SOME INEQUALITIES FOR AN INTEGRAL OPERATOR AND n-TIME DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper we establish some trapezoid type inequalities for the operator

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x \in (a,b)$$

in the case of functions $f : [a,b] \to \mathbb{C}$ whose n-derivatives $f^{(n)}$ are absolutely continuous on $[a,b]$. Several Hermite-Hadamard type inequalities are also provided.

1. Introduction

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Theorem 1. *Let $I \subset \mathbb{R}$ be a closed interval, $c \in I$ and let n be a positive integer. If $f : I \longrightarrow \mathbb{C}$ is such that the n-derivative $f^{(n)}$ is absolutely continuous on I, then for each $z \in I$*

$$f(z) = T_n(f;c,z) + R_n(f;c,z), \tag{1.1}$$

where $T_n(f;c,z)$ is Taylor's polynomial, i.e.,

$$T_n(f;c,z) := \sum_{k=0}^{n} \frac{(z-c)^k}{k!} f^{(k)}(c). \tag{1.2}$$

Note that $f^{(0)} := f$ and $0! := 1$ and the remainder is given by

$$R_n(f;c,z) := \frac{1}{n!}\int_c^z (z-t)^n f^{(n+1)}(t)\,dt. \tag{1.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

Assume that the function $f : (a,b) \to \mathbb{C}$ is Lebesgue integrable on (a,b). We consider the following operator [7]

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x \in (a,b). \tag{1.4}$$

We observe that if we take $x = \frac{a+b}{2}$, then we have

$$D_{a+,b-}f\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

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Moreover, if $f(a+) := \lim_{x\to a+} f(x)$ exists and is finite, then we have

$$\lim_{x\to a+} D_{a+,b-}f(x) = \frac{1}{2}\left[f(a+) + \frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

and if $f(b-) := \lim_{x\to b-} f(x)$ exists and is finite, then we have

$$\lim_{x\to b-} D_{a+,b-}f(x) = \frac{1}{2}\left[f(b-) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

So, if $f : [a,b] \to \mathbb{C}$ is Lebesgue integrable on $[a,b]$ and continuous at right in a and at left in b, then we can extend the operator on the whole interval by putting

$$D_{a+,b-}f(a) := \frac{1}{2}\left[f(a) + \frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

and

$$D_{a+,b-}f(b) := \frac{1}{2}\left[f(b) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

We say that the function $f : [a,b] \to \mathbb{C}$ is of *H-r-Hölder type* if

$$|f(t) - f(s)| \le H\,|t-s|^r$$

for any $t,\ s \in [a,b]$, where $H > 0$ and $r \in (0,1]$. If $r = 1$ and we put $H = L$, then we call the function of *L-Lipschitz type*.

In the recent paper [7] we obtained amongst other the following trapezoid type inequalities:

Theorem 2. *If f is of H-r-Hölder type on $[a,b]$ with $H > 0$ and $r \in (0,1]$, then for any $x \in (a,b)$ we have*

$$\left|D_{a+,b-}f(x) - \frac{f(a)+f(b)}{2}\right| \le \frac{1}{2(r+1)}H\left[(x-a)^r + (b-x)^r\right]. \tag{1.5}$$

In particular, if f is of L-Lipschitz type, then

$$\left|D_{a+,b-}f(x) - \frac{f(a)+f(b)}{2}\right| \le \frac{1}{4}L(b-a) \tag{1.6}$$

for any $x \in (a,b)$.

If we take in Theorem 2 $x = \frac{a+b}{2}$, then we get the following trapezoid type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \le \frac{1}{2^r(r+1)}H(b-a)^r. \tag{1.7}$$

In particular, if f is of *L-Lipschitz type,* then we get the result from [16]:

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \le \frac{1}{4}L(b-a). \tag{1.8}$$

Motivated by the above results, by the use of Taylor's formula with integral remainder (1.1), in this paper we establish a trapezoid type representation for the operator $D_{a+,b-}f(x)$, $x \in (a,b)$ in the case of functions $f : [a,b] \to \mathbb{C}$ whose n-derivatives $f^{(n)}$ are absolutely continuous on $[a,b]$. As applications, several trapezoid type inequalities are also provided. Moreover, several Hermite-Hadamard type inequalities are also established.

2. Some Trapezoid Type Identities

We have the following representation:

Theorem 3. *Let $I \subset \mathbb{R}$ be an interval, $[a,b] \subset I$ and $f : I \longrightarrow \mathbb{C}$ is such that the n-derivative $f^{(n)}$ is absolutely continuous on $[a,b]$. Then for any $x \in (a,b)$ we have the representation*

$$D_{a+,b-}f(x) = \sum_{k=0}^{n} \frac{1}{(k+1)!}\left[\frac{f^{(k)}(a)(x-a)^k + (-1)^k f^{(k)}(b)(b-x)^k}{2}\right] \tag{2.1}$$
$$+ \frac{1}{2n!}(x-a)^{n+1}\int_0^1\int_0^1 u^{n+1}s^n f^{(n+1)}(sa + (1-s)[(1-u)a + ux])\,dsdu$$
$$+ \frac{(-1)^{n+1}}{2n!}(b-x)^{n+1}\int_0^1\int_0^1 u^{n+1}s^n f^{(n+1)}((1-s)[ux + (1-u)b] + sb)\,dsdu.$$

