With the assumptions of Theorem 2 and if*

$$\|f\|_{\gamma,\infty} := \sup_{\xi\in\gamma}|f(\xi)| < \infty,$$

then

$$\begin{aligned}\|f(y)z - zf(x)\| &\leq \frac{1}{2\pi}\|yz-zx\|\,\|f\|_{\gamma,\infty}\int_\gamma \|R_y(\xi)\|\,\|R_x(\xi)\|\,|d\xi| \\ &\leq \frac{1}{2\pi}\|yz-zx\|\,\|f\|_{\gamma,\infty}\int_\gamma \frac{|d\xi|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}.\end{aligned} \tag{2.10}$$

In particular,

$$\begin{aligned}\|f(x)z - zf(x)\| &\leq \frac{1}{2\pi}\|xz-zx\|\,\|f\|_{\gamma,\infty}\int_\gamma \|R_x(\xi)\|^2\,|d\xi| \\ &\leq \frac{1}{2\pi}\|xz-zx\|\,\|f\|_{\gamma,\infty}\int_\gamma \frac{|d\xi|}{(|\xi|-\|x\|)^2}\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\|f(y) - f(x)\| &\leq \frac{1}{2\pi}\|y-x\|\,\|f\|_{\gamma,\infty}\int_\gamma \|R_y(\xi)\|\,\|R_x(\xi)\|\,|d\xi| \\ &\leq \frac{1}{2\pi}\|y-x\|\,\|f\|_{\gamma,\infty}\int_\gamma \frac{|d\xi|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}.\end{aligned} \tag{2.12}$$

Remark 2. *If we assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in* 0 *and of radius R, then by taking γ parametrized by $\xi(t) = Re^{2\pi it}$ where $t \in [0,1]$, then $d\xi(t) = 2\pi i Re^{2\pi it}dt$, $|d\xi(t)| = 2\pi R dt$, $|\xi| = R$ and by (2.10) we get*

$$\begin{aligned}&\|f(y)z - zf(x)\| \\ &\leq R\,\|yz-zx\|\int_0^1 \left|f\left(Re^{2\pi it}\right)\right|\left\|R_y\left(Re^{2\pi it}\right)\right\|\left\|R_x\left(Re^{2\pi it}\right)\right\|dt \\ &\leq \frac{R}{(R-\|y\|)(R-\|x\|)}\|yz-zx\|\int_0^1\left|f\left(Re^{2\pi it}\right)\right|dt.\end{aligned} \tag{2.13}$$

In particular,

$$\left\| f(x)z - zf(x) \right\| \le R\left\| xz - zx \right\| \int_0^1 \left| f\left(Re^{2\pi it} \right) \right| \left\| R_x\left(Re^{2\pi it} \right) \right\|^2 dt \tag{2.14}$$
$$\le \frac{R}{\left(R - \left\| x \right\| \right)^2} \left\| xz - zx \right\| \int_0^1 \left| f\left(Re^{2\pi it} \right) \right| dt$$

and

$$\begin{aligned} &\left\| f(y) - f(x) \right\| \\ &\le R\left\| y - x \right\| \int_0^1 \left| f\left(Re^{2\pi it} \right) \right| \left\| R_y\left(Re^{2\pi it} \right) \right\| \left\| R_x\left(Re^{2\pi it} \right) \right\| dt \\ &\le \frac{R}{\left(R - \left\| y \right\| \right)\left(R - \left\| x \right\| \right)} \left\| y - x \right\| \int_0^1 \left| f\left(Re^{2\pi it} \right) \right| dt. \end{aligned} \tag{2.15}$$

