SOME $f$-DIVERGENCE MEASURES RELATED TO JENSEN’S ONE

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ABSTRACT. In this paper we introduce some $f$-divergence measures that are related to the Jensen’s divergence introduced by Burbea and Rao in 1982. We establish their joint convexity and provide some inequalities between these measures and a combination of Csiszár’s $f$-divergence, $f$-midpoint divergence and $f$-integral divergence measures.

1. Introduction

Let $(X, A)$ be a measurable space satisfying $|A| > 2$ and $\mu$ be a $\sigma$-finite measure on $(X, A)$. Let $\mathcal{P}$ be the set of all probability measures on $(X, A)$ which are absolutely continuous with respect to $\mu$. For $P, Q \in \mathcal{P}$, let $p = \frac{dP}{d\mu}$ and $q = \frac{dQ}{d\mu}$ denote the Radon-Nikodym derivatives of $P$ and $Q$ with respect to $\mu$.

Two probability measures $P, Q \in \mathcal{P}$ are said to be orthogonal and we denote this by $Q \perp P$ if

$$P (\{q = 0\}) = Q (\{p = 0\}) = 1.$$  

Let $f : [0, \infty) \to (-\infty, \infty]$ be a convex function that is continuous at $0$, i.e., $f(0) = \lim_{u \to 0} f(u)$.

In 1963, I. Csiszár [4] introduced the concept of $f$-divergence as follows.

Definition 1. Let $P, Q \in \mathcal{P}$. Then

$$I_f (Q, P) = \int_X p(x) f \left( \frac{q(x)}{p(x)} \right) d\mu(x),$$

is called the $f$-divergence of the probability distributions $Q$ and $P$.

Remark 1. Observe that, the integrand in the formula (1.1) is undefined when $p(x) = 0$. The way to overcome this problem is to postulate for $f$ as above that

$$0f \left( \frac{q(x)}{0} \right) = q(x) \lim_{u \to 0} u f \left( \frac{1}{u} \right), \quad x \in X.$$  

We now give some examples of $f$-divergences that are well-known and often used in the literature (see also [3]).
1.1. The Class of $\chi^\alpha$-Divergences. The $f$-divergences of this class, which is generated by the function $\chi^\alpha$, $\alpha \in [1, \infty)$, defined by

$$\chi^\alpha(u) = |u - 1|^\alpha, \quad u \in [0, \infty)$$

have the form

$$I_f(Q, P) = \int_X p \left| \frac{q}{p} - 1 \right|^\alpha d\mu = \int_X p^{1-\alpha} |q - p|^\alpha d\mu.$$  \hspace{1cm} (1.3)

From this class only the parameter $\alpha = 1$ provides a distance in the topological sense, namely the total variation distance $V(Q, P) = \int_X |q - p| d\mu$. The most prominent special case of this class is, however, Karl Pearson’s $\chi^2$-divergence

$$\chi^2(Q, P) = \int_X \frac{q^2}{p} d\mu - 1$$

that is obtained for $\alpha = 2$.

1.2. Dichotomy Class. From this class, generated by the function $f_\alpha : [0, \infty) \to \mathbb{R}$

$$f_\alpha(u) = \begin{cases} u - 1 - \ln u & \text{for } \alpha = 0; \\ \frac{1}{\alpha(1-\alpha)} [\alpha u + 1 - \alpha - u^\alpha] & \text{for } \alpha \in \mathbb{R} \setminus \{0, 1\}; \\ 1 - u + u \ln u & \text{for } \alpha = 1; \end{cases}$$

only the parameter $\alpha = \frac{1}{2}$ \(f_{\frac{1}{2}}(u) = 2(\sqrt{u} - 1)^2\) provides a distance, namely, the Hellinger distance

$$H(Q, P) = \left[ \int_X (\sqrt{q} - \sqrt{p})^2 d\mu \right]^\frac{1}{2}.$$  \hspace{1cm}

Another important divergence is the Kullback-Leibler divergence obtained for $\alpha = 1$,

$$KL(Q, P) = \int_X q \ln \left(\frac{q}{p}\right) d\mu.$$  \hspace{1cm}

1.3. Matsushita’s Divergences. The elements of this class, which is generated by the function $\varphi_\alpha$, $\alpha \in (0, 1]$ given by

$$\varphi_\alpha(u) := |1 - u|^\frac{1}{\alpha}, \quad u \in [0, \infty),$$

are prototypes of metric divergences, providing the distances $[I_{\varphi_\alpha}(Q, P)]^\alpha$.

1.4. Puri-Vincze Divergences. This class is generated by the functions $\Phi_\alpha$, $\alpha \in [1, \infty)$ given by

$$\Phi_\alpha(u) := \frac{|1 - u|^\alpha}{(u + 1)^{\alpha-1}}, \quad u \in [0, \infty).$$

It has been shown in [26] that this class provides the distances $[I_{\Phi_\alpha}(Q, P)]^\frac{1}{\alpha}$. 
1.5. **Divergences of Arimoto-type.** This class is generated by the functions

\[
\Psi_\alpha(u) := \begin{cases} 
\frac{\alpha}{\alpha - 1} \left[ (1 + u^\alpha)^{\frac{1}{\alpha}} - 2^{\frac{1}{\alpha} - 1} (1 + u) \right] & \text{for } \alpha \in (0, \infty) \setminus \{1\}; \\
(1 + u) \ln 2 + u \ln u - (1 + u) \ln (1 + u) & \text{for } \alpha = 1; \\
\frac{1}{2} |1 - u| & \text{for } \alpha = \infty.
\end{cases}
\]

It has been shown in [28] that this class provides the distances \([I_{\Psi_\alpha}(Q, P)]_{\text{min}}(\alpha, \frac{1}{\alpha})\) for \(\alpha \in (0, \infty)\) and \(\frac{1}{2}V(Q, P)\) for \(\alpha = \infty\).

For \(f\) continuous convex on \([0, \infty)\) we obtain the \(*\)-conjugate function of \(f\) by

\[
f^*(u) = uf \left( \frac{1}{u} \right), \quad u \in (0, \infty)
\]

and

\[
f^*(0) = \lim_{u \to 0} f^*(u).
\]

It is also known that if \(f\) is continuous convex on \([0, \infty)\) then so is \(f^*\).

The following two theorems contain the most basic properties of \(f\)-divergences. For their proofs we refer the reader to Chapter 1 of [27] (see also [3]).

**Theorem 1 (Uniqueness and Symmetry Theorem).** Let \(f, f_1\) be continuous convex on \([0, \infty)\). We have

\[
I_{f_1}(Q, P) = I_f(Q, P),
\]

for all \(P, Q \in \mathcal{P}\) if and only if there exists a constant \(c \in \mathbb{R}\) such that

\[
f_1(u) = f(u) + c(u - 1),
\]

for any \(u \in [0, \infty)\).

**Theorem 2 (Range of Values Theorem).** Let \(f : [0, \infty) \to \mathbb{R}\) be a continuous convex function on \([0, \infty)\).

For any \(P, Q \in \mathcal{P}\), we have the double inequality

\[
(1.4) \quad f(1) \leq I_f(Q, P) \leq f(0) + f^*(0).
\]

(i) If \(P = Q\), then the equality holds in the first part of (1.4).

If \(f\) is strictly convex at 1, then the equality holds in the first part of (1.4) if and only if \(P = Q\);

(ii) If \(Q \perp P\), then the equality holds in the second part of (1.4).

If \(f(0) + f^*(0) < \infty\), then equality holds in the second part of (1.4) if and only if \(Q \perp P\).

