HERMITE-HADAMARD TRAPEZOID AND MID-POINT DIVERGENCES

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Abstract. In this paper we introduce the concepts of Hermite-Hadamard trapezoid and mid-point divergences that are closely related to the Jensen divergence considered by Burbea and Rao in 1982. The joint convexity of these divergences as well as several inequalities involving these measures are established. Various examples concerning the Csiszár, Lin-Wong and HH f-divergence measures are also given.

1. Introduction

For a function f defined on an interval I of the real line $\mathbb{R}$, by following the paper by Burbea & Rao [1], we consider the $\mathcal{J}$*-divergence* between the vectors x, $y\in I^n$ given by

$$\mathcal{J}_{n,f}(x,y):=\sum_{i=1}^{n}\left(\frac{1}{2}\left[f(x_i)+f(y_i)\right]-f\left(\frac{x_i+y_i}{2}\right)\right).$$

As important examples of such divergences, we can consider [1],

$$\mathcal{J}_{n,\alpha}(x,y):=\begin{cases}(\alpha-1)^{-1}\sum_{i=1}^{n}\left[\frac{1}{2}\left(x_i^{\alpha}+y_i^{\alpha}\right)-\left(\frac{x_i+y_i}{2}\right)^{\alpha}\right], & \alpha\neq 1\\ \\ \frac{1}{2}\sum_{i=1}^{n}\left[x_i\ln(x_i)+y_i\ln(y_i)-(x_i+y_i)\ln\left(\frac{x_i+y_i}{2}\right)\right], & \alpha=1.\end{cases}$$

If f is convex on I, then $\mathcal{J}_{n,f}(x,y)\geq 0$ for all $(x,y)\in I^n\times I^n$.

The following result concerning the joint convexity of $\mathcal{J}_{n,f}$ also holds:

Theorem 1 (Burbea-Rao, 1982 [1])**.** *Let f be a C^2 function on an interval I. Then $\mathcal{J}_{n,f}$ is convex (concave) on $I^n\times I^n$, if and only if f is convex (concave) and $\frac{1}{f''}$ is concave (convex) on I.*

We define the *Hermite-Hadamard trapezoid* and *mid-point divergences*

$$\mathcal{T}_{n,f}(x,y):=\sum_{i=1}^{n}\left(\frac{1}{2}\left[f(x_i)+f(y_i)\right]-\int_0^1 f\left((1-t)x_i+ty_i\right)dt\right)\tag{1.1}$$

and

$$\mathcal{M}_{n,f}(x,y):=\sum_{i=1}^{n}\left(\int_0^1 f\left((1-t)x_i+ty_i\right)dt-f\left(\frac{x_i+y_i}{2}\right)\right)\tag{1.2}$$

for all $(x,y)\in I^n\times I^n$.

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We observe that

$$\mathcal{J}_{n,f}(x,y)=\mathcal{T}_{n,f}(x,y)+\mathcal{M}_{n,f}(x,y) \tag{1.3}$$

for all $(x,y)\in I^{n}\times I^{n}$.

If f is convex on I, then by *Hermite-Hadamard inequalities*

$$\frac{f(a)+f(b)}{2}\geq\int_{0}^{1}f((1-t)a+tb)\,dt\geq f\left(\frac{a+b}{2}\right)$$

for all $a,\,b\in I$, we have the following fundamental facts

$$\mathcal{T}_{n,f}(x,y)\geq 0 \text{ and } \mathcal{M}_{n,f}(x,y)\geq 0 \tag{1.4}$$

for all $(x,y)\in I^{n}\times I^{n}$.

Using *Bullen's inequality*, see for instance [6, p. 2]

$$\begin{aligned}0&\leq\int_{0}^{1}f((1-t)a+tb)\,dt-f\left(\frac{a+b}{2}\right)\\&\leq\frac{f(a)+f(b)}{2}-\int_{0}^{1}f((1-t)a+tb)\,dt\end{aligned}$$

we also have

$$0\leq\mathcal{M}_{n,f}(x,y)\leq\mathcal{T}_{n,f}(x,y)\,. \tag{1.5}$$

Let us recall the following special means:

a) The *arithmetic mean*

$$A(a,b):=\frac{a+b}{2},\quad a,b>0,$$

b) The *geometric mean*

$$G(a,b):=\sqrt{ab};\quad a,b\geq 0,$$

c) The *harmonic mean*

$$H(a,b):=\frac{2}{\frac{1}{a}+\frac{1}{b}};\quad a,b>0,$$

d) The *identric mean*

$$I(a,b):=\begin{cases}\dfrac{1}{e}\left(\dfrac{b^{b}}{a^{a}}\right)^{\frac{1}{b-a}} & \text{if}\quad b\neq a\\ a & \text{if}\quad b=a\end{cases};\ a,b>0$$

e) The *logarithmic mean*

$$L(a,b):=\begin{cases}\dfrac{b-a}{\ln b-\ln a} & \text{if}\quad b\neq a\\ a & \text{if}\quad b=a\end{cases};\quad a,b>0$$

f) The *p-logarithmic mean*

$$L_{p}(a,b):=\begin{cases}\left(\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if}\quad b\neq a,\quad p\in\mathbb{R}\backslash\{-1,0\}\\ a & \text{if}\quad b=a\end{cases};\quad a,b>0.$$

If we put $L_0(a,b) := I(a,b)$ and $L_{-1}(a,b) := L(a,b)$, then it is well known that the function $\mathbb{R} \ni p \mapsto L_p(a,b)$ is monotonic increasing on $\mathbb{R}$.

We observe that for $p \in \mathbb{R}\setminus\{-1,0\}$ we have

$$\int_0^1 [(1-t)a+tb]^p \, dt = L_p^p(a,b), \quad \int_0^1 [(1-t)a+tb]^{-1} \, dt = L^{-1}(a,b)$$

and

$$\int_0^1 \ln[(1-t)a+tb] \, dt = \ln I(a,b).$$

Using these notations we can define the following divergences for $(x,y) \in I^n \times I^n$ where I is an interval of positive numbers:

$$\mathcal{T}_{n,p}(x,y) := \sum_{i=1}^{n} \left[A\left(x_i^p, y_i^p\right) - L_p^p(x_i,y_i)\right]$$

and

$$\mathcal{M}_{n,p}(x,y) := \sum_{i=1}^{n} \left[L_p^p(x_i,y_i) - A^p(x_i,y_i)\right]$$

for all $p \in \mathbb{R}\setminus\{-1,0\}$,

$$\mathcal{T}_{n,-1}(x,y) := \sum_{i=1}^{n} \left[H^{-1}(x_i,y_i) - L^{-1}(x_i,y_i)\right]$$

and

$$\mathcal{M}_{n,-1}(x,y) := \sum_{i=1}^{n} \left[L^{-1}(x_i,y_i) - A^{-1}(x_i,y_i)\right]$$

for $p=-1$ and

$$\mathcal{T}_{n,0}(x,y) := \ln\left[\prod_{i=1}^{n} \left(\frac{G(x_i,y_i)}{I(x_i,y_i)}\right)\right]$$

and

$$\mathcal{M}_{n,0}(x,y) := \ln\left[\prod_{i=1}^{n} \left(\frac{I(x_i,y_i)}{A(x_i,y_i)}\right)\right]$$

for $p=0$.