Moreover, if

$$\left\| f \right\|_{R,\infty} := \sup_{t\in[0,1]} \left| f\left(Re^{2\pi it} \right) \right| < \infty,$$

then we have the simpler inequality

$$\begin{aligned} &\left\| f(y)z - zf(x) \right\| \\ &\le R\left\| yz - zx \right\| \left\| f \right\|_{R,\infty} \int_0^1 \left\| R_y\left(Re^{2\pi it} \right) \right\| \left\| R_x\left(Re^{2\pi it} \right) \right\| dt \\ &\le \frac{R\left\| f \right\|_{R,\infty}}{\left(R - \left\| y \right\| \right)\left(R - \left\| x \right\| \right)} \left\| yz - zx \right\| \end{aligned} \tag{2.16}$$

and in particular

$$\left\| f(x)z - zf(x) \right\| \le R\left\| xz - zx \right\| \left\| f \right\|_{R,\infty} \int_0^1 \left\| R_x\left(Re^{2\pi it} \right) \right\|^2 dt \tag{2.17}$$
$$\le \frac{R\left\| f \right\|_{R,\infty}}{\left(R - \left\| x \right\| \right)^2} \left\| yz - zx \right\|$$

and

$$\left\| f(y) - f(x) \right\| \le R\left\| y - x \right\| \left\| f \right\|_{R,\infty} \int_0^1 \left\| R_y\left(Re^{2\pi it} \right) \right\| \left\| R_x\left(Re^{2\pi it} \right) \right\| dt \tag{2.18}$$
$$\le \frac{R\left\| f \right\|_{R,\infty}}{\left(R - \left\| y \right\| \right)\left(R - \left\| x \right\| \right)} \left\| y - x \right\|.$$

Corollary 2. *Let f, $g : D \subset \mathbb{C} \to \mathbb{C}$ be analytic functions on the domain D and x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D$ and γ a closed rectifiable path in D and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Then we have*

$$\begin{aligned} \left\| f(x)g(y) - g(y)f(x) \right\| &\le \frac{1}{4\pi^2} \left\| xy - yx \right\| \\ &\quad \times \int_\gamma \left| f(\xi) \right| \left\| R_x(\xi) \right\|^2 \left| d\xi \right| \int_\gamma \left| g(\xi) \right| \left\| R_y(\xi) \right\|^2 \left| d\xi \right| \\ &\le \frac{1}{4\pi^2} \left\| xy - yx \right\| \int_\gamma \frac{\left| f(\xi) \right| \left| d\xi \right|}{\left(\left| \xi \right| - \left\| x \right\| \right)^2} \int_\gamma \frac{\left| g(\xi) \right| \left| d\xi \right|}{\left(\left| \xi \right| - \left\| y \right\| \right)^2}. \end{aligned} \tag{2.19}$$

Moreover, if

$$\|f\|_{\gamma,\infty} := \sup_{\xi\in\gamma} |f(\xi)| < \infty, \quad \|g\|_{\gamma,\infty} := \sup_{\xi\in\gamma} |g(\xi)| < \infty,$$

then we have

$$\begin{aligned}
(2.20)\qquad & \|f(x)g(y) - g(y)f(x)\| \\
&\le \frac{1}{4\pi^2}\|f\|_{\gamma,\infty}\|g\|_{\gamma,\infty}\|xy-yx\|\int_\gamma \|R_x(\xi)\|^2|d\xi|\int_\gamma \|R_y(\xi)\|^2|d\xi| \\
&\le \frac{1}{4\pi^2}\|f\|_{\gamma,\infty}\|g\|_{\gamma,\infty}\|xy-yx\|\int_\gamma \frac{|d\xi|}{(|\xi|-\|x\|)^2}\int_\gamma \frac{|d\xi|}{(|\xi|-\|y\|)^2}.
\end{aligned}$$

Proof. From the first inequality in (2.6) for $z = g(y)$ we get

$$\|f(x)g(y) - g(y)f(x)\| \le \frac{1}{2\pi}\|xg(y) - g(y)x\|\int_\gamma |f(\xi)|\|R_x(\xi)\|^2|d\xi|.$$

From the same inequality we have

$$\|xg(y) - g(y)x\| = \|g(y)x - xg(y)\| \le \frac{1}{2\pi}\|xy-yx\|\int_\gamma |g(\xi)|\|R_y(\xi)\|^2|d\xi|.$$