The following result is a refinement of the second inequality in Theorem 2 (see [3, Theorem 3]).

**Theorem 3.** Let \(f\) be a continuous convex function on \([0, \infty)\) with \(f(1) = 0\) (\(f\) is normalised) and \(f(0) + f^*(0) < \infty\). Then

\[
(1.5) \quad 0 \leq I_f(Q, P) \leq \frac{1}{2} [f(0) + f^*(0)] V(Q, P)
\]

for any \(Q, P \in \mathcal{P}\).

For other inequalities for \(f\)-divergence see [2], [7]-[20].
2. Some Preliminary Facts

For a function $f$ defined on an interval $I$ of the real line $\mathbb{R}$, by following the paper by Burbea & Rao [1], we consider the $J$-divergence between the elements $t$, $s \in I$ given by

$$J_f(t, s) := \frac{1}{2} [f(t) + f(s)] - f\left(\frac{t + s}{2}\right).$$

As important examples of such divergences, we can consider [1],

$$J_\alpha(t, s) := \begin{cases} (\alpha - 1)^{-1} \left[\frac{1}{2} (t^\alpha + s^\alpha) - (\frac{t + s}{2})^\alpha\right], & \alpha \neq 1, \\ [t \ln(t) + s \ln(s) - (t + s) \ln\left(\frac{t + s}{2}\right)], & \alpha = 1. \end{cases}$$

If $f$ is convex on $I$, then $J_f(t, s) \geq 0$ for all $(t, s) \in I \times I$.

The following result concerning the joint convexity of $J_f$ also holds:

**Theorem 4** (Burbea-Rao, 1982 [1]). Let $f$ be a $C^2$ function on an interval $I$. Then $J_f$ is convex (concave) on $I \times I$, if and only if $f$ is convex (concave) and $\frac{1}{f''}$ is concave (convex) on $I$.

We define the Hermite-Hadamard trapezoid and mid-point divergences

(2.1) \[ T_f(t, s) := \frac{1}{2} [f(t) + f(s)] - \int_0^1 f((1 - \tau) t + \tau s) d\tau \]

and

(2.2) \[ M_f(t, s) := \int_0^1 f((1 - \tau) t + \tau s) d\tau - f\left(\frac{t + s}{2}\right) \]

for all $(t, s) \in I \times I$.

We observe that

(2.3) \[ J_f(t, s) = T_f(t, s) + M_f(t, s) \]

for all $(t, s) \in I \times I$.

If $f$ is convex on $I$, then by Hermite-Hadamard inequalities

$$\frac{f(a) + f(b)}{2} \geq \int_0^1 f((1 - \tau) a + \tau b) d\tau \geq f\left(\frac{a + b}{2}\right)$$

for all $a, b \in I$, we have the following fundamental facts

(2.4) \[ T_f(t, s) \geq 0 \text{ and } M_f(t, s) \geq 0 \]

for all $(t, s) \in I \times I$.

Using Bullen's inequality, see for instance [22, p. 2],

$$0 \leq \int_0^1 f((1 - \tau) a + \tau b) d\tau - f\left(\frac{a + b}{2}\right)$$

$$\leq \frac{f(a) + f(b)}{2} - \int_0^1 f((1 - \tau) a + \tau b) d\tau$$

we also have

(2.5) \[ 0 \leq M_f(t, s) \leq T_f(t, s). \]

Let us recall the following special means:
a) The arithmetic mean
\[ A(a, b) := \frac{a + b}{2}, \quad a, b > 0, \]

b) The geometric mean
\[ G(a, b) := \sqrt{ab}; \quad a, b \geq 0, \]

c) The harmonic mean
\[ H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}; \quad a, b > 0, \]

d) The identric mean
\[ I(a, b) := \begin{cases} \frac{b^e - a^e}{e(b - a)} & \text{if } b \neq a \quad ; \quad a, b > 0 \\ a & \text{if } b = a \end{cases} \]

e) The logarithmic mean
\[ L(a, b) := \begin{cases} \frac{b - a}{\ln b - \ln a} & \text{if } b \neq a \quad ; \quad a, b > 0 \\ a & \text{if } b = a \end{cases} \]

f) The p-logarithmic mean
\[ L_p(a, b) := \begin{cases} \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}} & \text{if } b \neq a, \quad p \in \mathbb{R} \setminus \{-1, 0\} \quad ; \quad a, b > 0. \\ a & \text{if } b = a \end{cases} \]

If we put \( L_0(a, b) := I(a, b) \) and \( L_{-1}(a, b) := L(a, b) \), then it is well known that the function \( \mathbb{R} \ni \tau \mapsto L_p(a, b) \) is monotonic increasing on \( \mathbb{R} \).

We observe that for \( p \in \mathbb{R} \setminus \{-1, 0\} \) we have
\[ \int_0^1 [(1 - \tau) a + \tau b]^p d\tau = L_p^p(a, b), \quad \int_0^1 [(1 - \tau) a + \tau b]^{-1} d\tau = L^{-1}(a, b) \]
and
\[ \int_0^1 \ln [(1 - \tau) a + \tau b] d\tau = \ln I(a, b). \]

Using these notations we can define the following divergences for \((t, s) \in I^n \times I^n\) where \( I \) is an interval of positive numbers:
\[ T_p(t, s) := A(t^p, s^p) - L_p^p(t, s) \]
and
\[ M_p(t, s) := L_p^p(t, s) - A^p(t, s) \]
for all \( p \in \mathbb{R} \setminus \{-1, 0\} \),
\[ T_{-1}(t, s) := H^{-1}(t, s) - L^{-1}(t, s) \]
and
\[ M_{-1}(t, s) := L^{-1}(t, s) - A^{-1}(t, s) \]
for $p = -1$ and
\[ T_0(t, s) := \ln \left( \frac{G(t, s)}{I(t, s)} \right) \]
and
\[ M_0(t, s) := \ln \left( \frac{I(t, s)}{A(t, s)} \right) \]
for $p = 0$.

Since the function $f(\tau) = \tau^p, \tau > 0$ is convex for $p \in (-\infty, 0) \cup (1, \infty)$, then we have
\[ T_p(t, s), M_p(t, s) \geq 0 \] for all $(t, s) \in I \times I$.

For $p \in (0, 1)$ the function $f(\tau) = \tau^p, \tau > 0$ and for $p = 0$, the function $f(\tau) = \ln \tau$ are concave, then we have for $p \in [0, 1)$ that
\[ T_p(t, s), M_p(t, s) \leq 0 \] for all $(t, s) \in I \times I$.

Finally for $p = 1$ we have both $T_1(t, s) = M_1(t, s) = 0$ for all $(t, s) \in I \times I$.

We need the following convexity result that is a consequence of Burbea-Rao’s theorem above:

**Lemma 1.** Let $f$ be a $C^2$ function on an interval $I$. Then $T_f$ and $M_f$ are convex (concave) on $I \times I$, if and only if $f$ is convex (concave) and $\frac{1}{\tau f}$ is concave (convex) on $I$.

**Proof.** If $T_f$ and $M_f$ are convex on $I \times I$ then the sum $T_f + M_f = J_f$ is convex on $I \times I$, which, by Burbea-Rao theorem implies that $f$ is convex and $\frac{1}{\tau f}$ is concave on $I$.

Now, if $f$ is convex and $\frac{1}{\tau f}$ is concave on $I$, then by the same theorem we have that the function $J_f : I \times I \rightarrow \mathbb{R}$
\[ J_f(t, s) := \frac{1}{2} [f(t) + f(s)] - f \left( \frac{t + s}{2} \right) \]
is convex.