Proof. Using Taylor's representation with the integral remainder (1.1) we can write the following two identities

$$f(y) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(y-a)^k + \frac{1}{n!}\int_a^y f^{(n+1)}(t)(y-t)^n\,dt \tag{2.2}$$

and

$$f(y) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(b)(b-y)^k + \frac{(-1)^{n+1}}{n!}\int_y^b f^{(n+1)}(t)(t-y)^n\,dt \tag{2.3}$$

for any y, a, $b \in \mathring{I}$.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0,1]$ that

$$\int_c^d h(t)\,dt = (d-c)\int_0^1 h((1-s)c + sd)\,ds.$$

Therefore,

$$\int_a^y f^{(n+1)}(t)(y-t)^n\,dt = (y-a)\int_0^1 f^{(n+1)}((1-s)a + sy)(y - (1-s)a - sy)^n\,ds$$
$$= (y-a)^{n+1}\int_0^1 f^{(n+1)}((1-s)a + sy)(1-s)^n\,ds$$

and

$$\int_y^b f^{(n+1)}(t)(t-y)^n\,dt = (b-y)\int_0^1 f^{(n+1)}((1-s)y + sb)((1-s)y + sb - y)^n\,ds$$
$$= (b-y)^{n+1}\int_0^1 f^{(n+1)}((1-s)y + sb)\,s^n ds.$$

The identities (2.2) and (2.3) can then be written as

$$f(y)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)(y-a)^{k}+\frac{1}{n!}(y-a)^{n+1}\int_{0}^{1}f^{(n+1)}((1-s)a+sy)(1-s)^{n}ds \tag{2.4}$$

and

$$f(y)=\sum_{k=0}^{n}\frac{(-1)^{k}}{k!}f^{(k)}(b)(b-y)^{k}+(-1)^{n+1}\frac{(b-y)^{n+1}}{n!}\int_{0}^{1}f^{(n+1)}((1-s)y+sb)s^{n}ds. \tag{2.5}$$

Now, for $x\in(a,b)$, if we integrate (2.4) on $[a,x]$ over y, then we get

$$\int_{a}^{x}f(y)\,dy=\sum_{k=0}^{n}\frac{1}{(k+1)!}f^{(k)}(a)(x-a)^{k+1}+\frac{1}{n!}\int_{a}^{x}(y-a)^{n+1}\left(\int_{0}^{1}f^{(n+1)}((1-s)a+sy)(1-s)^{n}ds\right)dy,$$

which gives

$$\frac{1}{x-a}\int_{a}^{x}f(y)\,dy=\sum_{k=0}^{n}\frac{1}{(k+1)!}f^{(k)}(a)(x-a)^{k}+\frac{1}{n!}\frac{1}{x-a}\int_{a}^{x}(y-a)^{n+1}\left(\int_{0}^{1}f^{(n+1)}((1-s)a+sy)(1-s)^{n}ds\right)dy. \tag{2.6}$$

Also, if we integrate (2.5) on $[x,b]$ over y, then we get

$$\int_{x}^{b}f(y)\,dy=\sum_{k=0}^{n}\frac{(-1)^{k}}{(k+1)!}f^{(k)}(b)(b-x)^{k+1}+\frac{(-1)^{n+1}}{n!}\int_{x}^{b}(b-y)^{n+1}\left(\int_{0}^{1}f^{(n+1)}((1-s)y+sb)s^{n}ds\right)dy,$$

which gives

$$\frac{1}{b-x}\int_{x}^{b}f(y)\,dy=\sum_{k=0}^{n}\frac{(-1)^{k}}{(k+1)!}f^{(k)}(b)(b-x)^{k}+\frac{(-1)^{n+1}}{n!}\frac{1}{b-x}\int_{x}^{b}(b-y)^{n+1}\left(\int_{0}^{1}f^{(n+1)}((1-s)y+sb)s^{n}ds\right)dy. \tag{2.7}$$

Now, if we make the change of variable $y=(1-u)a+ux$, $u\in[0,1]$, then

$$\int_{a}^{x}(y-a)^{n+1}\left(\int_{0}^{1}f^{(n+1)}((1-s)a+sy)(1-s)^{n}ds\right)dy$$
$$=(x-a)^{n+2}\int_{0}^{1}u^{n+1}\left(\int_{0}^{1}f^{(n+1)}((1-s)a+s[(1-u)a+ux])(1-s)^{n}ds\right)du$$

and by (2.6) we get

$$\frac{1}{x-a}\int_a^x f(y)\,dy=\sum_{k=0}^{n}\frac{1}{(k+1)!}f^{(k)}(a)(x-a)^k \tag{2.8}$$

$$+\frac{1}{n!}(x-a)^{n+1}\int_0^1 u^{n+1}\left(\int_0^1 f^{(n+1)}((1-s)a+s[(1-u)a+ux])(1-s)^n\,ds\right)du$$

$$=\sum_{k=0}^{n}\frac{1}{(k+1)!}f^{(k)}(a)(x-a)^k$$

$$+\frac{1}{n!}(x-a)^{n+1}\int_0^1 u^{n+1}\left(\int_0^1 f^{(n+1)}(sa+(1-s)[(1-u)a+ux])s^n\,ds\right)du$$

for $x\in(a,b)$, where for the last equality we replaced s by $1-s$.