Since the function $f(t)=t^p$, $t>0$ is convex for $p \in (-\infty,0)\cup(1,\infty)$, then we have

$$\mathcal{T}_{n,p}(x,y), \ \mathcal{M}_{n,p}(x,y) \geq 0 \tag{1.6}$$

for all $(x,y) \in I^n \times I^n$.

For $p \in (0,1)$ the function $f(t)=t^p$, $t>0$ and for $p=0$, the function $f(t)=\ln t$ are concave, then we have for $p \in [0,1)$ that

$$\mathcal{T}_{n,p}(x,y), \ \mathcal{M}_{n,p}(x,y) \leq 0 \tag{1.7}$$

for all $(x,y) \in I^n \times I^n$.

Finally for $p=1$ we have both $\mathcal{T}_{n,p}(x,y) = \mathcal{M}_{n,p}(x,y) = 0$ for all $(x,y) \in I^n \times I^n$.

In this paper we establish the joint convexity of the *Hermite-Hadamard trapezoid* and *mid-point divergences* $\mathcal{T}_{n,f}$ and $\mathcal{M}_{n,f}$ and also provide several inequalities involving these measures. Several examples concerning the Csiszár, Lin-Wong and HH f-divergence measures are also given.

2. General Results

We start with the following convexity result that is a consequence of Burbea-Rao theorem above:

Theorem 2. *Let f be a C^2 function on an interval I. Then $\mathcal{T}_{n,f}$ and $\mathcal{M}_{n,f}$ are convex (concave) on $I^n \times I^n$, if and only if f is convex (concave) and $\frac{1}{f''}$ is concave (convex) on I.*

Proof. If $\mathcal{T}_{n,f}$ and $\mathcal{M}_{n,f}$ are convex on $I^n \times I^n$ then the sum $\mathcal{T}_{n,f} + \mathcal{M}_{n,f} = \mathcal{J}_{n,f}$ is convex on $I^n \times I^n$, which, by Burbea-Rao theorem implies that f is convex and $\frac{1}{f''}$ is concave on I.

Now, if f is convex and $\frac{1}{f''}$ is concave on I, then by the same theorem we have that the function $\mathcal{J}_f : I \times I \to \mathbb{R}$

$$\mathcal{J}_f(x,y) := \frac{1}{2}[f(x) + f(y)] - f\left(\frac{x+y}{2}\right)$$

is convex.

Let x, y, u, $v \in I$. We define

$$\begin{aligned}
\varphi(t) &:= \mathcal{J}_f((1-t)(x,y) + t(u,v)) = \mathcal{J}_f(((1-t)x + tu, (1-t)y + tv)) \\
&= \frac{1}{2}[f((1-t)x + tu) + f((1-t)y + tv)] \\
&\quad - f\left(\frac{(1-t)x + tu + (1-t)y + tv}{2}\right) \\
&= \frac{1}{2}[f((1-t)x + tu) + f((1-t)y + tv)] \\
&\quad - f\left((1-t)\frac{x+y}{2} + t\frac{u+v}{2}\right)
\end{aligned}$$

for $t \in [0,1]$.

Let t_1, $t_2 \in [0,1]$ and α, $\beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of $\mathcal{J}_f$ we have

$$\begin{aligned}
&\varphi(\alpha t_1 + \beta t_2) \\
&= \mathcal{J}_f((1 - \alpha t_1 - \beta t_2)(x,y) + (\alpha t_1 + \beta t_2)(u,v)) \\
&= \mathcal{J}_f((\alpha + \beta - \alpha t_1 - \beta t_2)(x,y) + (\alpha t_1 + \beta t_2)(u,v)) \\
&= \mathcal{J}_f(\alpha(1-t_1)(x,y) + \beta(1-t_2)(x,y) + \alpha t_1(u,v) + \beta t_2(u,v)) \\
&= \mathcal{J}_f(\alpha[(1-t_1)(x,y) + t_1(u,v)] + \beta[(1-t_2)(x,y) + t_2(u,v)]) \\
&\leq \alpha\mathcal{J}_f((1-t_1)(x,y) + t_1(u,v)) + \beta\mathcal{J}_f((1-t_2)(x,y) + t_2(u,v)) \\
&= \alpha\varphi(t_1) + \beta\varphi(t_2),
\end{aligned}$$

which proves that φ is convex on $[0,1]$ for all x, y, u, $v \in I$.

Applying the Hermite-Hadamard inequality for φ we get

$$\frac{1}{2}[\varphi(0) + \varphi(1)] \geq \int_0^1 \varphi(t)\,dt \tag{2.1}$$

and since

$$\varphi(0) = \frac{1}{2}[f(x) + f(y)] - f\left(\frac{x+y}{2}\right),$$

$$\varphi(1) = \frac{1}{2}[f(u) + f(v)] - f\left(\frac{u+v}{2}\right)$$

and

$$\begin{aligned}\int_0^1 \varphi(t)\,dt = \frac{1}{2}&\left[\int_0^1 f((1-t)x + tu)\,dt + \int_0^1 f((1-t)y + tv)\,dt\right] \\ &- \int_0^1 f\left((1-t)\frac{x+y}{2} + t\frac{u+v}{2}\right)dt,\end{aligned}$$

hence by (2.1) we get

$$\begin{aligned}&\frac{1}{2}\left\{\frac{1}{2}[f(x) + f(y)] - f\left(\frac{x+y}{2}\right) + \frac{1}{2}[f(u) + f(v)] - f\left(\frac{u+v}{2}\right)\right\} \\ &\geq \frac{1}{2}\left[\int_0^1 f((1-t)x + tu)\,dt + \int_0^1 f((1-t)y + tv)\,dt\right] \\ &- \int_0^1 f\left((1-t)\frac{x+y}{2} + t\frac{u+v}{2}\right)dt.\end{aligned}$$

Re-arranging this inequality, we get

$$\begin{aligned}&\frac{1}{2}\left[\frac{f(x) + f(u)}{2} - \int_0^1 f((1-t)x + tu)\,dt\right] \\ &+ \frac{1}{2}\left[\frac{f(y) + f(v)}{2} - \int_0^1 f((1-t)y + tv)\,dt\right] \\ &\geq \frac{1}{2}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{u+v}{2}\right) - \int_0^1 f\left((1-t)\frac{x+y}{2} + t\frac{u+v}{2}\right)dt\right]\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{1}{2}[\mathcal{T}_f(x,u) + \mathcal{T}_f(y,v)] &\geq \mathcal{T}_f\left(\frac{x+y}{2}, \frac{u+v}{2}\right) \\ &= \mathcal{T}_f\left(\frac{1}{2}(x,u) + \frac{1}{2}(y,v)\right),\end{aligned}$$

for all $(x,u), (y,v) \in I \times I$, which shows that $\mathcal{T}_f$ is Jensen's convex on $I \times I$. Since $\mathcal{T}_f$ is continuous on $I \times I$, hence $\mathcal{T}_f$ is convex in the usual sense on $I \times I$. Further, by summing over i from 1 to n we deduce that $\mathcal{T}_{n,f}$ is convex on $I^n \times I^n$.