From these two inequalities we obtain

$$\begin{aligned}
&\|f(x)g(y) - g(y)f(x)\| \\
&\le \frac{1}{2\pi}\left(\frac{1}{2\pi}\|xy-yx\|\int_\gamma |g(\xi)|\|R_y(\xi)\|^2|d\xi|\right)\int_\gamma |f(\xi)|\|R_x(\xi)\|^2|d\xi| \\
&= \frac{1}{4\pi^2}\|xy-yx\|\int_\gamma |g(\xi)|\|R_y(\xi)\|^2|d\xi|\int_\gamma |f(\xi)|\|R_x(\xi)\|^2|d\xi|,
\end{aligned}$$

which proves the first part of (2.19).

The second part is obvious. □

Remark 3. *If we assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then*

$$\begin{aligned}
(2.21)\qquad & \|f(x)g(y) - g(y)f(x)\| \\
&\le R^2\|xy-yx\| \\
&\times \int_0^1 \left|f\left(Re^{2\pi it}\right)\right|\left\|R_x\left(Re^{2\pi it}\right)\right\|^2 dt\int_0^1 \left|g\left(Re^{2\pi it}\right)\right|\left\|R_y\left(Re^{2\pi it}\right)\right\|^2 dt \\
&\le \frac{R^2}{(|R|-\|x\|)^2(|R|-\|y\|)^2}\|xy-yx\| \\
&\times \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt\int_0^1 \left|g\left(Re^{2\pi it}\right)\right| dt.
\end{aligned}$$

Moreover, if

$$\|f\|_{R,\infty} := \sup_{t\in[0,1]}\left|f\left(Re^{2\pi it}\right)\right| < \infty, \quad \|g\|_{R,\infty} := \sup_{t\in[0,1]}\left|g\left(Re^{2\pi it}\right)\right| < \infty,$$

then we have

$$\begin{aligned}\left\|f(x)g(y)-g(y)f(x)\right\| &\le R^2\left\|f\right\|_{R,\infty}\left\|g\right\|_{R,\infty}\left\|xy-yx\right\| \\ &\quad\times\int_0^1\left\|R_x\left(Re^{2\pi it}\right)\right\|^2dt\int_0^1\left\|R_y\left(Re^{2\pi it}\right)\right\|^2dt \\ &\le\frac{R^2\left\|f\right\|_{R,\infty}\left\|g\right\|_{R,\infty}}{\left(|R|-\|x\|\right)^2\left(|R|-\|y\|\right)^2}\left\|xy-yx\right\|.\end{aligned}\tag{2.22}$$

When $g=f$ we have

$$\begin{aligned}&\left\|f(x)f(y)-f(y)f(x)\right\| \\ &\le R^2\left\|xy-yx\right\| \\ &\quad\times\int_0^1\left|f\left(Re^{2\pi it}\right)\right|\left\|R_x\left(Re^{2\pi it}\right)\right\|^2dt\int_0^1\left|f\left(Re^{2\pi it}\right)\right|\left\|R_y\left(Re^{2\pi it}\right)\right\|^2dt \\ &\le\frac{R^2}{\left(|R|-\|x\|\right)^2\left(|R|-\|y\|\right)^2}\left\|xy-yx\right\|\left(\int_0^1\left|f\left(Re^{2\pi it}\right)\right|dt\right)^2\end{aligned}\tag{2.23}$$

and

$$\begin{aligned}&\left\|f(x)f(y)-f(y)f(x)\right\| \\ &\le R^2\left\|f\right\|_{R,\infty}^2\left\|xy-yx\right\|\int_0^1\left\|R_x\left(Re^{2\pi it}\right)\right\|^2dt\int_0^1\left\|R_y\left(Re^{2\pi it}\right)\right\|^2dt \\ &\le\frac{R^2\left\|f\right\|_{R,\infty}^2}{\left(|R|-\|x\|\right)^2\left(|R|-\|y\|\right)^2}\left\|xy-yx\right\|.\end{aligned}\tag{2.24}$$

3. Some Related Results

We also have the following fact [6, p. 199]:

Lemma 2. *Let $\lambda\in\mathbb{C}$, $\lambda\neq0$ and $a,\,b\in\mathcal{B}$. If $\lambda\in\rho(ab)$, then $\lambda\in\rho(ba)$ and we have the equality*

$$\lambda R_{ba}(\lambda)=1+bR_{ab}(\lambda)\,a.\tag{3.1}$$

Also $\sigma(ab)\cup\{0\}=\sigma(ba)\cup\{0\}$.