Let $t, s, u, v \in I$. We define
\[ \varphi(\tau) := J_f \left( ((1 - \tau)(t, s) + (u, v)) \right) = J_f \left( (((1 - \tau)t + \tau u, (1 - \tau)s + \tau v) \right) \]
\[ = \frac{1}{2} \left[ f \left( (1 - \tau)t + \tau u \right) + f \left( (1 - \tau)s + \tau v \right) \right] \]
\[ - f \left( \frac{(1 - \tau)t + \tau u + (1 - \tau)s + \tau v}{2} \right) \]
\[ = \frac{1}{2} \left[ f \left( (1 - \tau)t + \tau u \right) + f \left( (1 - \tau)s + \tau v \right) \right] \]
\[ - f \left( \frac{(1 - \tau)t + s}{2} + \tau \frac{u + v}{2} \right) \]
for $\tau \in [0, 1]$.
Let $\tau_1, \tau_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of $J_f$ we have

$$
\varphi (\alpha \tau_1 + \beta \tau_2) = J_f ((1 - \alpha \tau_1 - \beta \tau_2) (t, s) + (\alpha \tau_1 + \beta \tau_2) (u, v)) = J_f ((\alpha + \beta - \alpha \tau_1 - \beta \tau_2) (t, s) + (\alpha \tau_1 + \beta \tau_2) (u, v)) = J_f (\alpha (1 - \tau_1) (t, s) + \beta (1 - \tau_2) (t, s) + \alpha \tau_1 (u, v) + \beta \tau_2 (u, v)) = J_f (\alpha ((1 - \tau_1) (t, s) + \tau_1 (u, v)) + \beta ((1 - \tau_2) (t, s) + \tau_2 (u, v))) \leq \alpha J_f ((1 - \tau_1) (t, s) + \tau_1 (u, v)) + \beta J_f ((1 - \tau_2) (t, s) + \tau_2 (u, v)) = \alpha \varphi (\tau_1) + \beta \varphi (\tau_2),
$$

which proves that $\varphi$ is convex on $[0, 1]$ for all $t, s, u, v \in I$.

Applying the Hermite-Hadamard inequality for $\varphi$ we get

$$
(2.8) \quad \frac{1}{2} |\varphi (0) + \varphi (1)| \geq \int_0^1 \varphi (\tau) d\tau
$$

and since

$$
\varphi (0) = \frac{1}{2} [f (t) + f (s)] - f \left( \frac{t + s}{2} \right),
$$

$$
\varphi (1) = \frac{1}{2} [f (u) + f (v)] - f \left( \frac{u + v}{2} \right)
$$

and

$$
\int_0^1 \varphi (\tau) d\tau = \frac{1}{2} \left[ \int_0^1 f ((1 - \tau) t + \tau u) d\tau + \int_0^1 f ((1 - \tau) s + \tau v) d\tau \right] - \int_0^1 f \left( (1 - \tau) \frac{t + s}{2} + \tau \frac{u + v}{2} \right) d\tau,
$$

hence by (2.8) we get

$$
\frac{1}{2} \left\{ \frac{1}{2} [f (t) + f (s)] - f \left( \frac{t + s}{2} \right) + \frac{1}{2} [f (u) + f (v)] - f \left( \frac{u + v}{2} \right) \right\} \geq \frac{1}{2} \left[ \int_0^1 f ((1 - \tau) t + \tau u) d\tau + \int_0^1 f ((1 - \tau) s + \tau v) d\tau \right] - \int_0^1 f \left( (1 - \tau) \frac{t + s}{2} + \tau \frac{u + v}{2} \right) d\tau.
$$

Re-arranging this inequality, we get

$$
\frac{1}{2} \left[ \frac{f (t) + f (u)}{2} - \int_0^1 f ((1 - \tau) t + \tau u) d\tau \right] + \frac{1}{2} \left[ \frac{f (s) + f (v)}{2} - \int_0^1 f ((1 - \tau) s + \tau v) d\tau \right] \geq \frac{1}{2} \left[ f \left( \frac{t + s}{2} \right) + f \left( \frac{u + v}{2} \right) - \int_0^1 f \left( (1 - \tau) \frac{t + s}{2} + \tau \frac{u + v}{2} \right) d\tau \right],
$$
which is equivalent to
\[
\frac{1}{2} [T_f(t, u) + T_f(s, v)] \geq T_f \left( \frac{t + s}{2}, \frac{u + v}{2} \right) \\
= T_f \left( \frac{1}{2} (t, u) + \frac{1}{2} (s, v) \right),
\]
for all \((t, u), (s, v) \in I \times I\), which shows that \(T_f\) is Jensen’s convex on \(I \times I\). Since \(T_f\) is continuous on \(I \times I\), hence \(T_f\) is convex in the usual sense on \(I \times I\).

Now, if we use the second Hermite-Hadamard inequality for \(\varphi\) on \([0, 1]\), we have

\[
(2.9) \quad \int_0^1 \varphi(\tau) \, d\tau \geq \varphi \left( \frac{1}{2} \right).
\]

Since
\[
\varphi \left( \frac{1}{2} \right) = \frac{1}{2} \left[ f \left( \frac{t + u}{2} \right) + f \left( \frac{s + v}{2} \right) \right] - f \left( \frac{1}{2} \left( \frac{t + u}{2} + \frac{1}{2} \right) \right)
\]
hence by (2.9) we have
\[
\frac{1}{2} \left[ \int_0^1 f \left( (1 - \tau) t + \tau u \right) \, d\tau + \int_0^1 f \left( (1 - \tau) s + \tau v \right) \, d\tau \right] - \int_0^1 f \left( (1 - \tau) \frac{t + s}{2} + \tau \frac{u + v}{2} \right) \, d\tau \\
\geq \frac{1}{2} \left[ f \left( \frac{t + u}{2} \right) + f \left( \frac{s + v}{2} \right) \right] - f \left( \frac{1}{2} \left( \frac{t + s}{2} + \frac{u + v}{2} \right) \right),
\]
which is equivalent to
\[
\frac{1}{2} \left[ \int_0^1 f \left( (1 - \tau) t + \tau u \right) \, d\tau - f \left( \frac{t + u}{2} \right) \right] \\
+ \frac{1}{2} \left[ \int_0^1 f \left( (1 - \tau) s + \tau v \right) \, d\tau - f \left( \frac{s + v}{2} \right) \right] \\
\geq \int_0^1 f \left( (1 - \tau) \frac{t + s}{2} + \tau \frac{u + v}{2} \right) \, d\tau - f \left( \frac{1}{2} \left( \frac{t + s}{2} + \frac{u + v}{2} \right) \right)
\]
that can be written as
\[
\frac{1}{2} [M_f(t, u) + M_f(s, v)] \geq M_f \left( \frac{t + s}{2}, \frac{u + v}{2} \right) \\
= M_f \left( \frac{1}{2} (t, u) + \frac{1}{2} (s, v) \right)
\]
for all \((t, u), (s, v) \in I \times I\), which shows that \(M_f\) is Jensen’s convex on \(I \times I\). Since \(M_f\) is continuous on \(I \times I\), hence \(M_f\) is convex in the usual sense on \(I \times I\). ☐

The following reverses of the Hermite-Hadamard inequality hold:
Lemma 2 (Dragomir, 2002 [10] and [11]). Let $h : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$. Then

\begin{equation}
0 \leq \frac{1}{8} \left[ h_+ \left( \frac{a + b}{2} \right) - h_- \left( \frac{a + b}{2} \right) \right] (b - a)
\end{equation}

(2.10)

\begin{align*}
&\leq \frac{h(a) + h(b)}{2} - \frac{1}{b - a} \int_a^b h(\tau) \, d\tau \\
&\leq \frac{1}{8} [h_- (b) - h_+ (a)] (b - a)
\end{align*}

and

\begin{equation}
0 \leq \frac{1}{8} \left[ h_+ \left( \frac{a + b}{2} \right) - h_- \left( \frac{a + b}{2} \right) \right] (b - a)
\end{equation}

(2.11)

\begin{align*}
&\leq \frac{1}{b - a} \int_a^b h(\tau) \, d\tau - h \left( \frac{a + b}{2} \right) \\
&\leq \frac{1}{8} [h_- (b) - h_+ (a)] (b - a).
\end{align*}

The constant $\frac{1}{8}$ is best possible in all inequalities from (2.10) and (2.11).