Now, if we use the second Hermite-Hadamard inequality for φ on $[0,1]$, we have

$$\int_0^1 \varphi(t)\,dt \geq \varphi\left(\frac{1}{2}\right). \tag{2.2}$$

Since

$$\varphi\left(\frac{1}{2}\right) = \frac{1}{2}\left[f\left(\frac{x+u}{2}\right) + f\left(\frac{y+v}{2}\right)\right] - f\left(\frac{1}{2}\frac{x+y}{2} + \frac{1}{2}\frac{u+v}{2}\right)$$

hence by (2.2) we have

$$\begin{aligned}
&\frac{1}{2}\left[\int_0^1 f\left((1-t)x+tu\right)dt+\int_0^1 f\left((1-t)y+tv\right)dt\right]\\
&-\int_0^1 f\left((1-t)\frac{x+y}{2}+t\frac{u+v}{2}\right)dt\\
&\geq \frac{1}{2}\left[f\left(\frac{x+u}{2}\right)+f\left(\frac{y+v}{2}\right)\right]-f\left(\frac{1}{2}\left(\frac{x+y}{2}+\frac{u+v}{2}\right)\right),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{1}{2}\left[\int_0^1 f\left((1-t)x+tu\right)dt-f\left(\frac{x+u}{2}\right)\right]\\
&+\frac{1}{2}\left[\int_0^1 f\left((1-t)y+tv\right)dt-f\left(\frac{y+v}{2}\right)\right]\\
&\geq \int_0^1 f\left((1-t)\frac{x+y}{2}+t\frac{u+v}{2}\right)dt-f\left(\frac{1}{2}\left(\frac{x+y}{2}+\frac{u+v}{2}\right)\right)
\end{aligned}$$

that can be written as

$$\begin{aligned}
\frac{1}{2}\left[\mathcal{M}_f(x,u)+\mathcal{M}_f(y,v)\right] &\geq \mathcal{M}_f\left(\frac{x+y}{2},\frac{u+v}{2}\right)\\
&=\mathcal{M}_f\left(\frac{1}{2}(x,u)+\frac{1}{2}(y,v)\right)
\end{aligned}$$

for all $(x,u),(y,v)\in I\times I$, which shows that $\mathcal{M}_f$ is Jensen's convex on $I\times I$. Since $\mathcal{M}_f$ is continuous on $I\times I$, hence $\mathcal{M}_f$ is convex in the usual sense on $I\times I$. Further, by summing over i from 1 to n we deduce that $\mathcal{M}_{n,f}$ is convex on $I^n\times I^n$. □

The following reverses of the Hermite-Hadamard inequality hold:

Lemma 1 (Dragomir, 2002 [4] and [5]). *Let $h:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned}
0 &\leq \frac{1}{8}\left[h_+\left(\frac{a+b}{2}\right)-h_-\left(\frac{a+b}{2}\right)\right](b-a)\\
&\leq \frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(x)\,dx\\
&\leq \frac{1}{8}\left[h_-(b)-h_+(a)\right](b-a)
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}
0 &\leq \frac{1}{8}\left[h_+\left(\frac{a+b}{2}\right)-h_-\left(\frac{a+b}{2}\right)\right](b-a)\\
&\leq \frac{1}{b-a}\int_a^b h(x)\,dx-h\left(\frac{a+b}{2}\right)\\
&\leq \frac{1}{8}\left[h_-(b)-h_+(a)\right](b-a).
\end{aligned}\tag{2.4}$$

The constant $\frac{1}{8}$ is best possible in all inequalities from (2.3) and (2.4).

We also have:

Theorem 3. *Let f be a C^1 convex function on an interval I. If $\mathring{I}$ is the interior of I, then for all $(x,y)\in\mathring{I}^n\times\mathring{I}^n$ we have*

$$0\leq\mathcal{M}_{n,f}(x,y)\leq\mathcal{T}_{n,f}(x,y)\leq\frac{1}{8}\mathcal{C}_{n,f'}(x,y) \tag{2.5}$$

where

$$\mathcal{C}_{n,f'}(x,y):=\sum_{i=1}^{n}\left[f'(x_i)-f'(y_i)\right](x_i-y_i). \tag{2.6}$$

Proof. Since for $b\neq a$

$$\frac{1}{b-a}\int_a^b f(x)\,dx=\int_0^1 f((1-t)a+tb)\,dt,$$

then from (2.3) we get

$$\frac{f(x_i)+f(y_i)}{2}-\int_0^1 f((1-t)x_i+ty_i)\,dt\leq\frac{1}{8}\left[f'(x_i)-f'(y_i)\right](x_i-y_i)$$

for all $i\in\{1,...,n\}$ and this inequality also holds if $x_i=y_i$.

By summing these inequalities over $i\in\{1,...,n\}$ we get the last inequality in (2.5). □

Remark 1. *If*

$$\gamma=\inf_{t\in\mathring{I}}f'(t)\ \text{ and }\ \Gamma=\sup_{t\in\mathring{I}}f'(t)$$

are finite, then

$$\mathcal{C}_{n,f'}(x,y)\leq(\Gamma-\gamma)\sum_{i=1}^{n}|x_i-y_i|$$

and by (2.5) we get the simpler upper bound

$$0\leq\mathcal{M}_{n,f}(x,y)\leq\mathcal{T}_{n,f}(x,y)\leq\frac{1}{8}(\Gamma-\gamma)\sum_{i=1}^{n}|x_i-y_i|.$$

Moreover, if $x_i,\ y_i\in[a,b]\subset\mathring{I}$ for all $i\in\{1,...,n\}$ and since f' is increasing on $\mathring{I}$, then we have the inequalities

$$0\leq\mathcal{M}_{n,f}(x,y)\leq\mathcal{T}_{n,f}(x,y)\leq\frac{1}{8}\left[f'(b)-f'(a)\right]\sum_{i=1}^{n}|x_i-y_i|. \tag{2.7}$$

Since $\mathcal{J}_{n,f}(x,y)=\mathcal{T}_{n,f}(x,y)+\mathcal{M}_{n,f}(x,y)$, hence

$$0\leq\mathcal{J}_{n,f}(x,y)\leq\frac{1}{4}\left[f'(b)-f'(a)\right]\sum_{i=1}^{n}|x_i-y_i|.$$

Corollary 1. *With the assumptions of Theorem 3 and if the derivative f' is Lipschitzian with the constant $K>0$, namely*

$$|f'(t)-f'(s)|\leq K|t-s|\ \text{ for all } t,\ s\in\mathring{I},$$

then we have the inequality

$$0\leq\mathcal{M}_{n,f}(x,y)\leq\mathcal{T}_{n,f}(x,y)\leq\frac{1}{8}Kd_2^2(x,y), \tag{2.8}$$

where $d_2(x,y)$ is the Euclidean distance between x and y, namely

$$d_2(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

Also, we have

$$0 \le \mathcal{J}_{n,f}(x,y) \le \frac{1}{4} K d_2^2(x,y).$$

3. Related Results

We have the following Jensen's type inequality:

Theorem 4. *Let f be a C^2 function on an interval I. If f is convex and $\frac{1}{f''}$ is concave on I, then for all $(x_i, y_i) \in I \times I$, $i \in \{1, ..., n\}$ and $p_i \ge 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$, we have*

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n} p_i \left[f'(x_i) - f'\left(\frac{x_i+y_i}{2}\right)\right](x_i - u) \\
&+ \frac{1}{2}\sum_{i=1}^{n} p_i \left[f'(y_i) - f'\left(\frac{x_i+y_i}{2}\right)\right](y_i - v) \\
&\ge \frac{1}{2}\sum_{i=1}^{n} p_i \left[f(x_i) + f(y_i)\right] - \sum_{i=1}^{n} p_i f\left(\frac{x_i+y_i}{2}\right) \\
&- \frac{1}{2}\left[f(u) + f(v)\right] + f\left(\frac{u+v}{2}\right) \\
&\ge \frac{1}{2}\left[f'(u) - f'\left(\frac{u+v}{2}\right)\right]\left(\sum_{i=1}^{n} p_i x_i - u\right) \\
&+ \frac{1}{2}\left[f'(v) - f'\left(\frac{u+v}{2}\right)\right]\left(\sum_{i=1}^{n} p_i y_i - v\right)
\end{aligned} \tag{3.1}$$

for all $(u,v) \in I \times I$.