We have the following identity for the generalized commutator:

Lemma 3. *For any elements a, b, c in the Banach algebra $\mathcal{B}$ and for any $n\geq1$ we have*

$$a^nc-cb^n=\sum_{i=0}^{n-1}a^{n-i-1}(ac-cb)\,b^i.\tag{3.2}$$

In particular, we have

$$a^nc-ca^n=\sum_{i=0}^{n-1}a^{n-i-1}(ac-ca)\,a^i\tag{3.3}$$

and

$$a^n-b^n=\sum_{i=0}^{n-1}a^{n-i-1}(a-b)\,b^i.\tag{3.4}$$

Proof. We prove it by induction over n. For $n = 1$ we obtain in both sides of (3.2) the same quantity $ac - cb$. Assume that for $k \geq 2$ we have that

$$a^k c - cb^k = \sum_{i=0}^{k-1} a^{k-i-1} (ac - cb) b^i$$

and let us prove that

$$a^{k+1} c - cb^{k+1} = \sum_{i=0}^{k} a^{k-i} (ac - cb) b^i.$$

We have

$$\begin{aligned}
\sum_{i=0}^{k} a^{k-i} (ac - cb) b^i &= \sum_{i=0}^{k-1} a^{k-i} (ac - cb) b^i + a^{k-k} (ac - cb) b^k \\
&= a \sum_{i=0}^{k-1} a^{k-i-1} (ac - cb) b^i + (ac - cb) b^k \\
&= a \left(a^k c - cb^k\right) + (ac - cb) b^k \text{ (by induction hypothesis)} \\
&= a^{k+1} c - acb^k + acb^k - cb^{k+1} = a^{k+1} c - cb^{k+1}
\end{aligned}$$

and the proof is completed. □

Corollary 3. *The following simple equality also holds*

$$(xy)^n x = x (yx)^n \tag{3.5}$$

for all $n \geq 0$ and x, y in the Banach algebra $\mathcal{B}$.

Proof. If we take $a = xy$, $b = yx$ and $c = x$ in (3.2), then we get

$$(xy)^n x - x (yx)^n = \sum_{i=0}^{n-1} (xy)^{n-i-1} (xyx - xyx) (yx)^i = 0,$$

which proves (3.5). □

Lemma 4. *Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and a, $b \in \mathcal{B}$. If $\lambda \in \rho(ab)$, then $\lambda \in \rho(ba)$ and*

$$aR_{ba}(\lambda) = R_{ab}(\lambda) a. \tag{3.6}$$

Proof. We have for $\lambda \in \rho(ab)$, $\lambda \neq 0$ that

$$R_{ab}(\lambda) a = \frac{1}{\lambda} \left(1 - \frac{ab}{\lambda}\right)^{-1} a = \frac{1}{\lambda} \left(\sum_{n=0}^{\infty} \left(\frac{ab}{\lambda}\right)^n\right) a = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(ab)^n a}{\lambda^n}.$$