We also have:

Lemma 3. Let $f$ be a $C^1$ convex function on an interval $I$. If $\bar{I}$ is the interior of $I$, then for all $(t, s) \in \bar{I} \times \bar{I}$ we have

\begin{equation}
0 \leq M_f (t, s) \leq T_f (t, s) \leq \frac{1}{8} C_{f'} (t, s)
\end{equation}

(2.12)

where

\begin{equation}
C_{f'} (t, s) := [f' (t) - f' (s)] (t - s).
\end{equation}

(2.13)

Proof. Since for $b \neq a$

\begin{equation*}
\frac{1}{b - a} \int_a^b f (t) \, dt = \int_0^1 f ((1 - t) a + t b) \, dt,
\end{equation*}

then from (2.10) we get

\begin{equation*}
\frac{f (t) + f (s)}{2} - \int_0^1 f ((1 - \tau) t + \tau s) \, d\tau \leq \frac{1}{8} [f' (t) - f' (s)] (t - s)
\end{equation*}

for all $(t, s) \in \bar{I} \times \bar{I}$. \hfill \Box

Remark 2. If

\begin{equation*}
\gamma = \inf_{t \in I} f' (t) \quad \text{and} \quad \Gamma = \sup_{t \in I} f' (t)
\end{equation*}

are finite, then

\begin{equation*}
C_{f'} (t, s) \leq (\Gamma - \gamma) |t - s|
\end{equation*}

and by (2.12) we get the simpler upper bound

\begin{equation*}
0 \leq M_f (t, s) \leq T_f (t, s) \leq \frac{1}{8} (\Gamma - \gamma) |t - s|.
\end{equation*}

Moreover, if $t, s \in [a, b] \subset \bar{I}$ and since $f'$ is increasing on $\bar{I}$, then we have the inequalities

\begin{equation}
0 \leq M_f (t, s) \leq T_f (t, s) \leq \frac{1}{8} [f' (b) - f' (a)] |t - s|.
\end{equation}

(2.14)
Since $\mathcal{J}_f(t, s) = \mathcal{T}_f(t, s) + \mathcal{M}_f(t, s)$, hence
\[0 \leq \mathcal{J}_f(t, s) \leq \frac{1}{4} |f'(b) - f'(a)| |t - s|.\]

**Corollary 1.** With the assumptions of Lemma 3 and if the derivative $f'$ is Lipschitzian with the constant $K > 0$, namely
\[|f'(t) - f'(s)| \leq K |t - s| \quad \text{for all} \quad t, s \in I,
\]
then we have the inequality
\[0 \leq \mathcal{M}_f(t, s) \leq \mathcal{T}_f(t, s) \leq \frac{1}{8} K (t - s)^2.\]

3. Main Results

Let $P, Q, W \in \mathcal{P}$ and $f : (0, \infty) \to \mathbb{R}$. We define the following $f$-divergence
\[\mathcal{J}_f(P, Q, W) := \int_X w(x) \mathcal{J}_f\left(\frac{p(x)}{w(x)}, \frac{q(x)}{w(x)}\right) d\mu(x)
\]
\[= \frac{1}{2} \left[\int_X w(x) f\left(\frac{p(x)}{w(x)}\right) d\mu(x) + \int_X w(x) f\left(\frac{q(x)}{w(x)}\right) d\mu(x)\right]
\[= \int_X w(x) f\left(\frac{p(x) + q(x)}{2w(x)}\right) d\mu(x).\]

If we consider the mid-point divergence measure $M_f$ defined by
\[M_f(Q, P, W) := \int_X f\left[\frac{q(x) + p(x)}{2w(x)}\right] w(x) d\mu(x),\]
for any $Q, P, W \in \mathcal{P}$, then from (3.1) we get
\[\mathcal{J}_f(P, Q, W) = \frac{1}{2} [I_f(P, W) + I_f(Q, W)] - M_f(Q, P, W).\]

We can also consider the integral divergence measure
\[A_f(Q, P, W) := \int_X \left(\int_0^1 f\left[\frac{(1 - t)q(x) + tp(x)}{w(x)}\right] dt\right) w(x) d\mu(x).\]

We introduce the related $f$-divergences
\[\mathcal{T}_f(P, Q, W) := \int_X w(x) \mathcal{T}_f\left(\frac{p(x)}{w(x)}, \frac{q(x)}{w(x)}\right) d\mu(x)
\]
\[= \frac{1}{2} [I_f(P, W) + I_f(Q, W)] - A_f(Q, P, W)\]
and
\[\mathcal{M}_f(P, Q, W) := \int_X w(x) \mathcal{M}_f\left(\frac{p(x)}{w(x)}, \frac{q(x)}{w(x)}\right) d\mu(x)
\]
\[= A_f(Q, P, W) - M_f(Q, P, W).\]

We observe that
\[\mathcal{J}_f(P, Q, W) = \mathcal{T}_f(P, Q, W) + \mathcal{M}_f(P, Q, W).\]

If $f$ is convex on $(0, \infty)$ then by the Hermite-Hadamard and Bullen’s inequalities we have the positivity properties
\[0 \leq \mathcal{M}_f(P, Q, W) \leq \mathcal{T}_f(P, Q, W).\]
and

$$0 \leq J_f(P, Q, W)$$

for $P, Q, W \in \mathcal{P}$.

We have the following result:

**Theorem 5.** Let $f$ be a $C^2$ function on an interval $(0, \infty)$. If $f$ is convex on $(0, \infty)$ and $\frac{1}{f'}$ is concave on $(0, \infty)$, then for all $W \in \mathcal{P}$, the mappings

$$\mathcal{P} \times \mathcal{P} \ni (P, Q) \mapsto J_f(P, Q, W), \quad M_f(P, Q, W), \quad T_f(P, Q, W)$$

are convex.

