HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR PERSPECTIVE FUNCTION

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Abstract. Let $f:(0,\infty)\to\mathbb{R}$ be a convex function on $(0,\infty)$. The associated two variables *perspective function* $P_f:(0,\infty)\times(0,\infty)\to\mathbb{R}$ is defined by

$$P_f(x,y):=xf\left(\frac{y}{x}\right).$$

In this paper we establish some basic and double integral inequalities for the perspective function P_f defined above. Some double integral inequalities in the case of rectangles, squares and circular sectors are also given.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right)\le\int_a^b f(x)dx\le(b-a)\frac{f(a)+f(b)}{2},\quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [7]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [7]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality. For a monograph devoted to this inequality see [6]. Related results can be also found in [4].

In 1990, [3] the author established the following refinement of Hermite-Hadamard inequality for double and triple integrals for the convex function $f:[a,b]\to\mathbb{R}$

$$\begin{aligned} f\left(\frac{a+b}{2}\right)&\le\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \\ &\le\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f\left((1-t)x+ty\right)dtdxdy\le\frac{1}{b-a}\int_a^b f(x)dx. \end{aligned} \tag{1.2}$$

More recently, [5] we obtained a different double integral inequality of Hermite-Hadamard type for the convex function $f:[a,b]\to\mathbb{R}$,

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{(d-c)^2}\int_c^d\int_c^d f\left(\frac{\alpha a+\beta b}{\alpha+\beta}\right)d\beta d\alpha\le\frac{f(a)+f(b)}{2} \tag{1.3}$$

where $0<c<d$.

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$$P_f(x,y):=xf\left(\frac{y}{x}\right). \tag{1.4}$$

In this paper we establish some basic and double integral inequalities for the perspective function P_f defined above. Some integral inequalities in the case of rectangles, squares and circular sectors are also given.

2. General Results

We start with the following fundamental fact.

Lemma 1. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function on $(0,\infty)$. Then the perspective function $P_f:(0,\infty)\times(0,\infty)\to\mathbb{R}$ defined by (1.4) is convex on $(0,\infty)\times(0,\infty)$.*

Proof. Let $(x_1,y_1),(x_2,y_2)\in(0,\infty)\times(0,\infty)$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$, then

$$\begin{aligned}
P_f(\alpha(x_1,y_1)+\beta(x_2,y_2)) &= P_f(\alpha x_1+\beta x_2,\alpha y_1+y_2\beta)\\
&=(\alpha x_1+\beta x_2)\,f\left(\frac{\alpha y_1+y_2\beta}{\alpha x_1+\beta x}\right)\\
&=(\alpha x_1+\beta x_2)\,f\left(\frac{\alpha x_1\frac{y_1}{x_1}+\beta x_2\frac{y_2}{x_2}}{\alpha x_1+\beta x_2}\right)\\
&\leq(\alpha x_1+\beta x_2)\left[\frac{\alpha x_1}{\alpha x_1+\beta x_2}f\left(\frac{y_1}{x_1}\right)+\frac{\beta x_2}{\alpha x_1+\beta x_2}f\left(\frac{y_2}{x_2}\right)\right]\\
&=\alpha x_1 f\left(\frac{y_1}{x_1}\right)+\beta x_2 f\left(\frac{y_2}{x_2}\right)=\alpha P_f(x_1,y_1)+\beta P_f(x_2,y_2),
\end{aligned}$$

which proves the joint convexity. $\square$

We have the following basic inequality for two values of the perspective function:

Theorem 1. *Let $f:(0,\infty)\to\mathbb{R}$ be a differentiable convex function on $(0,\infty)$. Then for all $(x,y),(u,v)\in(0,\infty)\times(0,\infty)$ we have the double inequality*

$$\begin{aligned}
f\left(\frac{y}{x}\right)(x-u)+f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right)&\geq P_f(x,y)-P_f(u,v)\\
&\geq f\left(\frac{v}{u}\right)(x-u)+f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right).
\end{aligned}\tag{2.1}$$

The inequality (2.1) is equivalent to the following two inequalities

$$P_f(x,y)\geq xf\left(\frac{v}{u}\right)+f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right) \tag{2.2}$$

and

$$P_f(u,v)\geq uf\left(\frac{y}{x}\right)-f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right) \tag{2.3}$$

for all $(x,y),(u,v)\in(0,\infty)\times(0,\infty)$.