By using (3.5) we have $(ab)^n a = a (ba)^n$ for all $n \geq 0$ and since, by Lemma 2, $\lambda \in \rho(ba)$, then

$$\begin{aligned}
\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(ab)^n a}{\lambda^n} &= \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{a (ba)^n}{\lambda^n} = \frac{1}{\lambda} a \left(\sum_{n=0}^{\infty} \frac{(ba)^n}{\lambda^n}\right) \\
&= \frac{1}{\lambda} a \sum_{n=0}^{\infty} \left(\frac{ba}{\lambda}\right)^n = \frac{1}{\lambda} a \left(1 - \frac{ba}{\lambda}\right)^{-1} = aR_{ba}(\lambda),
\end{aligned}$$

and the equality (3.6) is proved. □

Theorem 3. *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and a, $b \in \mathcal{B}$ with $\sigma(ab) \cup \{0\} \subset D$. Then we have*

$$af(ba) = f(ab)a \tag{3.7}$$

and

$$bf(ab) = f(ba)b. \tag{3.8}$$

Proof. Let γ be a closed rectifiable path in D and such that $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\} \subset \operatorname{ins}(\gamma)$. By using the identity (3.6) and the Riesz functional calculus, we have

$$\begin{aligned} af(ba) &= \frac{1}{2\pi i} a \int_\gamma f(\xi)(\xi - ba)^{-1} d\xi = \frac{1}{2\pi i} \int_\gamma f(\xi) a R_{ba}(\xi) d\xi = \\ &= \frac{1}{2\pi i} \int_\gamma f(\xi) R_{ab}(\xi) a d\xi = \left(\frac{1}{2\pi i} \int_\gamma f(\xi) R_{ab}(\xi) d\xi \right) a = \\ &= f(ab)a, \end{aligned}$$

and the identity (3.7) is proved. The identity (3.8) follows by (3.7). $\square$

Corollary 4. *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and a, $b \in \mathcal{B}$ with $\sigma(ab) \cup \{0\} \subset D$. If γ is a closed rectifiable path in D and such that $\sigma(ab) \cup \{0\} \subset \operatorname{ins}(\gamma)$, then*

$$\begin{aligned} &\|f(ab)a - af(ab)\| \\ &\le \frac{1}{2\pi} \|a\| \|ba - ab\| \\ &\times \min\left\{ \int_\gamma |f(\xi)| \|R_{ab}(\xi)\|^2 |d\xi|, \int_\gamma |f(\xi)| \|R_{ba}(\xi)\| \|R_{ab}(\xi)\| |d\xi| \right\} \\ &\le \frac{1}{2\pi} \|a\| \|ba - ab\| \\ &\times \min\left\{ \int_\gamma \frac{|f(\xi)| |d\xi|}{(|\xi| - \|ab\|)^2}, \int_\gamma \frac{|f(\xi)| |d\xi|}{(|\xi| - \|ba\|)(|\xi| - \|ab\|)} \right\}. \end{aligned} \tag{3.9}$$

Proof. Using the inequality (2.6) we get

$$\begin{aligned} \|f(ab)a - af(ab)\| &\le \frac{1}{2\pi} \left\|aba - a^2 b\right\| \int_\gamma |f(\xi)| \|R_{ab}(\xi)\|^2 |d\xi| \\ &\le \frac{1}{2\pi} \|a\| \|ba - ab\| \int_\gamma |f(\xi)| \|R_{ab}(\xi)\|^2 |d\xi| \\ &\le \frac{1}{2\pi} \|a\| \|ba - ab\| \int_\gamma \frac{|f(\xi)| |d\xi|}{(|\xi| - \|ab\|)^2}. \end{aligned} \tag{3.10}$$

From (3.7) we get

$$a(f(ba) - f(ab)) = af(ba) - af(ab) = f(ab)a - af(ab).$$

Taking the norm and using the inequality (2.5) we also have

$$\begin{aligned} \|f(ab)a - af(ab)\| &= \|a(f(ba) - f(ab))\| \leq \|a\| \|f(ba) - f(ab)\| \\ &\leq \frac{1}{2\pi} \|a\| \|ba - ab\| \int_\gamma |f(\xi)| \|R_{ba}(\xi)\| \|R_{ab}(\xi)\| |d\xi| \\ &\leq \frac{1}{2\pi} \|a\| \|ba - ab\| \int_\gamma \frac{|f(\xi)| |d\xi|}{(|\xi| - \|ba\|)(|\xi| - \|ab\|)}. \end{aligned} \tag{3.11}$$