Also, if we make the change of variable $y=(1-v)x+vb$, $v\in[0,1]$, then

$$\int_x^b (b-y)^{n+1}\left(\int_0^1 f^{(n+1)}((1-s)y+sb)s^n\,ds\right)dy$$

$$=(b-x)^{n+2}\int_0^1 (1-v)^{n+1}\left(\int_0^1 f^{(n+1)}((1-s)[(1-v)x+vb]+sb)s^n\,ds\right)dv$$

and by changing again the variable $u=1-v$, $v\in[0,1]$, we have

$$\int_x^b (b-y)^{n+1}\left(\int_0^1 f^{(n+1)}((1-s)y+sb)s^n\,ds\right)dy$$

$$=(b-x)^{n+2}\int_0^1 u^{n+1}\left(\int_0^1 f^{(n+1)}((1-s)[ux+(1-u)b]+sb)s^n\,ds\right)du.$$

From (2.7) we get

$$\frac{1}{b-x}\int_x^b f(y)\,dy=\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)!}f^{(k)}(b)(b-x)^k \tag{2.9}$$

$$+\frac{(-1)^{n+1}}{n!}(b-x)^{n+1}\int_0^1 u^{n+1}\left(\int_0^1 f^{(n+1)}((1-s)[ux+(1-u)b]+sb)s^n\,ds\right)du$$

for $x\in(a,b)$.

Therefore, by (2.8) and (2.9) we get

$$D_{a+,b-}f(x)=\frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]$$

$$=\frac{1}{2}\left[\sum_{k=0}^{n}\frac{1}{(k+1)!}f^{(k)}(a)(x-a)^k+\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)!}f^{(k)}(b)(b-x)^k\right]$$

$$+\frac{1}{2n!}(x-a)^{n+1}\int_0^1 u^{n+1}\left(\int_0^1 f^{(n+1)}(sa+(1-s)[(1-u)a+ux])s^n\,ds\right)du$$

$$+\frac{(-1)^{n+1}}{2n!}(b-x)^{n+1}\int_0^1 u^{n+1}\left(\int_0^1 f^{(n+1)}((1-s)[ux+(1-u)b]+sb)s^n\,ds\right)du,$$

which proves the desired result (2.1). □

The case when $x=\frac{a+b}{2}$ is of interest.

Corollary 1. *With the assumption of Theorem 3 we have*

$$\frac{1}{b-a}\int_a^b f(t)\,dt=\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2^{k+1}}\right](b-a)^k \tag{2.10}$$
$$+\frac{1}{2^{n+2}n!}(b-a)^{n+1}\int_0^1\int_0^1 u^{n+1}s^n\left[f^{(n+1)}\left(sa+(1-s)\left[(1-u)a+u\frac{a+b}{2}\right]\right)dsdu\right.$$
$$\left.+(-1)^{n+1}f^{(n+1)}\left((1-s)\left[u\frac{a+b}{2}+(1-u)b\right]+sb\right)\right]dsdu.$$

The proof follows by Theorem 3 on taking $x=\frac{a+b}{2}$.

Remark 1. *For $n=0$, we get from (2.1) that*

$$D_{a+,b-}f(x)=\frac{f(a)+f(b)}{2} \tag{2.11}$$
$$+\frac{1}{2}(x-a)\int_0^1\int_0^1 uf'(sa+(1-s)[(1-u)a+ux])\,dsdu$$
$$-\frac{1}{2}(b-x)\int_0^1\int_0^1 uf'((1-s)[ux+(1-u)b]+sb)\,dsdu$$

for $x\in(a,b)$ and, in particular

$$\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{f(a)+f(b)}{2} \tag{2.12}$$
$$+\frac{1}{4}(b-a)\int_0^1\int_0^1 u\left[f'\left(sa+(1-s)\left[(1-u)a+u\frac{a+b}{2}\right]\right)\right.$$
$$\left.-f'\left((1-s)\left[u\frac{a+b}{2}+(1-u)b\right]+sb\right)\right]dsdu.$$