The inequality (2.1) is also equivalent to the double inequality

$$f'\left(\frac{y}{x}\right)\left(y-\frac{xv}{u}\right)+xf\left(\frac{v}{u}\right)\geq P_f(x,y)\geq xf\left(\frac{v}{u}\right)+f'\left(\frac{v}{u}\right)\left(y-\frac{xv}{u}\right) \tag{2.4}$$

for all $(x,y),(u,v)\in(0,\infty)\times(0,\infty)$.

Proof. Observe that the following partial derivatives exist and for all $(x,y) \in (0,\infty)\times(0,\infty)$

$$\begin{aligned}\frac{\partial P_f(x,y)}{\partial x} &= \frac{d}{dx}\left(xf\left(\frac{y}{x}\right)\right) = f\left(\frac{y}{x}\right) + x\frac{d}{dx}\left(f\left(\frac{y}{x}\right)\right)\\ &= f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\frac{d}{dx}\left(\frac{y}{x}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial P_f(x,y)}{\partial y} &= \frac{d}{dy}\left(xf\left(\frac{y}{x}\right)\right) = x\frac{d}{dy}\left(f\left(\frac{y}{x}\right)\right)\\ &= xf'\left(\frac{y}{x}\right)\frac{d}{dy}\left(\frac{y}{x}\right) = f'\left(\frac{y}{x}\right).\end{aligned}$$

Also, for all $(u,v) \in (0,\infty)\times(0,\infty)$ we have

$$\frac{\partial P_f(u,v)}{\partial x} = f\left(\frac{v}{u}\right) - \frac{v}{u}f'\left(\frac{v}{u}\right)$$

and

$$\frac{\partial P_f(u,v)}{\partial y} = f'\left(\frac{v}{u}\right).$$

Since P_f is a convex function on $(0,\infty)\times(0,\infty)$, then for all $(x,y), (u,v) \in (0,\infty)\times(0,\infty)$ we have the gradient inequality

$$\begin{aligned}&\frac{\partial P_f(x,y)}{\partial x}(x-u) + \frac{\partial P_f(x,y)}{\partial y}(y-v)\\ &\geq P_f(x,y) - P_f(u,v)\\ &\geq \frac{\partial P_f(u,v)}{\partial x}(x-u) + \frac{\partial P_f(u,v)}{\partial y}(y-v),\end{aligned}$$

namely, by the calculations above,

$$\begin{aligned}&\left[f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)\right](x-u) + f'\left(\frac{y}{x}\right)(y-v)\\ &\geq P_f(x,y) - P_f(u,v)\\ &\geq \left[f\left(\frac{v}{u}\right) - \frac{v}{u}f'\left(\frac{v}{u}\right)\right](x-u) + f'\left(\frac{v}{u}\right)(y-v).\end{aligned} \tag{2.5}$$

Since

$$\begin{aligned}&\left[f\left(\frac{v}{u}\right) - \frac{v}{u}f'\left(\frac{v}{u}\right)\right](x-u) + f'\left(\frac{v}{u}\right)(y-v)\\ &= f\left(\frac{v}{u}\right)(x-u) + f'\left(\frac{v}{u}\right)(y-v) - \frac{v}{u}f'\left(\frac{v}{u}\right)(x-u)\\ &= f\left(\frac{v}{u}\right)(x-u) + f'\left(\frac{v}{u}\right)\left[y-v-\frac{v}{u}(x-u)\right]\\ &= f\left(\frac{v}{u}\right)(x-u) + f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right)\end{aligned}$$

and

$$\begin{aligned}
&\left[f\left(\frac{y}{x}\right)-\frac{y}{x}f'\left(\frac{y}{x}\right)\right](x-u)+f'\left(\frac{y}{x}\right)(y-v)\\
&=f\left(\frac{y}{x}\right)(x-u)-\frac{y}{x}f'\left(\frac{y}{x}\right)(x-u)+f'\left(\frac{y}{x}\right)(y-v)\\
&=f\left(\frac{y}{x}\right)(x-u)+f'\left(\frac{y}{x}\right)(y-v)-\frac{y}{x}f'\left(\frac{y}{x}\right)(x-u)\\
&=f\left(\frac{y}{x}\right)(x-u)+f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right),
\end{aligned}$$

hence by (2.5) we get (2.1).

Now, observe that

$$\begin{aligned}
&f\left(\frac{v}{u}\right)(x-u)+f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right)\\
&=f\left(\frac{v}{u}\right)x-f\left(\frac{v}{u}\right)u+f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right)\\
&=f\left(\frac{v}{u}\right)x-P_f(u,v)+f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right)
\end{aligned}$$

and by the second inequality in (2.1) we get

$$P_f(x,y)-P_f(u,v)\geq f\left(\frac{v}{u}\right)x-P_f(u,v)+f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right)$$

namely (2.2).