**Proof.** Let $(P_1, Q_1), (P_2, Q_2) \in \mathcal{P} \times \mathcal{P}$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. We have

$$J_f(\alpha (P_1, Q_1, W) + \beta (P_2, Q_2, W)) = J_f(\alpha P_1 + \beta P_2, \alpha Q_1 + \beta Q_2, W)$$

$$= \int_X w(x) J_f\left(\frac{\alpha p_1(x) + \beta p_2(x)}{w(x)}, \frac{\alpha q_1(x) + \beta q_2(x)}{w(x)}\right) d\mu(x)$$

$$= \int_X w(x) \left[\alpha J_f\left(\frac{p_1(x)}{w(x)}, \frac{q_1(x)}{w(x)}\right) + \beta J_f\left(\frac{p_2(x)}{w(x)}, \frac{q_2(x)}{w(x)}\right)\right] d\mu(x)$$

$$\leq \alpha \int_X w(x) J_f\left(\frac{p_1(x)}{w(x)}, \frac{q_1(x)}{w(x)}\right) d\mu(x) + \beta \int_X J_f\left(\frac{p_2(x)}{w(x)}, \frac{q_2(x)}{w(x)}\right) d\mu(x)$$

$$= \alpha J_f(P_1, Q_1, W) + \beta J_f(P_2, Q_2, W),$$

which proves the convexity of $\mathcal{P} \times \mathcal{P} \ni (P, Q) \mapsto J_f(P, Q, W)$ for all $W \in \mathcal{P}$.

The convexity of the other two mappings follows in a similar way and we omit the details. $\square$

**Theorem 6.** Let $f$ be a $C^1$ function on an interval $(0, \infty)$. If $f$ is convex on $(0, \infty)$, then for all $W \in \mathcal{P}$

$$0 \leq M_f(P, Q, W) \leq T_f(P, Q, W) \leq \frac{1}{8} \Delta_f'(Q, P, W)$$

where

$$\Delta_f'(Q, P, W) := \int_X \left[f'(\frac{q(x)}{w(x)}) - f'(\frac{p(x)}{w(x)})\right] (q(x) - p(x)) d\mu(x).$$

**Proof.** From the inequality (2.12) we have

$$\frac{1}{2} \left[f\left(\frac{p(x)}{w(x)}\right) + f\left(\frac{q(x)}{w(x)}\right)\right] - \int_0^1 f\left(\frac{(1-t)p(x) + tq(x)}{w(x)}\right) dt$$

$$\leq \frac{1}{8} \left[f'(\frac{p(x)}{w(x)}) - f'(\frac{q(x)}{w(x)})\right] \left(\frac{p(x)}{w(x)} - \frac{q(x)}{w(x)}\right)$$

for all $x \in X$. 

If we multiply by \( w(x) > 0 \) and integrate on \( X \) we get

\[
\frac{1}{2} \left[ I_f (P, W) + I_f (P, W) \right] - A_f (Q, P, W) \\
\leq \frac{1}{8} \int_X w(x) \left( f' \left( \frac{p(x)}{w(x)} \right) - f' \left( \frac{q(x)}{w(x)} \right) \right) \left( \frac{p(x)}{w(x)} - \frac{q(x)}{w(x)} \right) \, d\mu(x) \\
= \frac{1}{8} \int_X \left( f' \left( \frac{p(x)}{w(x)} \right) - f' \left( \frac{q(x)}{w(x)} \right) \right) \left( p(x) - q(x) \right) \, d\mu(x),
\]

which implies the desired inequality.

\[ \Box \]

**Corollary 2.** With the assumptions of Theorem 6 and if \( f' \) is Lipschitzian with the constant \( K > 0 \), namely

\[ |f'(s) - f'(t)| \leq K |s - t| \quad \text{for all } t, s \in (0, \infty), \]

then

\[ 0 \leq M_f (P, Q, W) \leq T_f (P, Q, W) \leq \frac{1}{8} K d_{\chi^2} (Q, P, W), \]

where

\[ d_{\chi^2} (Q, P, W) := \int_X \frac{(q(x) - p(x))^2}{w(x)} \, d\mu(x). \]

**Remark 3.** If there exists \( 0 < r < 1 < R < \infty \) such that the following condition holds

\[ ((r,R)) \quad r \leq \frac{q(x)}{w(x)} \leq \frac{p(x)}{w(x)} \leq R \quad \text{for } \mu\text{-a.e. } x \in X, \]

then

\[ 0 \leq M_f (P, Q, W) \leq T_f (P, Q, W) \leq \frac{1}{8} \int_X |f'(R) - f'(r)| \, d_1 (Q, P), \]

where

\[ d_1 (Q, P) := \int_X |q(x) - p(x)| \, d\mu(x). \]

Moreover, if \( f \) is twice differentiable and

\[ \|f''\|_{[r,R], \infty} := \sup_{t \in [r,R]} |f''(t)| < \infty \]

then

\[ 0 \leq M_f (P, Q, W) \leq T_f (P, Q, W) \leq \frac{1}{8} \|f''\|_{[r,R], \infty} d_{\chi^2} (Q, P, W). \]

We also have:

**Theorem 7.** Let \( f \) be a \( C^2 \) function on an interval \( (0, \infty) \). If \( f \) is convex on \( (0, \infty) \) and \( \frac{1}{f'} \) is concave on \( (0, \infty) \), then for all \( W \in \mathcal{P}, \)

\[ 0 \leq J_f (P, Q, W) \leq \frac{1}{2} \left[ \Psi_f' (P, Q, W) + \Psi_f' (Q, P, W) \right], \]

where

\[ \Psi_f' (P, Q, W) := \int_X \left( f' \left( \frac{p(x)}{w(x)} \right) - f' \left( \frac{q(x) + p(x)}{2w(x)} \right) \right) \left( p(x) - w(x) \right) \, d\mu(x). \]
Proof. It is well known that if the function of two independent variables $F : D \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is convex on the convex domain $D$ and has partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ on $D$ then for all $(t, s), (u, v) \in D$ we have the gradient inequalities

$$\frac{\partial F}{\partial x} (t, s) (t - u) + \frac{\partial F}{\partial y} (s - v) \geq F (t, s) - F (u, v)$$

and

$$\frac{\partial F}{\partial y} (t, s) (t - u) + \frac{\partial F}{\partial y} (s - v) \geq F (t, s) - F (u, v).$$

Now, if we take $F : (0, \infty) \times (0, \infty) \to \mathbb{R}$ given by

$$F (t, s) = \frac{1}{2} [f (t) + f (s)] - f \left( \frac{t + s}{2} \right)$$

and observe that

$$\frac{\partial F}{\partial x} (t, s) = \frac{1}{2} \left[ f' (t) - f' \left( \frac{t + s}{2} \right) \right]$$

and

$$\frac{\partial F}{\partial y} (t, s) = \frac{1}{2} \left[ f' (s) - f' \left( \frac{t + s}{2} \right) \right]$$

and since $F$ is convex on $(0, \infty) \times (0, \infty)$, then by (3.13) we get

$$\frac{1}{2} \left[ f' (t) - f' \left( \frac{t + s}{2} \right) \right] (t - u) + \frac{1}{2} \left[ f' (s) - f' \left( \frac{t + s}{2} \right) \right] (s - v)$$

$$\geq \frac{1}{2} [f (t) + f (s)] - f \left( \frac{t + s}{2} \right) - \frac{1}{2} [f (u) + f (v)] + f \left( \frac{u + v}{2} \right)$$

$$\geq \frac{1}{2} \left[ f' (u) - f' \left( \frac{u + v}{2} \right) \right] (t - u) + \frac{1}{2} \left[ f' (v) - f' \left( \frac{u + v}{2} \right) \right] (s - v).$$

If we take $u = v = 1$ in (3.14), then we have

$$\frac{1}{2} \left[ f' (t) - f' \left( \frac{t + s}{2} \right) \right] (t - 1) + \frac{1}{2} \left[ f' (s) - f' \left( \frac{t + s}{2} \right) \right] (s - 1)$$

$$\geq \frac{1}{2} [f (t) + f (s)] - f \left( \frac{t + s}{2} \right) \geq 0$$

for all $(t, s) \in (0, \infty) \times (0, \infty)$.