For $n=1$, we get from (2.1) that

$$D_{a+,b-}f(x)=\frac{f(a)+f(b)}{2}+\frac{f'(a)(x-a)-f'(b)(b-x)}{4} \tag{2.13}$$
$$+\frac{1}{2}(x-a)^2\int_0^1\int_0^1 u^2sf''(sa+(1-s)[(1-u)a+ux])\,dsdu$$
$$+\frac{1}{2}(b-x)^2\int_0^1\int_0^1 u^2sf''((1-s)[ux+(1-u)b]+sb)\,dsdu$$

for $x\in(a,b)$ and, in particular

$$\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{f(a)+f(b)}{2}-\frac{1}{8}(f'(b)-f'(a))(b-a) \tag{2.14}$$
$$+\frac{1}{8}(b-a)^2\int_0^1\int_0^1 u^2s\left[f''\left(sa+(1-s)\left[(1-u)a+u\frac{a+b}{2}\right]\right)\right.$$
$$\left.+f''\left((1-s)\left[u\frac{a+b}{2}+(1-u)b\right]+sb\right)\right]dsdu.$$

In [7] the first author obtained the following equality:

Lemma 1. *Assume that the function $f:(a,b)\to\mathbb{C}$ is Lebesgue integrable on (a,b) and $f(a+)$, $f(b-)$ exists and are finite. Then we have*

$$\int_a^b D_{a+,b-}f(x)\,dx=\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx. \tag{2.15}$$

Using this equality we can state the following corollary as well:

Corollary 2. *With the assumption of Theorem 3 we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx \\ =\sum_{k=0}^{n}\frac{1}{(k+1)!\,(k+1)}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right](b-a)^k \\ +\frac{1}{2n!}\frac{1}{b-a}\int_a^b (x-a)^{n+1} \\ \times\left(\int_0^1\int_0^1 s^n u^{n+1} f^{(n+1)}(sa+(1-s)[(1-u)a+ux])\,dsdu\right)dx \\ +\frac{(-1)^{n+1}}{2n!}\frac{1}{b-a}\int_a^b (b-x)^{n+1} \\ \times\left(\int_0^1\int_0^1 u^{n+1}s^n f^{(n+1)}((1-s)[ux+(1-u)b]+sb)\,dsdu\right)dx.\end{aligned} \tag{2.16}$$

Remark 2. *For $n=0$ we obtain*

$$\begin{aligned}\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx=\frac{f(a)+f(b)}{2} \\ +\frac{1}{2}\frac{1}{b-a}\int_a^b (x-a)\left(\int_0^1 u\left(\int_0^1 f'(sa+(1-s)[(1-u)a+ux])\,ds\right)du\right)dx \\ -\frac{1}{2}\frac{1}{b-a}\int_a^b (b-x)\left(\int_0^1 u\left(\int_0^1 f'((1-s)[ux+(1-u)b]+sb)\,ds\right)du\right)dx\end{aligned} \tag{2.17}$$

while for $n=1$, we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx \\ =\frac{f(a)+f(b)}{2}-\frac{f'(b)-f'(a)}{8}(b-a) \\ +\frac{1}{2}\frac{1}{b-a}\int_a^b (x-a)^2\left(\int_0^1\int_0^1 u^2 s f''(sa+(1-s)[(1-u)a+ux])\,dsdu\right)dx \\ +\frac{1}{2}\frac{1}{b-a}\int_a^b (b-x)^2\left(\int_0^1\int_0^1 u^2 s f''((1-s)[ux+(1-u)b]+sb)\,dsdu\right)dx.\end{aligned} \tag{2.18}$$

3. Some Trapezoid Type Inequalities

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}, \tag{3.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality.*

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [9], the recent survey paper [6], the research papers [1]-[2], [12]-[23] and the references therein.

The following result provides an inequality related to the second Hermite-Hadamard inequality in (3.1).

Theorem 4. *Let $I\subset\mathbb{R}$ be an interval, $[a,b]\subset I$ and $f:I\longrightarrow\mathbb{C}$ is such that the $2m+2$-derivative $f^{(2m+2)}$ is nonnegative on $[a,b]$, where $m\ge 0$, then for any $x\in(a,b)$ we have the trapezoid type inequality*

$$D_{a+,b-}f(x) \ge \sum_{k=0}^{2m+1}\frac{1}{(k+1)!}\left[\frac{f^{(k)}(a)(x-a)^k+(-1)^k f^{(k)}(b)(b-x)^k}{2}\right]. \tag{3.2}$$

In particular, we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \ge \sum_{k=0}^{2m+1}\frac{1}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2^{k+1}}\right](b-a)^k. \tag{3.3}$$

Proof. By the representation (2.1) we have

$$\begin{aligned}
D_{a+,b-}f(x) &= \sum_{k=0}^{2m+1}\frac{1}{(k+1)!}\left[\frac{f^{(k)}(a)(x-a)^k+(-1)^k f^{(k)}(b)(b-x)^k}{2}\right]\\
&\quad+\frac{1}{2(2m+1)!}(x-a)^{2m+2}\\
&\quad\times\int_0^1\int_0^1 u^{2m+2}s^{2m+1}f^{(2m+2)}(sa+(1-s)[(1-u)a+ux])\,dsdu\\
&\quad+\frac{1}{2(2m+1)!}(b-x)^{2m+2}\\
&\quad\times\int_0^1\int_0^1 u^{2m+2}s^{2m+1}f^{(2m+2)}((1-s)[ux+(1-u)b]+sb)\,dsdu\\
&\ge \sum_{k=0}^{2m+1}\frac{1}{(k+1)!}\left[\frac{f^{(k)}(a)(x-a)^k+(-1)^k f^{(k)}(b)(b-x)^k}{2}\right]
\end{aligned}$$

since the last two integrals are nonnegative due to the fact that $f^{(2m+2)}$ is nonnegative on $[a,b]$. □