Also,

$$\begin{aligned}
&f\left(\frac{y}{x}\right)(x-u)+f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right)\\
&=xf\left(\frac{y}{x}\right)-uf\left(\frac{y}{x}\right)+f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right)\\
&=P_f(x,y)-uf\left(\frac{y}{x}\right)+f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right)
\end{aligned}$$

and by the first inequality in (2.1) we have

$$P_f(x,y)-uf\left(\frac{y}{x}\right)+f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right)\geq P_f(x,y)-P_f(u,v),$$

namely (2.3).

The inequality (2.3) can also be written as

$$f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right)+uf\left(\frac{v}{u}\right)\geq uf\left(\frac{y}{x}\right).$$

By multiplying this inequality by x and dividing with u we get

$$f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{u}\right)+xf\left(\frac{v}{u}\right)\geq P_f(x,y),$$

which proves the last part of the theorem. □

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\left[f\left(\frac{y}{x}\right)-f'\left(\frac{y}{x}\right)\left(\frac{y+x}{x}\right)\right](x-y)\geq P_f(x,y)-P_f(y,x) \geq \left[f\left(\frac{x}{y}\right)-f'\left(\frac{x}{y}\right)\left(\frac{y+x}{y}\right)\right](x-y), \tag{2.6}$$

$$f'\left(\frac{y}{x}\right)(y-v)\geq P_f(x,y)-P_f(x,v)\geq f'\left(\frac{v}{x}\right)(y-v) \tag{2.7}$$

and

$$\left[f\left(\frac{y}{x}\right)-\frac{y}{x}f'\left(\frac{y}{x}\right)\right](x-u)\geq P_f(x,y)-P_f(u,y) \geq \left[f\left(\frac{y}{u}\right)-\frac{y}{u}f'\left(\frac{y}{u}\right)\right](x-u). \tag{2.8}$$

If f is normalized, namely $f(1)=0$, then

$$f\left(\frac{y}{x}\right)(x-u)+\frac{u}{x}f'\left(\frac{y}{x}\right)(y-x)\geq P_f(x,y)\geq f'(1)(y-x) \tag{2.9}$$

and

$$f'\left(\frac{y}{x}\right)(y-x)\geq P_f(x,y)\geq f'(1)(y-x). \tag{2.10}$$

Remark 1. *From the inequality (2.4) we have for $u=y$ and $v=x$, that*

$$f'\left(\frac{y}{x}\right)\left(\frac{y^2-x^2}{y}\right)+xf\left(\frac{x}{y}\right)\geq P_f(x,y)\geq xf\left(\frac{x}{y}\right)+f'\left(\frac{x}{y}\right)\left(\frac{y^2-x^2}{y}\right). \tag{2.11}$$

By taking $u=x$ in (2.4), we get

$$f'\left(\frac{y}{x}\right)(y-v)+xf\left(\frac{v}{x}\right)\geq P_f(x,y)\geq xf\left(\frac{v}{x}\right)+f'\left(\frac{v}{x}\right)(y-v). \tag{2.12}$$

Also, for $v=y$ in (2.4), we get

$$f'\left(\frac{y}{x}\right)\left(\frac{u-x}{u}\right)y+xf\left(\frac{y}{u}\right)\geq P_f(x,y)\geq xf\left(\frac{y}{u}\right)+f'\left(\frac{y}{u}\right)\left(\frac{u-x}{u}\right)y. \tag{2.13}$$

Consider the convex function $f(t)=-\ln t$, $t>0$. Then by the inequality (2.4) we get

$$\frac{x^2v}{yu}-x+x\ln\left(\frac{u}{v}\right)\geq x\ln\left(\frac{x}{y}\right)\geq x\ln\left(\frac{u}{v}\right)+x-\frac{yu}{v} \tag{2.14}$$

for all $(x,y),(u,v)\in(0,\infty)\times(0,\infty)$.

If we divide by $x>0$, then we get

$$\frac{xv}{yu}-1+\ln\left(\frac{u}{v}\right)\geq \ln\left(\frac{x}{y}\right)\geq \ln\left(\frac{u}{v}\right)+1-\frac{yu}{xv} \tag{2.15}$$

for all $(x,y),(u,v)\in(0,\infty)\times(0,\infty)$.

