SOME WEIGHTED INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this paper we establish some weighted integral inequalities of Čebyšev and Grüss' type for convex functions.

1. Introduction

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b]. \tag{1.1}$$

For the Lebesgue measurable functions $q,\ h,\ k:[a,b]\to\mathbb{R}$ we introduce the weighted *Čebyšev's functional*

$$\begin{aligned}\mathcal{C}_{[a,b]}(h,k;q)&:=\frac{1}{\int_a^b q(t)}\int_a^b q(t)h(t)k(t)\,dt\\&\quad-\frac{1}{\int_a^b q(t)}\int_a^b q(t)h(t)\,dt\frac{1}{\int_a^b q(t)}\int_a^b q(t)k(t)\,dt=\mathcal{C}_{[a,b]}(k,h;q)\end{aligned}$$

and the associated $(\smile)$-*Čebyšev's functional*

$$\begin{aligned}\breve{\mathcal{C}}_{[a,b]}(h,k;q)&:=\frac{1}{\int_a^b q(t)}\int_a^b \breve{q}(t)h(t)\breve{k}(t)\,dt\\&\quad-\frac{1}{\int_a^b q(t)}\int_a^b \breve{q}(t)h(t)\,dt\frac{1}{\int_a^b q(t)}\int_a^b \breve{q}(t)k(t)\,dt=\mathcal{C}_{[a,b]}(h,k;\breve{q}),\end{aligned}$$

provided that all the Lebesgue integrals exist on $[a,b]$ and $\int_a^b q(t)\neq 0$.

For $q\equiv 1$ we have the *unweighted functionals*

$$\mathcal{C}_{[a,b]}(h,k):=\frac{1}{b-a}\int_a^b h(t)k(t)\,dt-\frac{1}{b-a}\int_a^b h(t)\,dt\frac{1}{b-a}\int_a^b k(t)\,dt$$

and

$$\breve{\mathcal{C}}_{[a,b]}(h,k):=\frac{1}{b-a}\int_a^b h(t)\breve{k}(t)\,dt-\frac{1}{b-a}\int_a^b h(t)\,dt\frac{1}{b-a}\int_a^b k(t)\,dt.$$

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It is well known that, if the functions (h,k) are *synchronous* on $[a,b]$, namely

$$[h(x)-h(y)][k(x)-k(y)]\geq 0$$

for all x, $y\in[a,b]$ and q is nonnegative, then the following weighted *Čebyšev inequality* holds

$$0\leq \breve{\mathcal{C}}_{[a,b]}(h,k;q). \tag{1.2}$$

If there exists the constants m, M, n, N such that $m\leq h\leq M$ and $n\leq k\leq N$ almost everywhere on $[a,b]$, then we also have the weighted *Grüss inequality* (see for instance [4] for an extension to general Lebesgue integral and positive measure):

$$\breve{\mathcal{C}}_{[a,b]}(h,k;q)\leq \frac{1}{4}(M-m)(N-n), \tag{1.3}$$

with $\frac{1}{4}$ as best possible constant.

For other Čebyšev and Grüss' type inequalities see [1]-[9], [11]-[12] and [14]-[21].

In [13] the authors proved the following *Čebyšev type inequality:*

Theorem 1. *Let f, $g:[a,b]\to\mathbb{R}$ be both convex or concave on $[a,b]$ and $p:[a,b]\to[0,\infty)$ integrable an symmetric, namely $p(a+b-x)=p(x)$ for all $x\in[a,b]$, then*

$$\int_a^b p(x)\,dx\int_a^b p(x)f(x)\breve{g}(x)\,dx\geq\int_a^b p(x)f(x)\,dx\int_a^b p(x)g(x)\,dx. \tag{1.4}$$

If one of the functions is convex and the other concave, then the sign of inequality reverses in (1.4).