Remark 3. *For $m=0$ we obtain from Theorem 4 that*

$$D_{a+,b-}f(x) \ge \frac{f(a)+f(b)}{2}+\frac{f'(a)(x-a)-f'(b)(b-x)}{4} \tag{3.4}$$

for any $x \in (a,b)$ *and, see also* [5] *for a slightly more general version,*

$$\frac{1}{8}(b-a)\left(f'(b)-f'(a)\right) \geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \geq 0, \tag{3.5}$$

where f *is differentiable and convex on* $[a,b]$.

Corollary 3. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx \\ \geq \sum_{k=0}^{2m+1} \frac{1}{(k+1)!\,(k+1)}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right](b-a)^k.\end{aligned} \tag{3.6}$$

Remark 4. *If the function* f *is differentiable and convex on* $[a,b]$, *then for* $m=0$ *in (3.6) we get*

$$\begin{aligned}&\frac{1}{8}(b-a)\left(f'(b)-f'(a)\right) \\ &\geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx \geq 0.\end{aligned} \tag{3.7}$$

We use the ∞-norm of an essentially bounded function f on the interval $[c,d]$ defined by

$$\|f\|_{[c,d],\infty} := \operatorname*{essup}_{t\in[c,d]} |f(t)| < \infty,\ f \in L_\infty[c,d].$$

Theorem 5. *Let* $I \subset \mathbb{R}$ *be an interval,* $[a,b] \subset I$ *and* $f : I \longrightarrow \mathbb{C}$ *is such that the* n*-derivative* $f^{(n)}$ *is absolutely continuous on* $[a,b]$ *and* $f^{(n+1)} \in L_\infty[a,b]$. *Then for any* $x \in (a,b)$ *we have the inequality*

$$\begin{aligned}&\left| D_{a+,b-}f(x) - \sum_{k=0}^{n} \frac{1}{(k+1)!}\left[\frac{f^{(k)}(a)(x-a)^k + (-1)^k f^{(k)}(b)(b-x)^k}{2}\right]\right| \\ &\leq \frac{1}{2(n+2)!}\left[(x-a)^{n+1}\left\|f^{(n+1)}\right\|_{[a,x],\infty} + (b-x)^{n+1}\left\|f^{(n+1)}\right\|_{[x,b],\infty}\right] \\ &\leq \frac{1}{2(n+2)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]\left\|f^{(n+1)}\right\|_{[a,b],\infty}.\end{aligned} \tag{3.8}$$

In particular,

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \sum_{k=0}^{n}\frac{1}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2^{k+1}}\right](b-a)^k\right| \\ &\leq \frac{1}{2^{n+2}(n+2)!}\left[\left\|f^{(n+1)}\right\|_{[a,\frac{a+b}{2}],\infty} + \left\|f^{(n+1)}\right\|_{[\frac{a+b}{2},b],\infty}\right](b-a)^{n+1} \\ &\leq \frac{1}{2^{n+1}(n+2)!}\left\|f^{(n+1)}\right\|_{[a,b],\infty}(b-a)^{n+1}.\end{aligned} \tag{3.9}$$

Proof. By taking the modulus in the equality (2.1) we get

$$\left|D_{a+,b-}f(x)-\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[\frac{f^{(k)}(a)(x-a)^{k}+(-1)^{k}f^{(k)}(b)(b-x)^{k}}{2}\right]\right| \tag{3.10}$$
$$\leq \frac{1}{2n!}(x-a)^{n+1}\int_{0}^{1}\int_{0}^{1}u^{n+1}s^{n}\left|f^{(n+1)}(sa+(1-s)[(1-u)a+ux])\right|dsdu$$
$$+\frac{1}{2n!}(b-x)^{n+1}\int_{0}^{1}\int_{0}^{1}u^{n+1}s^{n}\left|f^{(n+1)}((1-s)[ux+(1-u)b]+sb)\right|dsdu$$
$$=: B(x,n).$$

Observe that $(1-u)a+ux\in[a,x]$ for any $u\in[0,1]$ and $sa+(1-s)[(1-u)a+ux]\in[a,x]$ for any $u,\ s\in[0,1]$. Therefore

$$\sup_{(s,u)\in[0,1]^{2}}\left|f^{(n+1)}(sa+(1-s)[(1-u)a+ux])\right|\leq\left\|f^{(n+1)}\right\|_{[a,x],\infty}$$

and

$$\int_{0}^{1}\int_{0}^{1}u^{n+1}s^{n}\left|f^{(n+1)}(sa+(1-s)[(1-u)a+ux])\right|dsdu$$
$$\leq\left\|f^{(n+1)}\right\|_{[a,x],\infty}\int_{0}^{1}\int_{0}^{1}u^{n+1}s^{n}dsdu$$
$$=\frac{1}{(n+1)(n+2)}\left\|f^{(n+1)}\right\|_{[a,x],\infty}.$$