If we take $t = \frac{p(x)}{w(x)}$ and $s = \frac{q(x)}{w(x)}$ in (3.15) then we obtain

$$\frac{1}{2} \left[ f' \left( \frac{p(x)}{w(x)} \right) - f' \left( \frac{q(x) + p(x)}{2w(x)} \right) \right] \left( \frac{p(x)}{w(x)} - 1 \right)$$

$$+ \frac{1}{2} \left[ f' \left( \frac{q(x)}{w(x)} \right) - f' \left( \frac{q(x) + p(x)}{2w(x)} \right) \right] \left( \frac{q(x)}{w(x)} - 1 \right)$$

$$\geq \frac{1}{2} \left[ f \left( \frac{p(x)}{w(x)} \right) + f \left( \frac{q(x)}{w(x)} \right) \right] - f \left( \frac{q(x) + p(x)}{2w(x)} \right) \geq 0.$$
By multiplying this inequality with \( w(x) > 0 \) we get
\[
0 \leq \frac{1}{2} \left[ w(x) f \left( \frac{p(x)}{w(x)} \right) + w(x) f \left( \frac{q(x)}{w(x)} \right) \right] - w(x) f \left( \frac{q(x) + p(x)}{2w(x)} \right)
\]
\[
\leq \frac{1}{2} \left[ f' \left( \frac{p(x)}{w(x)} \right) - f' \left( \frac{q(x) + p(x)}{2w(x)} \right) \right] (p(x) - w(x))
\]
\[
+ \frac{1}{2} \left[ f' \left( \frac{q(x)}{w(x)} \right) - f' \left( \frac{q(x) + p(x)}{2w(x)} \right) \right] (q(x) - w(x))
\]
for all \( x \in X \).

**Corollary 3.** With the assumptions of Theorem 6 and if \( f' \) is Lipschitzian with the constant \( K > 0 \), then
\[
(3.17) \quad 0 \leq J_f(P, Q, W)
\]
\[
\leq \frac{1}{4} K \int_X |p(x) - q(x)| \left[ \left| \frac{p(x)}{w(x)} - 1 \right| + \left| \frac{q(x)}{w(x)} - 1 \right| \right] d\mu(x).
\]

**Proof.** We have that
\[
\Psi_f(P, Q, W)
\]
\[
\leq \int_X \left| f' \left( \frac{p(x)}{w(x)} \right) - f' \left( \frac{q(x) + p(x)}{2w(x)} \right) \right| |p(x) - w(x)| d\mu(x)
\]
\[
\leq K \int_X \frac{p(x) - q(x) + p(x)}{2w(x)} |p(x) - w(x)| d\mu(x)
\]
\[
= K \int_X \frac{p(x) - q(x)}{2w(x)} |p(x) - w(x)| d\mu(x)
\]
\[
= \frac{1}{2} K \int_X \frac{|p(x) - q(x)|}{w(x)} |p(x) - w(x)| d\mu(x)
\]
\[
= \frac{1}{2} K \int_X \frac{|p(x) - q(x)|}{w(x)} |p(x) - w(x)| d\mu(x)
\]
and similarly
\[
\Psi_f(P, Q, W) \leq \frac{1}{2} K \int_X |p(x) - q(x)| \left| \frac{q(x)}{w(x)} - 1 \right| d\mu(x).
\]
Finally, by the use of (3.12) we get the desired result.

**Remark 4.** If there exist \( 0 < r < 1 < R < \infty \) such that the following condition \((r, R)\) holds and if \( f \) is twice differentiable and (3.10) is valid, then
\[
(3.18) \quad 0 \leq J_f(P, Q, W) \leq \frac{1}{4} \|f''\|_{[r, R], \infty}
\]
\[
\times \int_X |p(x) - q(x)| \left[ \left| \frac{p(x)}{w(x)} - 1 \right| + \left| \frac{q(x)}{w(x)} - 1 \right| \right] d\mu(x).
\]
Since
\[
\left| \frac{p(x)}{w(x)} - 1 \right|, \left| \frac{q(x)}{w(x)} - 1 \right| \leq \max \{R - 1, 1 - r\}
\]
and
\[
\left| \frac{p(x)}{w(x)} - \frac{q(x)}{w(x)} \right| \leq R - r,
\]
hence by (3.18) we get the simpler bound

\[ 0 \leq J_f (P, Q, W) \leq \frac{1}{2} \| f''_{[r, R]} \|_{[r, R]} (R - r) \max \{ R - 1, 1 - r \}. \]  

We also have:

**Theorem 8.** With the assumptions of Theorem 6 and if \( f' \) is Lipschitzian with the constant \( K > 0 \), then

\[ 0 \leq T_f (P, Q, W) \leq \frac{1}{6} K \int_X |p(x) - q(x)| \left[ \left| \frac{p(x)}{w(x)} - 1 \right| + \left| \frac{q(x)}{w(x)} - 1 \right| \right] d\mu(x). \]  

**Proof.** Let \((x, y), (u, v) \in (0, \infty) \times (0, \infty)\). If we take \( F : (0, \infty) \times (0, \infty) \to \mathbb{R} \) given by

\[ F(t, s) = \frac{f(t) + f(s)}{2} - \int_0^1 f((1 - \tau) t + \tau s) d\tau \]

then

\[ \frac{\partial F(t, s)}{\partial x} = \frac{1}{2} f'(t) - \int_0^1 (1 - \tau) f'((1 - \tau) t + \tau s) d\tau \]

\[ = \int_0^1 (1 - \tau) \left[ f'(t) - f'((1 - \tau) t + \tau s) \right] d\tau \]

and

\[ \frac{\partial F(t, s)}{\partial y} = \frac{1}{2} f'(s) - \int_0^1 \tau f'((1 - \tau) t + \tau s) d\tau \]

\[ = \int_0^1 \tau \left[ f'(s) - f'((1 - \tau) t + \tau s) \right] d\tau \]

and since \( F \) is convex on \((0, \infty) \times (0, \infty)\), then by (3.1) we get

\[ (t - u) \int_0^1 (1 - \tau) \left[ f'(t) - f'((1 - \tau) t + \tau s) \right] d\tau \]

\[ + (s - v) \int_0^1 \tau \left[ f'(s) - f'((1 - \tau) t + \tau s) \right] d\tau \]

\[ \geq \frac{f(t) + f(s)}{2} - \int_0^1 f((1 - \tau) t + \tau s) d\tau \]

\[ - \frac{f(u) + f(v)}{2} + \int_0^1 f((1 - \tau) u + \tau v) d\tau \]

\[ \geq (t - u) \int_0^1 (1 - \tau) \left[ f'(u) - f'((1 - \tau) u + \tau v) \right] d\tau \]

\[ + (s - v) \int_0^1 \tau \left[ f'(v) - f'((1 - \tau) u + \tau v) \right] d\tau \]

for all \((t, s), (u, v) \in (0, \infty) \times (0, \infty)\).
If we take \( u = v = 1 \) in (3.21), then we have

\[
(t - 1) \int_0^1 (1 - \tau) [f'(t) - f'((1 - \tau)t + \tau s)] d\tau \\
+ (s - 1) \int_0^1 \tau [f'(s) - f'((1 - \tau)t + \tau s)] d\tau \\
\geq \frac{f(t) + f(s)}{2} - \int_0^1 f((1 - \tau)t + \tau s) d\tau \geq 0
\]

for all \((u, v) \in (0, \infty) \times (0, \infty)\).