If $p\equiv 1$ then for two functions that have the same convexity one can get from (1.4) that

$$\begin{aligned}\breve{\mathcal{C}}_{[a,b]}(f,g)&=\frac{1}{b-a}\int_a^b f(x)\breve{g}(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\frac{1}{b-a}\int_a^b g(x)\,dx\\&\geq 0.\end{aligned} \tag{1.5}$$

Moreover, if in addition g is symmetric then $\breve{g}(x)=g(x)$ and from (1.5) one obtains [13, Corollary 4]

$$(b-a)\int_a^b f(x)g(x)\,dx\geq\int_a^b f(x)\,dx\int_a^b g(x)\,dx. \tag{1.6}$$

In the subsequent paper [10] the authors provided an upper bound for $\breve{\mathcal{C}}_{[a,b]}(f,g)$ as follows:

Theorem 2. *Let f, $g:[a,b]\to\mathbb{R}$ be both convex or concave on $[a,b]$, then*

$$\begin{aligned}0&\leq\breve{\mathcal{C}}_{[a,b]}(f,g)\\&\leq\frac{1}{4}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]\left[\frac{g(a)+g(b)}{2}-g\left(\frac{a+b}{2}\right)\right],\end{aligned} \tag{1.7}$$

where the constant $\frac{1}{4}$ is best possible.

One can observe that for $g=f$ we get the inequality

$$0\leq\breve{\mathcal{C}}_{[a,b]}(f,f)\leq\frac{1}{4}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]^2. \tag{1.8}$$

Motivated by the above results, in this paper we establish some new weighted integral inequalities of Čebyšev and Grüss' type for convex functions.

2. Main Results

We have the following fundamental relation between the Čebyšev's functional and the associated ($\smile$)-Čebyšev's functional:

Lemma 1. *For the Lebesgue measurable functions* p, f, $g : [a,b] \to \mathbb{R}$, *we have the following equalities*

$$\mathcal{C}_{[a,b]}\left(\breve{f},\breve{g};p\right)=\breve{\mathcal{C}}_{[a,b]}\left(f,g;p\right)=\breve{\mathcal{C}}_{[a,b]}\left(g,f;p\right), \tag{2.1}$$

provided the involved Lebesgue integrals exist.

In particular

$$\mathcal{C}_{[a,b]}\left(\breve{f},\breve{g}\right)=\breve{\mathcal{C}}_{[a,b]}\left(f,g\right)=\breve{\mathcal{C}}_{[a,b]}\left(g,f\right). \tag{2.2}$$

Proof. We observe that

$$\begin{aligned}
&\int_a^b p(t)\,\breve{f}(t)\,\breve{g}(t)\,dt\\
&=\frac{1}{4}\int_a^b p(t)\left[f(t)+f(a+b-t)\right]\left[g(t)+g(a+b-t)\right]dt\\
&=\frac{1}{4}\left[\int_a^b p(t)\,f(t)\,g(t)\,dt+\int_a^b p(t)\,f(a+b-t)\,g(t)\,dt\right.\\
&\left.+\int_a^b p(t)\,f(t)\,g(a+b-t)\,dt+\int_a^b p(t)\,f(a+b-t)\,g(a+b-t)\,dt\right].
\end{aligned}$$

By using the change of variable $s=a+b-t$, $t\in[a,b]$ we have

$$\int_a^b p(t)\,f(a+b-t)\,g(t)\,dt=\int_a^b p(a+b-t)\,f(t)\,g(a+b-t)\,dt$$

and

$$\int_a^b p(t)\,f(a+b-t)\,g(a+b-t)\,dt=\int_a^b p(a+b-t)\,f(t)\,g(t)\,dt.$$

Therefore

$$\int_a^b p(t)\,f(t)\,g(t)\,dt+\int_a^b p(a+b-t)\,f(t)\,g(t)\,dt=2\int_a^b \breve{p}(t)\,f(t)\,g(t)\,dt$$

and

$$\begin{aligned}
&\int_a^b p(t)\,f(a+b-t)\,g(t)\,dt+\int_a^b p(t)\,f(t)\,g(a+b-t)\,dt\\
&=\int_a^b p(a+b-t)\,f(t)\,g(a+b-t)\,dt+\int_a^b p(t)\,f(t)\,g(a+b-t)\,dt\\
&=2\int_a^b \breve{p}(t)\,f(t)\,g(a+b-t)\,dt.
\end{aligned}$$