On making use of (3.10) and (3.11) we get the desired inequality (3.9). □

Remark 4. *If we assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the domain D and $a, b \in \mathcal{B}$ with $\sigma(ab) \cup \{0\} \subset D \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then*

$$\|f(ab)a - af(ab)\| \leq \frac{R\|a\| \|ba - ab\|}{(R - \|ab\|)\max\{R - \|ab\|, R - \|ba\|\}} \|f\|_{R,\infty} \tag{3.12}$$

provided that

$$\|f\|_{R,\infty} := \sup_{t \in [0,1]} \left|f\left(Re^{2\pi i t}\right)\right| < \infty.$$

Corollary 5. *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and $a, b \in \mathcal{B}$ with $\sigma(ab) \cup \{0\} \subset D$. If γ is a closed rectifiable path in D and such that $\sigma(ab) \cup \{0\} \subset \operatorname{ins}(\gamma)$, then*

$$\begin{aligned} &\|af(ba) - f(ba)a + bf(ab) - f(ab)b\| \\ &\leq \frac{1}{2\pi} \|a - b\| \|ab - ba\| \int_\gamma |f(\xi)| \|R_{ab}(\xi)\| \|R_{ba}(\xi)\| |d\xi| \\ &\leq \frac{1}{2\pi} \|a - b\| \|ab - ba\| \int_\gamma \frac{|f(\xi)| |d\xi|}{(|\xi| - \|ba\|)(|\xi| - \|ab\|)}. \end{aligned} \tag{3.13}$$

Proof. From (3.7) we have

$$af(ba) - f(ab)b = f(ab)a - f(ab)b = f(ab)(a - b)$$

while from (3.8) we get

$$f(ba)a - bf(ab) = f(ba)a - f(ba)b = f(ba)(a - b),$$

which implies

$$af(ba) - f(ab)b - f(ba)a + bf(ab) = f(ab)(a - b) - f(ba)(a - b)$$

namely

$$af(ba) - f(ba)a + bf(ab) - f(ab)b = [f(ab) - f(ba)](a - b). \tag{3.14}$$

By taking the norm in (3.14), we get

$$\begin{aligned} \|af(ba) - f(ba)a + bf(ab) - f(ab)b\| &= \|[f(ab) - f(ba)](a - b)\| \\ &\leq \|f(ab) - f(ba)\| \|a - b\|. \end{aligned} \tag{3.15}$$

From the inequality (2.7) we have

$$\begin{aligned} \|f(ab) - f(ba)\| &\leq \frac{1}{2\pi} \|ab - ba\| \int_\gamma |f(\xi)| \|R_{ab}(\xi)\| \|R_{ba}(\xi)\| |d\xi| \\ &\leq \frac{1}{2\pi} \|ab - ba\| \int_\gamma \frac{|f(\xi)| |d\xi|}{(|\xi| - \|ba\|)(|\xi| - \|ab\|)} \end{aligned}$$

and by (3.15) we deduce the desired result (3.13). $\square$

Remark 5. *If we assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the domain D and $a, b \in \mathcal{B}$ with $\sigma(ab) \cup \{0\} \subset D \subset \overline{D}(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then*

$$\|af(ba) - f(ba)a + bf(ab) - f(ab)b\| \leq \frac{R\|a-b\|\|ab-ba\|\|f\|_{R,\infty}}{(R-\|ba\|)(R-\|ab\|)} \tag{3.16}$$

provided that

$$\|f\|_{R,\infty} := \sup_{t\in[0,1]} \left|f\left(Re^{2\pi it}\right)\right| < \infty.$$