Similarly, we have

$$\int_{0}^{1}\int_{0}^{1}u^{n+1}s^{n}\left|f^{(n+1)}((1-s)[ux+(1-u)b]+sb)\right|dsdu$$
$$\leq\frac{1}{(n+1)(n+2)}\left\|f^{(n+1)}\right\|_{[x,b],\infty}.$$

Therefore

$$B(x,n)$$
$$\leq\frac{1}{2n!}(x-a)^{n+1}\frac{1}{(n+1)(n+2)}\left\|f^{(n+1)}\right\|_{[a,x],\infty}$$
$$+\frac{1}{2n!}(b-x)^{n+1}\frac{1}{(n+1)(n+2)}\left\|f^{(n+1)}\right\|_{[x,b],\infty}$$
$$=\frac{1}{2(n+2)!}\left[(x-a)^{n+1}\left\|f^{(n+1)}\right\|_{[a,x],\infty}+(b-x)^{n+1}\left\|f^{(n+1)}\right\|_{[x,b],\infty}\right]$$
$$\leq\frac{1}{2(n+2)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]\max\left\{\left\|f^{(n+1)}\right\|_{[a,x],\infty},\left\|f^{(n+1)}\right\|_{[x,b],\infty}\right\}$$
$$=\frac{1}{2(n+2)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]\left\|f^{(n+1)}\right\|_{[a,b],\infty}$$

and the inequality (3.8) is thus proved. □

Remark 5. *If we take in (3.8) $n=0$, then we get*

$$\left|D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}\right| \leq \frac{1}{4}\left[(x-a)\|f'\|_{[a,x],\infty}+(b-x)\|f'\|_{[x,b],\infty}\right] \leq \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty}. \tag{3.11}$$

for any $x\in(a,b)$, and in particular

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right| \leq \frac{1}{8}\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right](b-a) \leq \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty}. \tag{3.12}$$

If we take in (3.8) $n=1$, then we get

$$\begin{aligned}&\left|D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}-\frac{1}{4}\left[f'(a)(x-a)-f'(b)(b-x)\right]\right| \\ &\leq \frac{1}{12}\left[(x-a)^2\|f''\|_{[a,x],\infty}+(b-x)^2\|f''\|_{[x,b],\infty}\right] \\ &\leq \frac{1}{6}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty}\end{aligned} \tag{3.13}$$

for any $x\in(a,b)$, and in particular

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}-\frac{1}{8}\left[f'(a)-f'(b)\right](b-a)\right| \\ &\leq \frac{1}{48}\left[\|f''\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|f''\|_{\left[\frac{a+b}{2},b\right],\infty}\right](b-a)^2 \leq \frac{1}{24}(b-a)^2\|f''\|_{[a,b],\infty}.\end{aligned} \tag{3.14}$$

Corollary 4. *With the assumptions of Theorem 5 we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx\right. \\ &\left.\quad-\sum_{k=0}^{n}\frac{1}{(k+1)!\,(k+1)}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right](b-a)^k\right| \\ &\leq \frac{1}{2(n+2)!}\left[\frac{1}{b-a}\int_a^b (x-a)^{n+1}\left\|f^{(n+1)}\right\|_{[a,x],\infty}dx\right. \\ &\quad\left.+\frac{1}{b-a}\int_a^b (b-x)^{n+1}\left\|f^{(n+1)}\right\|_{[x,b],\infty}dx\right] \\ &\leq \frac{1}{(n+2)!\,(n+2)}\left\|f^{(n+1)}\right\|_{[a,b],\infty}(b-a)^{n+1}.\end{aligned} \tag{3.15}$$

If we take $n=0$ in (3.15), then we get

$$\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \tag{3.16}$$
$$\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b (x-a)\|f'\|_{[a,x],\infty}\,dx + \frac{1}{b-a}\int_a^b (b-x)\|f'\|_{[x,b],\infty}\,dx\right]$$
$$\leq \frac{1}{4}\|f'\|_{[a,b],\infty}(b-a)$$

while for $n=1$ we get

$$\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx - \frac{f(a)+f(b)}{2} - \frac{1}{8}\left[f'(a)-f'(b)\right](b-a)\right| \tag{3.17}$$
$$\leq \frac{1}{12}\left[\frac{1}{b-a}\int_a^b (x-a)^2\|f''\|_{[a,x],\infty}\,dx + \frac{1}{b-a}\int_a^b (b-x)^2\|f''\|_{[x,b],\infty}\,dx\right]$$
$$\leq \frac{1}{18}\|f''\|_{[a,b],\infty}(b-a)^2.$$

4. Some Other Inequalities

To prove some more inequalities involving the operator $D_{a+,b-}$ we need the following lemmas:

Lemma 2 ([13, p. 21])**.** *Suppose g is a real-valued function on $[a,b]$ that satisfies $g'''(x)\geq 0$ on $[a,b]$. If g''' is continuous on $[a,b]$ then*

$$\frac{1}{b-a}\int_a^b g(x)\,dx \leq g\left(\frac{a+b}{2}\right) + \frac{1}{12}(b-a)\left[g'(b)-g'\left(\frac{a+b}{2}\right)\right] \tag{4.1}$$

and

$$\frac{1}{b-a}\int_a^b g(x)\,dx \geq g\left(\frac{a+b}{2}\right) + \frac{1}{12}(b-a)\left[g'\left(\frac{a+b}{2}\right)-g'(a)\right]. \tag{4.2}$$

If $g'''(x)\leq 0$ on $[a,b]$ instead, then (4.1) and (4.2) hold with the inequality signs reversed.

and

Lemma 3 (Theorem 1.4 of [11])**.** *Let φ be continuous on $[a,b]$, twice differentiable on (a,b). Suppose w and p are continuous on $[a,b]$ and $p(x)\geq 0$ with $\int_a^b p(x)\,dx > 0$.*

(a) *If $m=\inf_{x\in(a,b)}\varphi''(x)$ exists, then*

$$G := \frac{\int_a^b p(x)\varphi(w(x))\,dx}{\int_a^b p(x)\,dx} - \varphi\left(\frac{\int_a^b p(x)w(x)\,dx}{\int_a^b p(x)\,dx}\right) \geq \frac{1}{2}mV, \tag{4.3}$$

where

$$V := \frac{\int_a^b p(x)w^2(x)\,dx}{\int_a^b p(x)\,dx} - \left(\frac{\int_a^b p(x)w(x)\,dx}{\int_a^b p(x)\,dx}\right)^2$$

is the variance of $w(X)$.

(b) *If* $M=\sup_{x\in(a,b)}\varphi''(x)$ *exists, then*

$$G\leq\frac{1}{2}MV. \tag{4.4}$$

The inequalities (4.3) and (4.4) are particular cases of the Jensen's type inequalities for positive linear functionals obtained in 2002, [3].

We have the following result:

Theorem 6. *Suppose* f'' *exists on* $[a,b]$ *and* $m=\inf_{x\in(a,b)}f''(x)$, $M=\sup_{x\in(a,b)}\varphi''(x)$ *exist, then*

$$\frac{1}{18}m(b-a)^2\leq\frac{1}{b-a}\int_a^b D_{a+,b-}f(x)\,dx-f\left(\frac{a+b}{2}\right)\leq\frac{1}{18}M(b-a)^2. \tag{4.5}$$

Proof. A change of variable gives

$$\int_a^b\ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)dx=\frac{b-a}{2}\int_0^1(-\ln u-\ln(1-u))\,du=b-a.$$

We also have

$$\begin{aligned}&\int_a^b f(x)\ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)dx\\&=(b-a)\int_0^1\frac{1}{2}(-\ln u-\ln(1-u))\,f(ua+(1-u)b)\,du.\end{aligned}$$

It is easy verified that $p(u):=\frac{1}{2}(-\ln u-\ln(1-u))$, $u\in[0,1]$ is a probability density function with mean

$$\int_0^1 up(u)\,du=\frac{1}{2}$$

and variance

$$V=\int_0^1u^2p(u)\,du-\frac{1}{4}=\frac{1}{9}.$$

The inequality (4.5) follows immediately by Lemma 3 upon letting $p(u)=\frac{1}{2}(-\ln u-\ln(1-u))$, $w(u)=u$ and $\varphi(u)=f(ua+(1-u)b)$. □

Next, we show how improved bounds for the integrals in Theorem 6 can often be found if we have functions f satisfying $f'''(x)\geq0$ on $[a,b]$.

Theorem 7. *Suppose that* f'' *is continuous on* $[a,b]$. *Let*

$$M=\sup_{t\in[a,b]}f''(t),\ M_1=\sup_{t\in\left[\frac{a+b}{2},b\right]}f''(t),\ M_2=\sup_{t\in\left[a,\frac{a+b}{2}\right]}f''(t)$$

and

$$m=\inf_{t\in[a,b]}f''(t),\ m_1=\inf_{t\in\left[\frac{a+b}{2},b\right]}f''(t),\ m_2=\inf_{t\in\left[a,\frac{a+b}{2}\right]}f''(t),$$

then

$$\begin{aligned}(4.6)\quad & \frac{1}{192}(m_1+m_2)(b-a)^2+\frac{1}{72}m(b-a)^2 \\ & \leq \frac{1}{b-a}\int_a^b D_{a+,b-}f(x)\,dx-\frac{1}{2}\left[f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)\right] \\ & \leq \frac{1}{192}(M_1+M_2)(b-a)^2+\frac{1}{72}M(b-a)^2.\end{aligned}$$

Proof. We shall prove only the second inequality in (4.6). The proof of the first inequality is similar and is omitted.