If we take \( t = \frac{p(x)}{w(x)} \) and \( s = \frac{q(x)}{w(x)} \) in (3.22) then we get

\[
(3.23) \quad \left( \frac{p(x)}{w(x)} - 1 \right) \int_0^1 (1 - \tau) \left[ f' \left( \frac{p(x)}{w(x)} \right) - f' \left( (1 - \tau) \frac{p(x)}{w(x)} + \tau \frac{q(x)}{w(x)} \right) \right] d\tau \\
+ \left( \frac{q(x)}{w(x)} - 1 \right) \int_0^1 \tau \left[ f' \left( \frac{q(x)}{w(x)} \right) - f' \left( (1 - \tau) \frac{p(x)}{w(x)} + \tau \frac{q(x)}{w(x)} \right) \right] d\tau \\
\geq \frac{f \left( \frac{p(x)}{w(x)} \right)}{2} + \frac{f \left( \frac{q(x)}{w(x)} \right)}{2} - \int_0^1 f \left( (1 - \tau) \frac{p(x)}{w(x)} + \tau \frac{q(x)}{w(x)} \right) d\tau \geq 0.
\]

Since \( f' \) is Lipschitzian with the constant \( K > 0 \), hence

\[
0 \leq \frac{f \left( \frac{p(x)}{w(x)} \right)}{2} + \frac{f \left( \frac{q(x)}{w(x)} \right)}{2} - \int_0^1 f \left( (1 - \tau) \frac{p(x)}{w(x)} + \tau \frac{q(x)}{w(x)} \right) d\tau \\
\leq \left| \frac{p(x)}{w(x)} - 1 \right| \int_0^1 (1 - \tau) \left| f' \left( \frac{p(x)}{w(x)} \right) - f' \left( (1 - \tau) \frac{p(x)}{w(x)} + \tau \frac{q(x)}{w(x)} \right) \right| d\tau \\
+ \left| \frac{q(x)}{w(x)} - 1 \right| \int_0^1 \tau \left| f' \left( \frac{q(x)}{w(x)} \right) - f' \left( (1 - \tau) \frac{p(x)}{w(x)} + \tau \frac{q(x)}{w(x)} \right) \right| d\tau \\
\leq K \left| \frac{p(x)}{w(x)} - 1 \right| \left| \frac{p(x)}{w(x)} - \frac{q(x)}{w(x)} \right| \int_0^1 (1 - \tau) \tau d\tau \\
+ K \left| \frac{q(x)}{w(x)} - 1 \right| \left| \frac{p(x)}{w(x)} - \frac{q(x)}{w(x)} \right| \int_0^1 (1 - \tau) \tau d\tau \\
= \frac{1}{6} K \left| \frac{p(x)}{w(x)} - \frac{q(x)}{w(x)} \right| \left[ \left| \frac{p(x)}{w(x)} - 1 \right| + \left| \frac{q(x)}{w(x)} - 1 \right| \right].
\]

If we multiply this inequality by \( w(x) > 0 \) and integrate, then we get the desired result (3.20). \( \square \)

**Corollary 4.** If there exist \( 0 < r < 1 < R < \infty \) such that the condition \( (r, R) \) holds and if \( f \) is twice differentiable and (3.10) is valid, then

\[
(3.24) \quad 0 \leq T_f (P, Q, W) \leq \frac{1}{3} \| f'' \|_{[r, R], \infty} (R - r) \max \{ R - 1, 1 - r \}.
\]

Finally, we also have:
Theorem 9. With the assumptions of Theorem 6 and if $f'$ is Lipschitzian with the constant $K > 0$, then

$$0 \leq M_f (P, Q, W) \leq \frac{1}{8} K \int_X |p(x) - q(x)| \left[ \left| \frac{p(x)}{w(x)} - 1 \right| + \left| \frac{q(x)}{w(x)} - 1 \right| \right] d\mu(x).$$

Proof. Let $(t, s), (u, v) \in (0, \infty) \times (0, \infty)$. If we take $F : (0, \infty) \times (0, \infty) \to \mathbb{R}$ given by

$$F(t, s) = \int_0^1 f((1 - \tau) t + \tau s) d\tau - f \left( \frac{t + s}{2} \right)$$

then

$$\frac{\partial F(t, s)}{\partial x} = \int_0^1 (1 - \tau) f'((1 - \tau) t + \tau s) d\tau - \frac{1}{2} f' \left( \frac{t + s}{2} \right)$$

$$= \int_0^1 (1 - \tau) \left[ f'((1 - \tau) t + \tau s) - f' \left( \frac{t + s}{2} \right) \right] d\tau,$$

$$\frac{\partial F(t, s)}{\partial y} = \int_0^1 \tau f'((1 - \tau) t + \tau s) d\tau - \frac{1}{2} f' \left( \frac{t + s}{2} \right)$$

$$= \int_0^1 \tau \left[ f'((1 - \tau) t + \tau s) - f' \left( \frac{t + s}{2} \right) \right] d\tau$$

and since $F$ is convex on $(0, \infty) \times (0, \infty)$, then by (3.1) we get

$$0 \leq (t - u) \left[ \int_0^1 (1 - \tau) \left[ f'((1 - \tau) t + \tau s) - f' \left( \frac{t + s}{2} \right) \right] d\tau \right]$$

$$+ (s - v) \left[ \int_0^1 \tau \left[ f'((1 - \tau) t + \tau s) - f' \left( \frac{t + s}{2} \right) \right] d\tau \right]$$

$$\geq \int_0^1 f((1 - \tau) t + \tau s) d\tau - f \left( \frac{t + s}{2} \right)$$

$$- \int_0^1 f((1 - \tau) u + \tau v) d\tau + f \left( \frac{u + v}{2} \right)$$

$$\geq (t - u) \left[ \int_0^1 (1 - \tau) \left[ f'((1 - \tau) u + \tau v) - f' \left( \frac{u + v}{2} \right) \right] d\tau \right]$$

$$+ (s - v) \int_0^1 \tau \left[ f'((1 - \tau) u + \tau v) - f' \left( \frac{u + v}{2} \right) \right] d\tau.$$
If we take \( t = \frac{p(x)}{w(x)} \) and \( s = \frac{q(x)}{w(x)} \) in (3.27) then we get

\[
(3.28) \quad 0 \leq \int_0^1 f \left( (1 - \tau) \frac{p(x)}{w(x)} + \tau \frac{q(x)}{w(x)} \right) d\tau - f \left( \frac{p(x) + q(x)}{2w(x)} \right) \\
\leq \frac{p(x)}{w(x)} - 1 \\
\times \left[ \int_0^1 (1 - \tau) \left| f' \left( (1 - \tau) \frac{p(x)}{w(x)} + \tau \frac{q(x)}{w(x)} \right) - f' \left( \frac{p(x) + q(x)}{2w(x)} \right) \right| d\tau \right] \\
\times \left[ \int_0^1 \tau \left| f' \left( (1 - \tau) \frac{p(x)}{w(x)} + \tau \frac{q(x)}{w(x)} \right) - f' \left( \frac{p(x) + q(x)}{2w(x)} \right) \right| d\tau \right] \\
\leq K \left[ \frac{p(x)}{w(x)} - 1 \right] \left| \frac{p(x)}{w(x)} - \frac{q(x)}{w(x)} \right| \int_0^1 (1 - \tau) \left| \tau - \frac{1}{2} \right| d\tau \\
+ K \left[ \frac{q(x)}{w(x)} - 1 \right] \left| \frac{p(x)}{w(x)} - \frac{q(x)}{w(x)} \right| \int_0^1 (1 - \tau) \left| \tau - \frac{1}{2} \right| d\tau.
\]

Since
\[
\int_0^1 (1 - \tau) \left| \tau - \frac{1}{2} \right| d\tau = \frac{1}{8},
\]

hence
\[
0 \leq \int_0^1 f \left( (1 - \tau) \frac{p(x)}{w(x)} + \tau \frac{q(x)}{w(x)} \right) d\tau - f \left( \frac{p(x) + q(x)}{2w(x)} \right) \\
\leq \frac{1}{8} K \left| \frac{p(x)}{w(x)} - \frac{q(x)}{w(x)} \right| \left[ \left| \frac{p(x)}{w(x)} - 1 \right| + \left| \frac{q(x)}{w(x)} - 1 \right| \right]
\]

for all \( x \in X \).