In particular,

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n} p_i \left[f'(x_i) - f'\left(\frac{x_i+y_i}{2}\right)\right]\left(x_i - \sum_{j=1}^{n} p_j x_j\right) \\
&+ \frac{1}{2}\sum_{i=1}^{n} p_i \left[f'(y_i) - f'\left(\frac{x_i+y_i}{2}\right)\right]\left(y_i - \sum_{j=1}^{n} p_j y_j\right) \\
&\ge \frac{1}{2}\sum_{i=1}^{n} p_i \left[f(x_i) + f(y_i)\right] - \sum_{i=1}^{n} p_i f\left(\frac{x_i+y_i}{2}\right) \\
&- \frac{1}{2}\left[f\left(\sum_{i=1}^{n} p_i x_i\right) + f\left(\sum_{i=1}^{n} p_i y_i\right)\right] + f\left(\frac{\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} p_i y_i}{2}\right) \\
&\ge 0.
\end{aligned} \tag{3.2}$$

Proof. It is well known that if the function of two independent variables $F : D \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is convex on the convex domain D and has partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ on D then for all $(x, y), (u, v) \in D$ we have the gradient inequalities

$$\begin{aligned}
&\frac{\partial F(x,y)}{\partial x}(x-u)+\frac{\partial F(x,y)}{\partial y}(y-v) \\
&\geq F(x,y)-F(u,v) \\
&\geq \frac{\partial F(u,v)}{\partial x}(x-u)+\frac{\partial F(u,v)}{\partial y}(y-v).
\end{aligned} \tag{3.3}$$

Now, if we take $F : I \times I \to \mathbb{R}$ given by

$$F(x,y)=\frac{1}{2}[f(x)+f(y)]-f\left(\frac{x+y}{2}\right)$$

and observe that

$$\frac{\partial F(x,y)}{\partial x}=\frac{1}{2}\left[f'(x)-f'\left(\frac{x+y}{2}\right)\right]$$

and

$$\frac{\partial F(x,y)}{\partial y}=\frac{1}{2}\left[f'(y)-f'\left(\frac{x+y}{2}\right)\right]$$

and since F is convex on $I \times I$, then by (3.3) we get

$$\begin{aligned}
&\frac{1}{2}\left[f'(x)-f'\left(\frac{x+y}{2}\right)\right](x-u)+\frac{1}{2}\left[f'(y)-f'\left(\frac{x+y}{2}\right)\right](y-v) \\
&\geq \frac{1}{2}[f(x)+f(y)]-f\left(\frac{x+y}{2}\right)-\frac{1}{2}[f(u)+f(v)]+f\left(\frac{u+v}{2}\right) \\
&\geq \frac{1}{2}\left[f'(u)-f'\left(\frac{u+v}{2}\right)\right](x-u)+\frac{1}{2}\left[f'(v)-f'\left(\frac{u+v}{2}\right)\right](y-v).
\end{aligned} \tag{3.4}$$

Moreover, if $(x_i, y_i) \in I \times I$, $i \in \{1, ..., n\}$ then by (3.4) we get

$$\begin{aligned}
&\frac{1}{2}\left[f'(x_i)-f'\left(\frac{x_i+y_i}{2}\right)\right](x_i-u)+\frac{1}{2}\left[f'(y_i)-f'\left(\frac{x_i+y_i}{2}\right)\right](y_i-v) \\
&\geq \frac{1}{2}[f(x_i)+f(y_i)]-f\left(\frac{x_i+y_i}{2}\right)-\frac{1}{2}[f(u)+f(v)]+f\left(\frac{u+v}{2}\right) \\
&\geq \frac{1}{2}\left[f'(u)-f'\left(\frac{u+v}{2}\right)\right](x_i-u)+\frac{1}{2}\left[f'(v)-f'\left(\frac{u+v}{2}\right)\right](y_i-v)
\end{aligned} \tag{3.5}$$

for all $i \in \{1, ..., n\}$ and $(u, v) \in I \times I$.

Let $p_i \geq 0$ for all $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$. If we multiply (3.5) by $p_i \geq 0$ and sum over i from 1 to n, then we get the desired result (3.1). $\square$

Corollary 2. *With the assumptions of Theorem 4 we have*

$$
\begin{aligned}
(3.6)\qquad & \frac{1}{2}\sum_{i=1}^{n}\left[f'(x_i)-f'\left(\frac{x_i+y_i}{2}\right)\right]\left(x_i-\frac{1}{n}\sum_{j=1}^{n}x_j\right)\\
&+\frac{1}{2}\sum_{i=1}^{n}\left[f'(y_i)-f'\left(\frac{x_i+y_i}{2}\right)\right]\left(y_i-\frac{1}{n}\sum_{j=1}^{n}y_j\right)\\
&\geq \mathcal{J}_{n,f}(x,y)\\
&-\frac{1}{2}n\left[f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)+f\left(\frac{1}{n}\sum_{i=1}^{n}y_i\right)\right]+nf\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i+y_i}{2}\right)\right)\\
&\geq 0.
\end{aligned}
$$

Similar results hold for the *Hermite-Hadamard trapezoid* and *mid-point divergences.*

Theorem 5. *Let f be a C^2 function on an interval I. If f is convex and $\frac{1}{f''}$ is concave on I, then for all $(x_i, y_i) \in I \times I$, $i \in \{1, ..., n\}$ and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, we have*

$$
\begin{aligned}
(3.7)\qquad & \sum_{i=1}^{n}p_i(x_i-u)\int_0^1(1-t)\left[f'(x_i)-f'((1-t)x_i+ty_i)\right]dt\\
&+\sum_{i=1}^{n}p_i(y_i-v)\int_0^1 t\left[f'(y_i)-f'((1-t)x_i+ty_i)\right]dt\\
&\geq \frac{\sum_{i=1}^{n}p_if(x_i)+\sum_{i=1}^{n}p_if(y_i)}{2}-\sum_{i=1}^{n}p_i\int_0^1 f((1-t)x_i+ty_i)\,dt\\
&-\frac{f(u)+f(v)}{2}+\int_0^1 f((1-t)u+tv)\,dt\\
&\geq \sum_{i=1}^{n}p_i(x_i-u)\int_0^1(1-t)\left[f'(u)-f'((1-t)u+tv)\right]dt\\
&+\sum_{i=1}^{n}p_i(y_i-v)\int_0^1 t\left[f'(v)-f'((1-t)u+tv)\right]dt
\end{aligned}
$$

for all $(u, v) \in I \times I$.