By using these identities we get

$$\begin{aligned}\int_a^b p(t)\breve{f}(t)\breve{g}(t)\,dt &= \frac{1}{4}\left[2\int_a^b \breve{p}(t) f(t) g(t)\,dt + 2\int_a^b \breve{p}(t) f(t) g(a+b-t)\,dt\right] \\ &= \frac{1}{2}\left[\int_a^b \breve{p}(t) f(t) g(t)\,dt + \int_a^b \breve{p}(t) f(t) g(a+b-t)\,dt\right] \\ &= \int_a^b \breve{p}(t) f(t)\breve{g}(t)\,dt.\end{aligned}$$

Since

$$\begin{aligned}\int_a^b p(t)\breve{f}(t)\,dt &= \int_a^b p(t)\frac{f(t)+f(a+b-t)}{2}dt \\ &= \frac{1}{2}\left[\int_a^b p(t) f(t)\,dt + \int_a^b p(t) f(a+b-t)\,dt\right] \\ &= \frac{1}{2}\left[\int_a^b p(t) f(t)\,dt + \int_a^b p(a+b-t) f(t)\,dt\right] = \int_a^b \breve{p}(t) f(t)\,dt\end{aligned}$$

and, similarly

$$\int_a^b p(t)\breve{g}(t)\,dt = \int_a^b \breve{p}(t) g(t)\,dt,$$

hence

$$\begin{aligned}&\int_a^b p(t)\int_a^b p(t)\breve{f}(t)\breve{g}(t)\,dt - \int_a^b p(t)\breve{f}(t)\,dt\int_a^b p(t)\breve{g}(t)\,dt \\ &= \int_a^b p(t)\int_a^b \breve{p}(t) f(t)\breve{g}(t)\,dt - \int_a^b \breve{p}(t) f(t)\,dt\int_a^b \breve{p}(t) g(t)\,dt,\end{aligned}$$

which proves the first equality in (2.1).

The second equality follows by the symmetry of the Čebyšev functional. $\square$

Lemma 2 ([10, Lemma 2.2])**.** *Let $f:[a,b]\to\mathbb{R}$ be convex (concave) on $[a,b]$, then $\breve{f}$ is nonincreasing (nondecreasing) on $\left[a,\frac{a+b}{2}\right]$ and nondecreasing (nonincreasing) on $\left[a,\frac{a+b}{2}\right]$.*

We have the following weighted integral inequality:

Theorem 3. *Assume that $f,g:[a,b]\to\mathbb{R}$ are both convex or concave on $[a,b]$ and $p:[a,b]\to[0,\infty)$ with $\int_a^b p(t)\,dt>0$. Then we have*

$$\begin{aligned}(2.3)\quad 0 &\le \breve{\mathcal{C}}_{[a,b]}(f,g;p) \\ &\le \frac{1}{4}\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right]\left[\frac{g(a)+g(b)}{2} - g\left(\frac{a+b}{2}\right)\right] \\ &\le \frac{1}{64}\left[f'_-(b)-f'_+(a)\right]\left[g'_-(b)-g'_+(a)\right](b-a)^2\end{aligned}$$

where

$$\check{\mathcal{C}}_{[a,b]}(f,g;p)=\frac{1}{\int_a^b p(t)}\int_a^b \check{p}(t)\,f(t)\,\check{g}(t)\,dt \\ -\frac{1}{\int_a^b p(t)}\int_a^b \check{p}(t)\,f(t)\,dt\frac{1}{\int_a^b p(t)}\int_a^b \check{p}(t)\,g(t)\,dt.$$

Proof. Since the set of differentiable convex functions defined on (a,b) is dense in the class of all convex functions defined on (a,b) in the uniform convergence topology, we can assume without loosing the generality that f and g are differentiable convex on (a,b). This imply that $\check{f}$ and $\check{g}$ are differentiable convex and nonincreasing on $\left[a,\frac{a+b}{2}\right]$ and nondreasing on $\left[a,\frac{a+b}{2}\right]$.