Let $M_1(x)=\sup_{t\in[a,x]}f''(t)$, $M_2(x)=\sup_{t\in[x,b]}f''(t)$ for $x\in[a,b]$. By Lemma 3 we have

$$\begin{aligned}(4.7)\qquad \frac{1}{x-a}\int_a^x f(t)\,dt & \leq f\left(\frac{a+x}{2}\right)+\frac{1}{2}M_1(x)\frac{(x-a)^2}{12} \\ & \leq f\left(\frac{a+x}{2}\right)+\frac{1}{24}M(x-a)^2\end{aligned}$$

and

$$\begin{aligned}(4.8)\qquad \frac{1}{b-x}\int_x^b f(t)\,dt & \leq f\left(\frac{x+b}{2}\right)+\frac{1}{2}M_2(x)\frac{(b-x)^2}{12} \\ & \leq f\left(\frac{x+b}{2}\right)+\frac{1}{24}M(b-x)^2.\end{aligned}$$

So, addition of (4.7) and (4.8) gives

$$D_{a+,b-}f(x)\leq\frac{1}{2}\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)+\frac{1}{24}M\left[(x-a)^2+(b-x)^2\right]\right]$$

for $x\in(a,b)$.

Integration gives

$$\begin{aligned}(4.9)\qquad \int_a^b D_{a+,b-}f(x)\,dx & \leq\frac{1}{2}\int_a^b\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right]dx \\ & \quad+\frac{1}{48}M\int_a^b\left[(x-a)^2+(b-x)^2\right]dx.\end{aligned}$$

By Lemma 3 we also have

$$(4.10)\qquad \int_a^b f\left(\frac{a+x}{2}\right)dx\leq(b-a)\left[f\left(\frac{3a+b}{4}\right)+\frac{1}{96}M_1(b-a)^2\right]$$

and

$$(4.11)\qquad \int_a^b f\left(\frac{x+b}{2}\right)dx\leq(b-a)\left[f\left(\frac{a+3b}{4}\right)+\frac{1}{96}M_2(b-a)^2\right].$$

Thus, (4.9)-(4.11) give the upper bound in (4.6). □

The next theorem provides more bounds for the integral $\frac{1}{b-a}\int_a^b D_{a+,b-}f(x)\,dx$ in the case that f is 3-convex:

Theorem 8. *Suppose that f''' is continuous and nonnegative on $[a,b]$. Then*

$$\begin{aligned}(4.12)\quad & \frac{1}{72} f''(a)(b-a)^2 \\ & + \frac{1}{48}(b-a)\left[f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{a+b}{2}\right) + f'\left(\frac{3a+b}{4}\right) - f'(a)\right] \\ & \le \frac{1}{b-a}\int_a^b D_{a+,b-} f(x)\,dx - \frac{1}{2}\left[f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right)\right] \\ & \le \frac{1}{48}(b-a)\left[f'\left(\frac{a+b}{2}\right) - f'\left(\frac{3a+b}{4}\right) + f'(b) - f'\left(\frac{a+3b}{4}\right)\right] \\ & \qquad + \frac{1}{72} f''(b)(b-a)^2 .\end{aligned}$$

Proof. We shall prove only the inequality for the upper bound. The other inequality is very similar and is omitted.

In the proof of Theorem 7 it was shown that

$$\begin{aligned}(4.13)\quad & D_{a+,b-} f(x) \\ & \le \frac{1}{2}\left\{f\left(\frac{a+x}{2}\right) + f\left(\frac{x+b}{2}\right) + \frac{1}{24} M\left[(x-a)^2 + (b-x)^2\right]\right\}\end{aligned}$$

for $x \in (a,b)$.

Now, apply Lemma 2 to both

$$\int_a^b f\left(\frac{a+x}{2}\right) dx \text{ and } \int_a^b f\left(\frac{x+b}{2}\right) dx$$

to get

$$\begin{aligned}(4.14)\quad & \int_a^b f\left(\frac{a+x}{2}\right) dx \\ & \le (b-a) f\left(\frac{3a+b}{4}\right) + \frac{1}{24}(b-a)^2\left[f'\left(\frac{a+b}{2}\right) - f'\left(\frac{3a+b}{4}\right)\right]\end{aligned}$$

and

$$\begin{aligned}(4.15)\quad & \int_a^b f\left(\frac{x+b}{2}\right) dx \\ & \le (b-a) f\left(\frac{a+3b}{4}\right) + \frac{1}{24}(b-a)^2\left[f'(b) - f'\left(\frac{a+3b}{4}\right)\right].\end{aligned}$$

Integrating both sides of (4.13) and using (4.14)-(4.15) we obtain the desired result. □

Remark 6. *It is easily seen that Theorems 6, 7 and 8 provide bounds which are exact (zero error) in the case that f is a polynomial of degree 2 or less. Numerical experiments show that the bounds of Theorem 7 and 8 are better than the bounds of Theorem 6 in the cases where all three theorems are applicable. Of course, the bounds of Theorem 6 are more easily computed.*

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