In particular,

$$\sum_{i=1}^{n} p_i \left(x_i - \sum_{j=1}^{n} p_j x_j \right) \int_0^1 (1-t) \left[f'(x_i) - f'((1-t)x_i + t y_i) \right] dt \tag{3.8}$$
$$+ \sum_{i=1}^{n} p_i \left(y_i - \sum_{j=1}^{n} p_j y_j \right) \int_0^1 t \left[f'(y_i) - f'((1-t)x_i + t y_i) \right] dt$$
$$\geq \frac{\sum_{i=1}^{n} p_i f(x_i) + \sum_{i=1}^{n} p_i f(y_i)}{2} - \sum_{i=1}^{n} p_i \int_0^1 f((1-t)x_i + t y_i) \, dt$$
$$- \frac{f\left(\sum_{j=1}^{n} p_j x_j\right) + f\left(\sum_{j=1}^{n} p_j y_j\right)}{2} + \int_0^1 f\left(\sum_{j=1}^{n} p_j \left[(1-t) x_j + t y_j \right] \right) dt$$
$$\geq 0.$$

Proof. Let $(x, y), (u, v) \in I \times I$. If we take $F : I \times I \to \mathbb{R}$ given by

$$F(x, y) = \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty) \, dt$$

then

$$\begin{aligned} \frac{\partial F(x, y)}{\partial x} &= \frac{1}{2} f'(x) - \int_0^1 (1-t) f'((1-t)x + ty) \, dt \\ &= \int_0^1 (1-t) \left[f'(x) - f'((1-t)x + ty) \right] dt \end{aligned}$$

and

$$\begin{aligned} \frac{\partial F(x, y)}{\partial y} &= \frac{1}{2} f'(y) - \int_0^1 t f'((1-t)x + ty) \, dt \\ &= \int_0^1 t \left[f'(y) - f'((1-t)x + ty) \right] dt \end{aligned}$$

and since F is convex on $I \times I$, then by (3.3) we get

$$\int_0^1 (1-t) \left[f'(x) - f'((1-t)x + ty) \right] dt \, (x - u) \tag{3.9}$$
$$+ \int_0^1 t \left[f'(y) - f'((1-t)x + ty) \right] dt \, (y - v)$$
$$\geq \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty) \, dt$$
$$- \frac{f(u) + f(v)}{2} + \int_0^1 f((1-t)u + tv) \, dt$$
$$\geq \int_0^1 (1-t) \left[f'(u) - f'((1-t)u + tv) \right] dt \, (x - u)$$
$$+ \int_0^1 t \left[f'(v) - f'((1-t)u + tv) \right] dt \, (y - v).$$

Therefore, if $(x_i, y_i) \in I \times I$, $i \in \{1, ..., n\}$ then by (3.9) we get

$$
\begin{aligned}
(3.10) \qquad & (x_i - u) \int_0^1 (1-t)\left[f'(x_i) - f'((1-t)x_i + ty_i)\right] dt \\
& + (y_i - v) \int_0^1 t\left[f'(y_i) - f'((1-t)x_i + ty_i)\right] dt \\
& \geq \frac{f(x_i) + f(y_i)}{2} - \int_0^1 f((1-t)x_i + ty_i)\, dt \\
& - \frac{f(u) + f(v)}{2} + \int_0^1 f((1-t)u + tv)\, dt \\
& \geq (x_i - u) \int_0^1 (1-t)\left[f'(u) - f'((1-t)u + tv)\right] dt \\
& + (y_i - v) \int_0^1 t\left[f'(v) - f'((1-t)u + tv)\right] dt
\end{aligned}
$$

for all $i \in \{1, ..., n\}$ and $(u, v) \in I \times I$.

Let $p_i \geq 0$ for all $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$. If we multiply (3.10) by $p_i \geq 0$ and sum over i from 1 to n, then we get the desired result (3.7). □

Corollary 3. *With the assumptions of Theorem 4 we have*

$$
\begin{aligned}
(3.11) \qquad & \sum_{i=1}^n \left(x_i - \frac{1}{n}\sum_{j=1}^n x_j\right) \int_0^1 (1-t)\left[f'(x_i) - f'((1-t)x_i + ty_i)\right] dt \\
& + \sum_{i=1}^n \left(y_i - \frac{1}{n}\sum_{j=1}^n y_j\right) \int_0^1 t\left[f'(y_i) - f'((1-t)x_i + ty_i)\right] dt \\
& \geq \mathcal{T}_{n,f}(x, y) \\
& - n\frac{f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) + f\left(\frac{1}{n}\sum_{i=1}^n y_i\right)}{2} + n\int_0^1 f\left(\frac{1}{n}\sum_{i=1}^n \left[(1-t)x_i + ty_i\right]\right) dt \\
& \geq 0.
\end{aligned}
$$

We also have:

Theorem 6. *Let f be a C^2 function on an interval I. If f is convex and $\frac{1}{f''}$ is concave on I, then for all $(x_i, y_i) \in I \times I$, $i \in \{1, ..., n\}$ and $p_i \geq 0$, $i \in \{1, ..., n\}$*

with $\sum_{i=1}^{n} p_i = 1$, *we have*

$$\begin{aligned}
(3.12)\qquad & \sum_{i=1}^{n} p_i (x_i - u) \int_0^1 (1-t) \left[f'((1-t)x_i + ty_i) - f'\left(\frac{x_i+y_i}{2}\right)\right] dt \\
& + \sum_{i=1}^{n} p_i (y_i - v) \int_0^1 t \left[f'((1-t)x_i + ty_i) - f'\left(\frac{x_i+y_i}{2}\right)\right] dt \\
& \geq \sum_{i=1}^{n} p_i \int_0^1 f((1-t)x_i + ty_i)\, dt - \sum_{i=1}^{n} p_i f\left(\frac{x_i+y_i}{2}\right) \\
& - \int_0^1 f((1-t)u + tv)\, dt + f\left(\frac{u+v}{2}\right) \\
& \geq \sum_{i=1}^{n} p_i (x_i - u) \int_0^1 (1-t) \left[f'((1-t)u + tv) - f'\left(\frac{u+v}{2}\right)\right] dt \\
& + \sum_{i=1}^{n} p_i (y_i - v) \int_0^1 t \left[f'((1-t)u + tv) - f'\left(\frac{u+v}{2}\right)\right] dt
\end{aligned}$$

for all $(u,v) \in I \times I$.

In particular,

$$\begin{aligned}
(3.13)\quad & \sum_{i=1}^{n} p_i \left(x_i - \sum_{j=1}^{n} p_j x_j \right) \int_0^1 (1-t) \left[f'((1-t)x_i + ty_i) - f'\left(\frac{x_i+y_i}{2}\right)\right] dt \\
& + \sum_{i=1}^{n} p_i \left(y_i - \sum_{j=1}^{n} p_j y_j \right) \int_0^1 t \left[f'((1-t)x_i + ty_i) - f'\left(\frac{x_i+y_i}{2}\right)\right] dt \\
& \geq \sum_{i=1}^{n} p_i \int_0^1 f((1-t)x_i + ty_i)\, dt - \sum_{i=1}^{n} p_i f\left(\frac{x_i+y_i}{2}\right) \\
& - \int_0^1 f\left(\sum_{j=1}^{n} p_i [(1-t)x_i + ty_i] \right) dt + f\left(\sum_{j=1}^{n} p_i \left(\frac{x_i+y_i}{2}\right)\right) \\
& \geq 0.
\end{aligned}$$