For any $x,\,y\in[a,b]$ with $x\neq y$, by Cauchy mean value theorem, there exists a c between x and y such that

$$\left[\check{f}(x)-\check{f}(y)\right](\check{g})'(c)=\left[\check{g}(x)-\check{g}(y)\right]\left(\check{f}\right)'(c), \tag{2.4}$$

which implies that

$$\left[\check{f}(x)-\check{f}(y)\right]\left[\check{g}(x)-\check{g}(y)\right](\check{g})'(c)=\left[\check{g}(x)-\check{g}(y)\right]^2\left(\check{f}\right)'(c).$$

Since $\check{f}$ and $\check{g}$ are differentiable and nonincreasing on $\left[a,\frac{a+b}{2}\right]$ and nondreasing on $\left[a,\frac{a+b}{2}\right]$ then $\left(\check{f}\right)'(c)$ and $(\check{g})'(c)$ have the same sign which implies that

$$\left[\check{f}(x)-\check{f}(y)\right]\left[\check{g}(x)-\check{g}(y)\right]\geq 0. \tag{2.5}$$

For $x=y$ the inequality (2.5) also holds, so $\left(\check{f},\check{g}\right)$ are synchronous on $[a,b]$.

Using the weighted Čebyšev's inequality (1.2) for $\left(\check{f},\check{g}\right)$ and p we get

$$0\leq\mathcal{C}_{[a,b]}\left(\check{f},\check{g};p\right)=\check{\mathcal{C}}_{[a,b]}(f,g;p),\ \text{by Lemma 1}$$

and the first inequality in (2.3) is proved.

Now, if we use the weighted Grüss' inequality (1.3) for the functions $\check{f}$, $\check{g}$ that satisfy the bounds

$$f\left(\frac{a+b}{2}\right)\leq\check{f}\leq\frac{f(a)+f(b)}{2} \tag{2.6}$$

and

$$g\left(\frac{a+b}{2}\right)\leq\check{g}\leq\frac{g(a)+g(b)}{2}, \tag{2.7}$$

we get

$$\mathcal{C}_{[a,b]}\left(\check{f},\check{g};p\right) \\ \leq\frac{1}{4}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]\left[\frac{g(a)+g(b)}{2}-g\left(\frac{a+b}{2}\right)\right]$$

and by Lemma 1 we deduce the second inequality in (2.3).

In paper [8] we proved between others that for any convex function $h:[a,b]\to\mathbb{R}$ we have the inequality

$$0\leq \frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\leq \frac{1}{4}\left[h'_-(b)-h'_+(a)\right](b-a)$$

with the constant $\frac{1}{4}$ as best possible.

By using this inequality for f and g we deduce the last part of (2.3). □

Remark 1. *If p is symmetrical on $[a,b]$, then by (2.3) we get*

$$\begin{aligned}0\leq \breve{\mathcal{C}}_{[a,b]}(f,g;p)&=\frac{1}{\int_a^b p(t)}\int_a^b p(t)\,f(t)\,\breve{g}(t)\,dt\\&\quad-\frac{1}{\int_a^b p(t)}\int_a^b p(t)\,f(t)\,dt\frac{1}{\int_a^b p(t)}\int_a^b p(t)\,g(t)\,dt\end{aligned}$$

and therefore we recapture the inequality (1.5).

The second inequality in (2.3) gives a weighted generalization of the inequality (1.7).

Corollary 1. *Assume that $f:[a,b]\to\mathbb{R}$ is either convex or concave on $[a,b]$ and $p:[a,b]\to[0,\infty)$ with $\int_a^b p(t)\,dt>0$. Then we have*

$$0\leq \breve{\mathcal{C}}_{[a,b]}(f,f;p)\leq \frac{1}{4}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]^2\leq \frac{1}{64}\left[f'_-(b)-f'_+(a)\right]^2 \tag{2.8}$$

where

$$\breve{\mathcal{C}}_{[a,b]}(f,f;p)=\frac{1}{\int_a^b p(t)}\int_a^b \breve{p}(t)\,f(t)\,\breve{f}(t)\,dt-\left(\frac{1}{\int_a^b p(t)}\int_a^b \breve{p}(t)\,f(t)\,dt\right)^2.$$