If we multiply this inequality by \( w(x) > 0 \) and integrate, then we get the desired result (3.20).

\[\square\]

**Corollary 5.** If there exist \( 0 < r < 1 < R < \infty \) such that the condition \((r, R)\) holds and if \( f \) is twice differentiable and (3.10) is valid, then

\[
(3.29) \quad 0 \leq \mathcal{M}_f(P, Q, W) \leq \frac{1}{4} \|f''\|_{[r, R], \infty} (R - r) \max \{R - 1, 1 - r\}.
\]
Some f-Divergence Measures Related to Jensen’s One

4. Some Examples

The Dichotomy Class of f-divergences are generated by the functions \( f_\alpha : [0, \infty) \to \mathbb{R} \) defined as

\[
 f_\alpha (u) = \begin{cases} 
 u - 1 - \ln u & \text{for } \alpha = 0; \\
 \frac{1}{\alpha (1 - \alpha)} \left[ \alpha u + 1 - \alpha - u^\alpha \right] & \text{for } \alpha \in \mathbb{R} \setminus \{0, 1\}; \\
 1 - u + u \ln u & \text{for } \alpha = 1.
\end{cases}
\]

Observe that

\[
 f_\alpha'' (u) = \begin{cases} 
 \frac{1}{u^2} & \text{for } \alpha = 0; \\
 u^{\alpha - 2} & \text{for } \alpha \in \mathbb{R} \setminus \{0, 1\}; \\
 \frac{1}{u} & \text{for } \alpha = 1.
\end{cases}
\]

In this family of functions only the functions \( f_\alpha \) with \( \alpha \in [1, 2) \) are both convex and with \( \frac{1}{f_\alpha''} \) concave on \((0, \infty)\).

We have

\[
 I_{f_\alpha} (P, W) = \int_X w(x) f_\alpha \left( \frac{p(x)}{w(x)} \right) d\mu(x)
\]

\[
 = \begin{cases} 
 \frac{1}{\alpha (\alpha - 1)} \left[ \int_X w^{1-\alpha} (x) p^\alpha (x) d\mu(x) - 1 \right], & \alpha \in (1, 2), \\
 \int_X p(x) \ln \left( \frac{p(x)}{w(x)} \right) d\mu(x), & \alpha = 1,
\end{cases}
\]

and

\[
 M_{f_\alpha} (Q, P, W) = \int_X f \left[ \frac{q(x) + p(x)}{2w(x)} \right] w(x) d\mu(x)
\]

\[
 = \begin{cases} 
 \frac{1}{\alpha (\alpha - 1)} \left[ \int_X \left[ \frac{q(x) + p(x)}{2} \right]^\alpha w^{1-\alpha} (x) d\mu(x) - 1 \right], & \alpha \in (1, 2), \\
 \int_X \left[ \frac{q(x) + p(x)}{2w(x)} \right] \ln \left[ \frac{q(x) + p(x)}{2w(x)} \right] d\mu(x), & \alpha = 1.
\end{cases}
\]

We also have

\[
 \int_0^1 [(1 - t) a + tb] \ln [(1 - t) a + tb] \, dt
= \frac{1}{4} (b + a) \ln I (a^2, b^2) = \frac{1}{2} A (a, b) \ln I (a^2, b^2).
\]

Therefore

\[
 A_{f_\alpha} (Q, P, W) := \int_X \left( \int_0^1 f \left[ \frac{(1 - t) q(x) + tp(x)}{w(x)} \right] dt \right) w(x) d\mu(x)
\]

\[
 = \begin{cases} 
 \frac{1}{\alpha (\alpha - 1)} \left[ \int_X L_\alpha \left( \frac{q(x)}{w(x)}, \frac{p(x)}{w(x)} \right) w(x) d\mu(x) - 1 \right], & \alpha \in (1, 2), \\
 \frac{1}{2} \int_X A \left( \frac{q(x)}{w(x)}, \frac{p(x)}{w(x)} \right) \ln I \left( \left( \frac{q(x)}{w(x)} \right)^2, \left( \frac{p(x)}{w(x)} \right)^2 \right) w(x) d\mu(x), & \alpha = 1.
\end{cases}
\]
We have
\[
\mathcal{J}_{f_{\alpha}}(P,Q,W) = \frac{1}{2} [I_{f_{\alpha}}(P,W) + I_{f_{\alpha}}(Q,W)] - M_{f_{\alpha}}(Q,P,W),
\]
\[
\mathcal{T}_{f_{\alpha}}(P,Q,W) = \frac{1}{2} [I_{f_{\alpha}}(P,W) + I_{f_{\alpha}}(Q,W)] - A_{f_{\alpha}}(Q,P,W)
\]
and
\[
M_{f_{\alpha}}(P,Q,W) = A_{f_{\alpha}}(Q,P,W) - M_{f_{\alpha}}(Q,P,W).
\]

According to Theorem 5, for all \( \alpha \in [1,2] \), the mappings
\( \mathcal{P} \times \mathcal{P} \ni (P,Q) \mapsto \mathcal{J}_{f_{\alpha}}(P,Q,W), A_{f_{\alpha}}(P,Q,W), T_{f_{\alpha}}(P,Q,W) \)
are convex for all \( W \in \mathcal{P} \).

If \( 0 < r < 1 < R \), then
\[
\|f''_{\alpha}\|_{[r,R],\infty} = \sup_{t \in [r,R]} f''_{\alpha}(t) = \frac{1}{r^{2-\alpha}} \text{ for } \alpha \in [1,2].
\]

If there exists \( 0 < r < 1 < R < \infty \) such that the following condition holds
\[
(r,R) = \left( \frac{q(x)}{w(x)}, \frac{p(x)}{w(x)} \right) \leq R \text{ for } \mu\text{-a.e. } x \in X,
\]
then by (3.19), (3.24) and (3.29) we get
\[
0 \leq \mathcal{J}_{f_{\alpha}}(P,Q,W) \leq \frac{1}{2} \|f''_{\alpha}\|_{[r,R],\infty} (R - r) \max \{R - 1, 1 - r\},
\]
\[
0 \leq \mathcal{T}_{f_{\alpha}}(P,Q,W) \leq \frac{1}{3} \frac{(R - r)}{r^{2-\alpha}} \max \{R - 1, 1 - r\}
\]
and
\[
0 \leq M_{f_{\alpha}}(P,Q,W) \leq \frac{1}{4} \frac{(R - r)}{r^{2-\alpha}} \max \{R - 1, 1 - r\},
\]
for all \( \alpha \in [1,2] \) and \( W \in \mathcal{P} \).

The interested reader may apply the above general results for other particular divergences of interest generated by the convex functions provided in the introduction. We omit the details.

References


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