Proof. Let $(x,y), (u,v) \in I \times I$. If we take $F : I \times I \to \mathbb{R}$ given by

$$F(x,y) = \int_0^1 f((1-t)x + ty)\, dt - f\left(\frac{x+y}{2}\right)$$

then

$$\begin{aligned}
\frac{\partial F(x,y)}{\partial x} &= \int_0^1 (1-t) f'((1-t)x + ty)\, dt - \frac{1}{2} f'\left(\frac{x+y}{2}\right) \\
&= \int_0^1 (1-t) \left[f'((1-t)x + ty) - f'\left(\frac{x+y}{2}\right)\right] dt
\end{aligned}$$

and

$$\frac{\partial F(x,y)}{\partial y} = \int_0^1 tf'((1-t)x+ty)\,dt - \frac{1}{2}f'\left(\frac{x+y}{2}\right) = \int_0^1 t\left[f'((1-t)x+ty) - f'\left(\frac{x+y}{2}\right)\right]dt.$$

The rest of the proof follows in the similar way to the one from above and we omit the details. □

Corollary 4. *With the assumptions of Theorem 4 we have*

$$\begin{aligned}(3.14)\quad & \sum_{i=1}^n \left(x_i - \frac{1}{n}\sum_{j=1}^n x_j\right)\int_0^1 (1-t)\left[f'((1-t)x_i+ty_i) - f'\left(\frac{x_i+y_i}{2}\right)\right]dt \\ & + \sum_{i=1}^n \left(y_i - \frac{1}{n}\sum_{j=1}^n y_j\right)\int_0^1 t\left[f'((1-t)x_i+ty_i) - f'\left(\frac{x_i+y_i}{2}\right)\right]dt \\ & \geq \mathcal{M}_{n,f}(x,y) \\ & - n\int_0^1 f\left(\frac{1}{n}\sum_{j=1}^n [(1-t)x_i+ty_i]\right)dt + nf\left(\frac{1}{n}\sum_{j=1}^n \left(\frac{x_i+y_i}{2}\right)\right) \\ & \geq 0.\end{aligned}$$

4. Some Results for f-Divergences

Consider the probability distributions p, q. Assume that $f:(0,\infty)\to\mathbb{R}$ is convex and define the *Csiszár's f-divergence measure* [2] and [3]

$$\mathcal{C}_{n,f}(p,q) := \sum_{i=1}^n p_i f\left(\frac{q_i}{p_i}\right)$$

and the *Lin-Wong f-divergence measure* [9]

$$\mathcal{LW}_{n,f}(p,q) := \sum_{i=1}^n p_i f\left(\frac{q_i+p_i}{2p_i}\right).$$

If $f:(0,\infty)\to\mathbb{R}$ is a C^2 convex function and such that $\frac{1}{f''}$ is concave on $(0,\infty)$, then we get from (3.2) for $x_i = \frac{q_i}{p_i}$ and $y_i = 1$, $i\in\{1,...,n\}$ that

$$\frac{1}{2}\sum_{i=1}^n \left[f'\left(\frac{q_i}{p_i}\right) - f'\left(\frac{q_i+p_i}{2p_i}\right)\right](q_i-p_i) \geq \frac{1}{2}\sum_{i=1}^n p_i\left[f\left(\frac{q_i}{p_i}\right)+f(1)\right] - \sum_{i=1}^n p_i f\left(\frac{q_i+p_i}{2p_i}\right) \geq 0,$$

namely

$$(4.1)\quad 0 \leq \frac{1}{2}\left[\mathcal{C}_{n,f}(p,q)+f(1)\right] - \mathcal{LW}_{n,f}(p,q) \leq \frac{1}{2}\sum_{i=1}^n \left[f'\left(\frac{q_i}{p_i}\right) - f'\left(\frac{q_i+p_i}{2p_i}\right)\right](q_i-p_i)$$

for any probability distributions p, q.

If there exists $0 < r < 1 < R < \infty$ and $\frac{q_i}{p_i} \in [r, R]$ for any $i \in \{1, ..., n\}$ and for some $K > 0$ we have

$$|f'(s) - f'(t)| \le K |s - t| \tag{4.2}$$

for any s, $t \in [r, R]$, then

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\left[f'\left(\frac{q_i}{p_i}\right) - f'\left(\frac{q_i + p_i}{2p_i}\right)\right](q_i - p_i) \\
&\le \frac{1}{2}\sum_{i=1}^{n}\left|f'\left(\frac{q_i}{p_i}\right) - f'\left(\frac{q_i + p_i}{2p_i}\right)\right||q_i - p_i| \\
&\le \frac{1}{2}K\sum_{i=1}^{n}\left|\frac{q_i}{p_i} - \frac{q_i + p_i}{2p_i}\right||q_i - p_i| = \frac{1}{4}K\sum_{i=1}^{n}\frac{(q_i - p_i)^2}{p_i} \\
&= \frac{1}{4}K\sum_{i=1}^{n}\frac{q_i^2 - 2p_iq_i + p_i^2}{p_i} = \frac{1}{4}K\left(\sum_{i=1}^{n}\frac{q_i^2}{p_i} - 1\right) = D_{\chi^2}(p, q)
\end{aligned}$$

where $D_{\chi^2}(p, q)$ is the well known χ^2-*divergence.*

By utilising the inequality (4.1) we get

$$0 \le \frac{1}{2}\left[\mathcal{C}_{n,f}(p, q) + f(1)\right] - \mathcal{LW}_{n,f}(p, q) \le \frac{1}{4}KD_{\chi^2}(p, q). \tag{4.3}$$

Since f is a C^2 convex function on $[r, R]$ then we can take $K = \max_{t\in[r,R]}|f''(t)|$ in the inequality (4.3).

In the same paper [9], the authors introduced the *Hermite-Hadamard (HH) f-divergence* by

$$\mathcal{D}^{f}_{n,HH}(p, q) := \sum_{i=1}^{n} p_i \frac{\int_1^{\frac{q_i}{p_i}} f(t)\,dt}{\frac{q_i}{p_i} - 1} = \sum_{i=1}^{n} p_i \int_0^1 f\left((1 - t)\frac{q_i}{p_i} + t\right)dt.$$

If $f : (0, \infty) \to \mathbb{R}$ is a C^2 convex function and such that $\frac{1}{f''}$ is concave on $(0, \infty)$, then we get from (3.8) for $x_i = \frac{q_i}{p_i}$ and $y_i = 1$, $i \in \{1, ..., n\}$ that namely

$$\begin{aligned}
0 &\le \frac{1}{2}\left[\mathcal{C}_{n,f}(p, q) + f(1)\right] - \mathcal{D}^{f}_{n,HH}(p, q) \\
&\le \sum_{i=1}^{n}(q_i - p_i)\int_0^1 (1 - t)\left[f'\left(\frac{q_i}{p_i}\right) - f'\left((1 - t)\frac{q_i}{p_i} + t\right)\right]dt
\end{aligned} \tag{4.4}$$

for any probability distributions p, q.