3. Related Results

We can improve the second inequality in (2.3) as follows:

Theorem 4. *Assume that f, $g:[a,b]\to\mathbb{R}$ are both convex on $[a,b]$ and $p:[a,b]\to[0,\infty)$ with $\int_a^b p(t)\,dt>0$.*

$$
\begin{aligned}
(3.1)\quad 0\leq \breve{\mathcal{C}}_{[a,b]}(f,g;p) &\leq \left(\frac{f(a)+f(b)}{2}-\frac{1}{\int_a^b p(t)\,dt}\int_a^b \breve{p}(t)\,f(t)\,dt\right)^{1/2}\\
&\quad\times\left(\frac{1}{\int_a^b p(t)\,dt}\int_a^b \breve{p}(t)\,f(t)\,dt-f\left(\frac{a+b}{2}\right)\right)^{1/2}\\
&\quad\times\left(\frac{g(a)+g(b)}{2}-\frac{1}{\int_a^b p(t)\,dt}\int_a^b \breve{p}(t)\,g(t)\,dt\right)^{1/2}\\
&\quad\times\left(\frac{1}{\int_a^b p(t)\,dt}\int_a^b \breve{p}(t)\,g(t)\,dt-g\left(\frac{a+b}{2}\right)\right)^{1/2}\\
&\leq\frac{1}{4}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]\left[\frac{g(a)+g(b)}{2}-g\left(\frac{a+b}{2}\right)\right]\\
&\leq\frac{1}{64}\left[f_-'(b)-f_+'(a)\right]\left[g_-'(b)-g_+'(a)\right](b-a)^2.
\end{aligned}
$$

Proof. We employ the following well known inequality that follows from the weighted Korkine's identity [15, p. 296] and the weighted double integral Cauchy-Bunyakovsky-Schwarz inequality

$$
\left[\mathcal{C}_{[a,b]}(h,k;q)\right]^2\leq\left[\mathcal{C}_{[a,b]}(h,h;q)\right]\left[\mathcal{C}_{[a,b]}(k,k;q)\right], \tag{3.2}
$$

see also [5].

We also have the identity [5, p. 399]

$$
\begin{aligned}
&\mathcal{C}_{[a,b]}(h,h;q)\\
&=\left(\Phi-\frac{1}{\int_a^b q(t)\,dt}\int_a^b q(t)\,h(t)\,dt\right)\left(\frac{1}{\int_a^b q(t)}\int_a^b q(t)\,h(t)\,dt-\phi\right)\\
&-\frac{1}{\int_a^b q(t)\,dt}\int_a^b q(t)\left[\Phi-h(t)\right]\left[h(t)-\phi\right]dt
\end{aligned}
$$

and if $[\Phi-h(t)][h(t)-\phi]\geq 0$ and $q(t)\geq 0$ for $t\in[a,b]$, then

$$
\begin{aligned}
(3.3)\quad &\mathcal{C}_{[a,b]}(h,h;q)\leq\\
&\left(\Phi-\frac{1}{\int_a^b q(t)\,dt}\int_a^b q(t)\,h(t)\,dt\right)\left(\frac{1}{\int_a^b q(t)\,dt}\int_a^b q(t)\,h(t)\,dt-\phi\right)\\
&\qquad\leq\frac{1}{4}(\Phi-\phi)^2.
\end{aligned}
$$

From (3.2) we have

$$
\left[\mathcal{C}_{[a,b]}\left(\breve{f},\breve{g};p\right)\right]^2\leq\left[\mathcal{C}_{[a,b]}\left(\breve{f},\breve{f};p\right)\right]\left[\mathcal{C}_{[a,b]}(\breve{g},\breve{g};p)\right], \tag{3.4}
$$

and since $\breve{f}$ and $\breve{g}$ satisfy the conditions (2.6) and (2.7), hence