4. Some Examples

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function*. For $n = 0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}. \tag{4.1}$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t - \nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n = 0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Let $\alpha \in \mathbb{R}$ and $R > 0$. If we change the variable $\theta = 2\pi t$, then $dt = \frac{1}{2\pi}d\theta$ and

$$\begin{aligned}
\int_0^1 \exp[\alpha R\cos(2\pi t)]\,dt &= \frac{1}{2\pi}\int_0^{2\pi}\exp[\alpha R\cos\theta]\,d\theta \\
&= \frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi \exp[\alpha R\cos\theta]\,d\theta + \frac{1}{\pi}\int_\pi^{2\pi}\exp[\alpha R\cos\theta]\,d\theta\right) \\
&= \frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi \exp[\alpha R\cos\theta]\,d\theta + \frac{1}{\pi}\int_0^{\pi}\exp[-\alpha R\cos\theta]\,d\theta\right) \\
&= \frac{1}{2}\left(I_0(\alpha R) + I_0(-\alpha R)\right) = I_0(\alpha R) \text{ by (4.1).}
\end{aligned}$$

Consider the inequality (2.21) in the form

$$\begin{aligned}
\|f(x)g(y) - g(y)f(x)\| &\leq \frac{R^2}{(|R|-\|x\|)^2(|R|-\|y\|)^2}\|xy-yx\| \\
&\quad\times \int_0^1 \left|f\left(Re^{2\pi it}\right)\right|dt \int_0^1 \left|g\left(Re^{2\pi it}\right)\right|dt,
\end{aligned} \tag{4.2}$$

where $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R.

Consider the exponential function $f(a) = \exp(\alpha a)$, $a \in \mathcal{B}$ and $\alpha \in \mathbb{R}$. Assume that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(\alpha R e^{2\pi i t}\right)\right| = \left|\exp\left[\alpha R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[\alpha R\cos(2\pi t)\right].$$

Now, if we write the inequality (4.2) for the functions $f(x) = \exp(\alpha x)$, $x \in \mathcal{B}$, $\alpha \in \mathbb{R}$ and $g(y) = \exp(\beta y)$, $y \in \mathcal{B}$, $\beta \in \mathbb{R}$ then we get

$$\begin{aligned}
&\|\exp(\alpha x)\exp(\beta y) - \exp(\beta y)\exp(\alpha x)\| \\
&\leq \frac{R^2}{(|R| - \|x\|)^2 (|R| - \|y\|)^2}\|xy - yx\| \\
&\times \int_0^1 \exp\left[\alpha R\cos(2\pi t)\right] dt \int_0^1 \exp\left[\beta R\cos(2\pi t)\right] dt,
\end{aligned}$$

and since

$$\int_0^1 \exp\left[\alpha R\cos(2\pi t)\right] dt = I_0(\alpha R) \text{ and } \int_0^1 \exp\left[\beta R\cos(2\pi t)\right] dt = I_0(\beta R),$$

hence we get the following inequality of interest for the *exponential commutator*

$$\begin{aligned}
&\|\exp(\alpha x)\exp(\beta y) - \exp(\beta y)\exp(\alpha x)\| \\
&\leq \frac{R^2}{(|R| - \|x\|)^2 (|R| - \|y\|)^2}\|xy - yx\| I_0(\alpha R) I_0(\beta R)
\end{aligned} \tag{4.3}$$

for all $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$.

From the equality (3.7) we also have the equality

$$x\exp(yx) = \exp(xy)\,x \tag{4.4}$$

for all $x, y \in \mathcal{B}$.

Moreover, if x is invertible, then we get from (4.4) that

$$\exp(yx) = x^{-1}\exp(xy)\,x \tag{4.5}$$

for all $y \in \mathcal{B}$.

Finally, from the inequality (3.16) written for the exponential function, we get

$$\|xf(yx) - f(yx)\,x + yf(xy) - f(xy)\,y\| \leq \frac{R\exp(R)\|x - y\|\|xy - yx\|}{(R - \|yx\|)(R - \|xy\|)} \tag{4.6}$$

for $x, y \in \mathcal{B}$ with $\|xy\|, \|yx\| < R$.

The interested reader may apply some of the above inequalities for other analytic functions such as sin, cos, sinh, cosh. The details are omitted.

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