If there exists $0<r<1<R<\infty$ and $\frac{q_i}{p_i}\in[r,R]$ for any $i\in\{1,...,n\}$ and for some $K>0$ we have the condition (4.2) then

$$\begin{aligned}
&\sum_{i=1}^{n}(q_i-p_i)\int_0^1(1-t)\left[f'\left(\frac{q_i}{p_i}\right)-f'\left((1-t)\frac{q_i}{p_i}+t\right)\right]dt\\
&\leq\sum_{i=1}^{n}|q_i-p_i|\int_0^1(1-t)\left|f'\left(\frac{q_i}{p_i}\right)-f'\left((1-t)\frac{q_i}{p_i}+t\right)\right|dt\\
&\leq K\sum_{i=1}^{n}|q_i-p_i|\int_0^1(1-t)\left|\frac{q_i}{p_i}-(1-t)\frac{q_i}{p_i}-t\right|dt\\
&=K\sum_{i=1}^{n}\frac{(q_i-p_i)^2}{p_i}\int_0^1(1-t)\,tdt=\frac{1}{6}K\sum_{i=1}^{n}\frac{(q_i-p_i)^2}{p_i}=\frac{1}{6}KD_{\chi^2}(p,q).
\end{aligned}$$

Therefore, if $\frac{q_i}{p_i}\in[r,R]$ for any $i\in\{1,...,n\}$ and $K=\max_{t\in[r,R]}|f''(t)|$, then

$$0\leq\frac{1}{2}\left[\mathcal{C}_{n,f}(p,q)+f(1)\right]-\mathcal{D}^f_{n,HH}(p,q)\leq\frac{1}{6}KD_{\chi^2}(p,q). \tag{4.5}$$

If $f:(0,\infty)\to\mathbb{R}$ is a C^2 convex function and such that $\frac{1}{f''}$ is concave on $(0,\infty)$, then we get from (3.13) for $x_i=\frac{q_i}{p_i}$ and $y_i=1$, $i\in\{1,...,n\}$ that

$$\begin{aligned}
&\sum_{i=1}^{n}p_i\left(\frac{q_i}{p_i}-1\right)\int_0^1(1-t)\left[f'\left((1-t)\frac{q_i}{p_i}+t\right)-f'\left(\frac{\frac{q_i}{p_i}+1}{2}\right)\right]dt\\
&\geq\sum_{i=1}^{n}p_i\int_0^1 f\left((1-t)\frac{q_i}{p_i}+t\right)dt-\sum_{i=1}^{n}p_i f\left(\frac{\frac{q_i}{p_i}+1}{2}\right)\geq 0,
\end{aligned}$$

namely

$$\begin{aligned}
0 &\leq \mathcal{D}^f_{n,HH}(p,q)-\mathcal{LW}_{n,f}(p,q)\\
&\leq \sum_{i=1}^{n}(q_i-p_i)\int_0^1(1-t)\left[f'\left((1-t)\frac{q_i}{p_i}+t\right)-f'\left(\frac{q_i+p_i}{2p_i}\right)\right]dt
\end{aligned} \tag{4.6}$$

for any probability distributions p, q.

If there exists $0<r<1<R<\infty$ and $\frac{q_i}{p_i}\in[r,R]$ for any $i\in\{1,...,n\}$ and for some $K>0$ we have the condition (4.2) then

$$\begin{aligned}
&\sum_{i=1}^{n}(q_i-p_i)\int_0^1(1-t)\left[f'\left((1-t)\frac{q_i}{p_i}+t\right)-f'\left(\frac{q_i+p_i}{2p_i}\right)\right]dt\\
&\leq\sum_{i=1}^{n}|q_i-p_i|\int_0^1(1-t)\left|f'\left((1-t)\frac{q_i}{p_i}+t\right)-f'\left(\frac{q_i+p_i}{2p_i}\right)\right|dt\\
&\leq K\sum_{i=1}^{n}|q_i-p_i|\int_0^1(1-t)\left|(1-t)\frac{q_i}{p_i}+t-\frac{q_i+p_i}{2p_i}\right|dt\\
&=K\sum_{i=1}^{n}\frac{(q_i-p_i)^2}{p_i}\int_0^1(1-t)\left|t-\frac{1}{2}\right|dt=\frac{1}{8}KD_{\chi^2}(p,q).
\end{aligned}$$

Therefore if $\frac{q_i}{p_i} \in [r, R]$ for any $i \in \{1, ..., n\}$ and $K = \max_{t\in[r,R]} |f''(t)|$, then

$$0 \leq \mathcal{D}^f_{n,HH}(p, q) - \mathcal{LW}_{n,f}(p, q) \leq \frac{1}{8} K D_{\chi^2}(p, q). \tag{4.7}$$

5. Some Examples

Consider the power function $f_\alpha : [0, \infty) \to \mathbb{R}$, $f_\alpha(t) = (\alpha - 1)^{-1} t^\alpha$ with $\alpha \in (1, 2]$. This function is convex on $[0, \infty)$ and $\frac{1}{f''_\alpha}$ is concave on $(0, \infty)$ and therefore

$$\mathcal{J}_{n,\alpha}(x, y) := (\alpha - 1)^{-1} \sum_{i=1}^{n} [A(x_i^\alpha, y_i^\alpha) - A^\alpha(x_i, y_i)] \tag{5.1}$$

is jointly convex on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, where $\mathbb{R}_+ := [0, \infty)$.

The *Hermite-Hadamard trapezoid* and *mid-point divergences* associated to f_α are

$$\mathcal{T}_{n,\alpha}(x, y) := (\alpha - 1)^{-1} \sum_{i=1}^{n} [A(x_i^\alpha, y_i^\alpha) - L_\alpha^\alpha(x_i, y_i)] \tag{5.2}$$

and

$$\mathcal{M}_{n,f}(x, y) := (\alpha - 1)^{-1} \sum_{i=1}^{n} [L_\alpha^\alpha(x_i, y_i) - A^\alpha(x_i, y_i)] \tag{5.3}$$

for $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$.

According to Theorem 2, these divergences are jointly convex on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$. From Theorem 3 we have the inequalities

$$0 \leq \mathcal{M}_{n,\alpha}(x, y) \leq \mathcal{T}_{n,\alpha}(x, y) \leq \frac{1}{8}\alpha^2 (\alpha - 1)^{-1} \sum_{i=1}^{n} L_{\alpha-1}^{\alpha-1}(x_i, y_i)(x_i - y_i)^2 \tag{5.4}$$

for $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$.

If $[a, b] \subset \mathbb{R}^n_+$ and $(x, y) \in [a, b]^n \times [a, b]^n$ then by (2.7) we have

$$0 \leq \mathcal{M}_{n,f}(x, y) \leq \mathcal{T}_{n,f}(x, y) \leq \frac{1}{8}(\alpha - 1)^{-1} \alpha \left(b^{\alpha-1} - a^{\alpha-1}\right) \sum_{i=1}^{n} |x_i - y_i|. \tag{5.5}$$

We have for $[a, b] \subset \mathbb{R}^n_{++} := (0, \infty)$ that

$$K := \max_{t\in[a,b]} f''_\alpha(t) = (\alpha - 1)^{-1} \alpha (\alpha - 1) \max_{t\in[a,b]} t^{\alpha-2} = \frac{\alpha}{a^{2-\alpha}}$$

and by the inequality (2.8) we have

$$0 \leq \mathcal{M}_{n,f}(x, y) \leq \mathcal{T}_{n,f}(x, y) \leq \frac{\alpha}{8a^{2-\alpha}} d_2^2(x, y). \tag{5.6}$$

For f_α we have

$$\mathcal{C}_{n,f_\alpha}(p, q) := (\alpha - 1)^{-1} \sum_{i=1}^{n} p_i^{1-\alpha} q_i^\alpha,$$

$$\mathcal{LW}_{n,f_\alpha}(p, q) := (\alpha - 1)^{-1} \sum_{i=1}^{n} p_i^{1-\alpha} \left(\frac{q_i + p_i}{2}\right)^\alpha$$

and

$$\mathcal{D}^{f_\alpha}_{n,HH}(p, q) := (\alpha - 1)^{-1} \sum_{i=1}^{n} p_i \frac{\int_1^{\frac{q_i}{p_i}} f_\alpha(t)\, dt}{\frac{q_i}{p_i} - 1} = (\alpha - 1)^{-1} \sum_{i=1}^{n} p_i L_\alpha\left(\frac{q_i}{p_i}, 1\right).$$