$$\begin{aligned}\mathcal{C}_{[a,b]}\left(\breve{f},\breve{f};p\right) &\leq \left(\frac{f(a)+f(b)}{2}-\frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,\breve{f}(t)\,dt\right)\\ &\quad\times\left(\frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,\breve{f}(t)\,dt - f\left(\frac{a+b}{2}\right)\right)\\ &\leq \frac{1}{4}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]^2,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\mathcal{C}_{[a,b]}\left(\breve{g},\breve{g};p\right) &\leq \left(\frac{g(a)+g(b)}{2}-\frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,\breve{g}(t)\,dt\right)\\ &\quad\times\left(\frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,\breve{g}(t)\,dt - g\left(\frac{a+b}{2}\right)\right)\\ &\leq \frac{1}{4}\left[\frac{g(a)+g(b)}{2}-g\left(\frac{a+b}{2}\right)\right]^2.\end{aligned} \tag{3.6}$$

By taking into account that

$$\int_a^b p(t)\,\breve{f}(t)\,dt=\int_a^b \breve{p}(t)\,f(t)\,dt \text{ and } \int_a^b p(t)\,\breve{g}(t)\,dt=\int_a^b \breve{p}(t)\,g(t)\,dt$$

and by making use of (3.5), (3.6) and the representation Lemma 1 we deduce the desired result (3.1). □

Remark 2. *If p is symmetrical on $[a,b]$, then by (3.1) we get*

$$\begin{aligned}0\leq \breve{\mathcal{C}}_{[a,b]}(f,g;p) &\leq \left(\frac{f(a)+f(b)}{2}-\frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,f(t)\,dt\right)^{1/2}\\ &\quad\times\left(\frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,f(t)\,dt - f\left(\frac{a+b}{2}\right)\right)^{1/2}\\ &\quad\times\left(\frac{g(a)+g(b)}{2}-\frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,g(t)\,dt\right)^{1/2}\\ &\quad\times\left(\frac{1}{\int_a^b p(t)\,dt}\int_a^b p(t)\,g(t)\,dt - g\left(\frac{a+b}{2}\right)\right)^{1/2}\\ &\leq \frac{1}{4}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]\left[\frac{g(a)+g(b)}{2}-g\left(\frac{a+b}{2}\right)\right]\\ &\leq \frac{1}{64}\left[f'_-(b)-f'_+(a)\right]\left[g'_-(b)-g'_+(a)\right](b-a)^2.\end{aligned} \tag{3.7}$$

Corollary 2. *Assume that* $f:[a,b]\to\mathbb{R}$ *is either convex or concave on* $[a,b]$ *and* $p:[a,b]\to[0,\infty)$ *with* $\int_a^b p(t)\,dt>0$*. Then we have*

$$
\begin{aligned}
(3.8)\quad 0 &\le \breve{\mathcal{C}}_{[a,b]}(f,f;p)\\
&\le \left(\frac{f(a)+f(b)}{2}-\frac{1}{\int_a^b p(t)\,dt}\int_a^b \breve{p}(t)\,f(t)\,dt\right)\\
&\quad\times\left(\frac{1}{\int_a^b p(t)\,dt}\int_a^b \breve{p}(t)\,f(t)\,dt-f\left(\frac{a+b}{2}\right)\right)\\
&\le \frac{1}{4}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]^2\le\frac{1}{64}\left[f'_-(b)-f'_+(a)\right]^2.
\end{aligned}
$$

The following case for unweighted inequalities is of interest:

Corollary 3. *Let* $f,\ g:[a,b]\to\mathbb{R}$ *be both convex on* $[a,b]$*, then*

$$
\begin{aligned}
(3.9)\quad 0 &\le \breve{\mathcal{C}}_{[a,b]}(f,g)\le\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)^{1/2}\\
&\quad\times\left(\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right)^{1/2}\\
&\quad\times\left(\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right)^{1/2}\\
&\quad\times\left(\frac{1}{b-a}\int_a^b g(t)\,dt-g\left(\frac{a+b}{2}\right)\right)^{1/2}\\
&\le\frac{1}{4}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]\left[\frac{g(a)+g(b)}{2}-g\left(\frac{a+b}{2}\right)\right]\\
&\le\frac{1}{64}\left[f'_-(b)-f'_+(a)\right]\left[g'_-(b)-g'_+(a)\right](b-a)^2.
\end{aligned}
$$

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