Let $0<r<1<R<\infty$. If $\frac{q_i}{p_i}\in[r,R]$ for any $i\in\{1,...,n\}$, then from the inequality (4.3) we get

$$0\leq\frac{1}{2}\left[\mathcal{C}_{n,f_\alpha}(p,q)+(\alpha-1)^{-1}\right]-\mathcal{LW}_{n,f_\alpha}(p,q)\leq\frac{1}{4}\frac{\alpha}{r^{2-\alpha}}D_{\chi^2}(p,q)\,,\tag{5.7}$$

from (4.5) we have

$$0\leq\frac{1}{2}\left[\mathcal{C}_{n,f_\alpha}(p,q)+(\alpha-1)^{-1}\right]-\mathcal{D}^{f_\alpha}_{n,HH}(p,q)\leq\frac{1}{6}\frac{\alpha}{r^{2-\alpha}}D_{\chi^2}(p,q)\tag{5.8}$$

while from (4.7) we obtain

$$0\leq\mathcal{D}^{f_\alpha}_{n,HH}(p,q)-\mathcal{LW}_{n,f_\alpha}(p,q)\leq\frac{1}{8}\frac{\alpha}{r^{2-\alpha}}D_{\chi^2}(p,q)\,.\tag{5.9}$$

Consider now the function $f_1:(0,\infty)\to\mathbb{R}$, $f_1(t)=t\ln t$. The function f_1 is convex on $(0,\infty)$ and $\frac{1}{f_1''}$ is concave on $(0,\infty)$. Then the function

$$\mathcal{J}_{n,1}(x,y):=\frac{1}{2}\sum_{i=1}^n\left[x_i\ln(x_i)+y_i\ln(y_i)-(x_i+y_i)\ln\left(\frac{x_i+y_i}{2}\right)\right]$$

is jointly convex on $\mathbb{R}^n_{++}\times\mathbb{R}^n_{++}$.

Observe that

$$\begin{aligned}\frac{1}{b-a}\int_a^b t\ln t dt&=\frac{1}{2}\frac{1}{b-a}\int_a^b\ln t d\left(t^2\right)\\&=\frac{1}{2}\frac{1}{b-a}\left[t^2\ln t\Big|_a^b-\int_a^b tdt\right]\\&=\frac{1}{2}\frac{1}{b-a}\left[b^2\ln b-a^2\ln a-\frac{b^2-a^2}{2}\right]\\&=\frac{1}{2}\frac{1}{b-a}\left[\frac{b^2\ln b^2-a^2\ln a^2}{2}-\frac{b^2-a^2}{2}\right]\\&=\frac{1}{2}\frac{1}{b-a}\frac{b^2-a^2}{2}\left[\frac{b^2\ln b^2-a^2\ln a^2}{b^2-a^2}-1\right]\\&=\frac{1}{4}(b+a)\ln I\left(a^2,b^2\right),\end{aligned}$$

where I is the *identric mean.*

Therefore

$$\begin{aligned}\mathcal{T}_{n,1}(x,y)&:=\sum_{i=1}^n\left(\frac{1}{2}\left[f(x_i)+f(y_i)\right]-\int_0^1 f((1-t)x_i+ty_i)\,dt\right)\\&=\frac{1}{2}\sum_{i=1}^n\left[\frac{x_i\ln\left(x_i^2\right)+y_i\ln\left(y_i^2\right)}{2}-A(x_i,y_i)\ln I\left(x_i^2,y_i^2\right)\right]\\&=\frac{1}{2}\sum_{i=1}^n\left[A\left(x_i\ln\left(x_i^2\right),y_i\ln\left(y_i^2\right)\right)-A(x_i,y_i)\ln I\left(x_i^2,y_i^2\right)\right]\end{aligned}\tag{5.10}$$

and

$$\begin{aligned}\mathcal{M}_{n,1}(x,y) &:= \sum_{i=1}^{n}\left(A(x_i,y_i)\ln I\left(x_i^2,y_i^2\right) - A(x_i,y_i)\ln A(x_i,y_i)\right)\\ &= \sum_{i=1}^{n} A(x_i,y_i)\left[\ln I\left(x_i^2,y_i^2\right) - \ln A(x_i,y_i)\right]\end{aligned}$$

for $(x,y)\in\mathbb{R}_{++}^n\times\mathbb{R}_{++}^n$.

According to Theorem 2, these divergences are jointly convex on $\mathbb{R}_{++}^n\times\mathbb{R}_{++}^n$. From Theorem 3 we have the inequalities

$$0\le\mathcal{M}_{n,1}(x,y)\le\mathcal{T}_{n,1}(x,y)\le\frac{1}{8}\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{L(x_i,y_i)}. \tag{5.11}$$

From the inequality (2.7) we have

$$0\le\mathcal{M}_{n,1}(x,y)\le\mathcal{T}_{n,1}(x,y)\le\frac{1}{8}(\ln b-\ln a)\sum_{i=1}^{n}|x_i-y_i| \tag{5.12}$$

for $(x,y)\in[a,b]^n\times[a,b]^n$, where $[a,b]\subset(0,\infty)$.

We also have from (2.8) that

$$0\le\mathcal{M}_{n,1}(x,y)\le\mathcal{T}_{n,1}(x,y)\le\frac{1}{8}\frac{b-a}{ba}d_2^2(x,y) \tag{5.13}$$

for $(x,y)\in[a,b]^n\times[a,b]^n$, where $[a,b]\subset(0,\infty)$.

Consider the divergences

$$\mathcal{C}_{n,f_1}(p,q):=\sum_{i=1}^{n}q_i\ln\left(\frac{q_i}{p_i}\right),$$

Kullback-Leibler divergence [7],

$$\mathcal{LW}_{n,f_1}(p,q):=\sum_{i=1}^{n}\frac{q_i+p_i}{2}\ln\left(\frac{q_i+p_i}{2p_i}\right),$$

Lin-Wong divergence measure [8].

and

$$\mathcal{D}_{n,HH}^{f_1}(p,q):=\sum_{i=1}^{n}p_i\frac{\int_1^{\frac{q_i}{p_i}}t\ln t\,dt}{\frac{q_i}{p_i}-1}=\frac{1}{2}\sum_{i=1}^{n}A(q_i,p_i)\ln I\left(\left(\frac{q_i}{p_i}\right)^2,1\right).$$

Let $0<r<1<R<\infty$. If $\frac{q_i}{p_i}\in[r,R]$ for any $i\in\{1,...,n\}$, then from the inequality (4.3) we get

$$0\le\frac{1}{2}\mathcal{C}_{n,f_1}(p,q)-\mathcal{LW}_{n,f_1}(p,q)\le\frac{1}{4r}D_{\chi^2}(p,q), \tag{5.14}$$

from (4.5) we have

$$0\le\frac{1}{2}\mathcal{C}_{n,f_1}(p,q)-\mathcal{D}_{n,HH}^{f_1}(p,q)\le\frac{1}{6r}D_{\chi^2}(p,q), \tag{5.15}$$

while from (4.7) we obtain

$$0\le\mathcal{D}_{n,HH}^{f_1}(p,q)-\mathcal{LW}_{n,f_1}(p,q)\le\frac{1}{8r}D_{\chi^2}(p,q). \tag{5.16}$$

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