Also, consider the convex function $f(t)=t\ln t$, $t>0$. Then by the inequality (2.4) we have

$$\left(\ln\left(\frac{y}{x}\right)+1\right)\left(y-\frac{xv}{u}\right)+\frac{xv}{u}\ln\left(\frac{v}{u}\right)\geq y\ln\left(\frac{y}{x}\right) \geq \frac{xv}{u}\ln\left(\frac{v}{u}\right)+\left(\ln\left(\frac{v}{u}\right)+1\right)\left(y-\frac{xv}{u}\right)$$

namely, by division with $y > 0$,

$$\begin{aligned}
(2.16) \qquad & \left(\ln\left(\frac{y}{x}\right)+1\right)\left(1-\frac{xv}{yu}\right)+\frac{xv}{yu}\ln\left(\frac{v}{u}\right) \\
& \geq \ln\left(\frac{y}{x}\right) \\
& \geq \frac{xv}{yu}\ln\left(\frac{v}{u}\right)+\left(\ln\left(\frac{v}{u}\right)+1\right)\left(1-\frac{xv}{yu}\right)
\end{aligned}$$

for all $(x, y), (u, v) \in (0, \infty) \times (0, \infty)$.

3. Double Integral Inequalities

Consider G a closed and bounded subset of $(0, \infty) \times (0, \infty)$. Define

$$A_G := \int\int_G dxdy$$

the *area* of G and $(\overline{x_G}, \overline{y_G})$ the *centre of mass* for G, where

$$\overline{x_G} := \frac{1}{A_G}\int\int_G xdxdy, \ \overline{y_G} := \frac{1}{A_G}\int\int_G ydxdy.$$

Observe that if $f : (0, \infty) \to \mathbb{R}$ is convex and G a closed and bounded subset of $(0, \infty) \times (0, \infty)$, then the double integral

$$\int\int_G P_f(x, y)\, dxdy = \int\int_G xf\left(\frac{y}{x}\right) dxdy$$

exists.

We have the following main result:

Theorem 2. *If $f : (0, \infty) \to \mathbb{R}$ is differentiable convex on $(0, \infty)$ and G a closed and bounded subset of $(0, \infty) \times (0, \infty)$, then*

$$\begin{aligned}
(3.1) \qquad & \frac{1}{A_G}\left[\int\int_G f'\left(\frac{y}{x}\right) ydxdy - \frac{v}{u}\int\int_G f'\left(\frac{y}{x}\right) xdxdy\right] + \overline{x_G} f\left(\frac{v}{u}\right) \\
& \geq \frac{1}{A_G}\int\int_G P_f(x, y)\, dxdy \geq \overline{x_G} f\left(\frac{v}{u}\right) + \left(\overline{y_G} - \overline{x_G}\frac{v}{u}\right) f'\left(\frac{v}{u}\right)
\end{aligned}$$

for all $(u, v) \in G$.

Proof. By taking the integral in the inequality (2.4) over (x, y) on G, we get

$$\begin{aligned}
(3.2) \qquad & \int\int_G f'\left(\frac{y}{x}\right)\left(y - \frac{xv}{u}\right) dxdy + \int\int_G xf\left(\frac{v}{u}\right) dxdy \\
& \geq \int\int_G P_f(x, y)\, dxdy \\
& \geq \int\int_G xf\left(\frac{v}{u}\right) dxdy + \int\int_G f'\left(\frac{v}{u}\right)\left(y - \frac{xv}{u}\right).
\end{aligned}$$

Observe that

$$\begin{aligned}
& \int\int_G f'\left(\frac{y}{x}\right)\left(y - \frac{xv}{u}\right) dxdy \\
& = \int\int_G f'\left(\frac{y}{x}\right) ydxdy - \frac{v}{u}\int\int_G f'\left(\frac{y}{x}\right) xdxdy,
\end{aligned}$$

$$\int\int_G xf\left(\frac{v}{u}\right)dxdy = f\left(\frac{v}{u}\right)\int\int_G xdxdy = \overline{x_G}A_G f\left(\frac{v}{u}\right)$$

and

$$\int\int_G f'\left(\frac{v}{u}\right)\left(y-\frac{xv}{u}\right)dxdy = f'\left(\frac{v}{u}\right)\int\int_G\left(y-\frac{xv}{u}\right)dxdy$$
$$= A_G\left(\overline{y_G}-\overline{x_G}\frac{v}{u}\right)f'\left(\frac{v}{u}\right).$$

By replacing these values in (3.2) and dividing by the area A_G we obtain the desired result (3.1). □

Corollary 2. *With the assumptions of Theorem 2 we have*

$$\leq \frac{1}{A_G}\int\int_G P_f(x,y)\,dxdy - \overline{x_G}f\left(\frac{\overline{y_G}}{\overline{x_G}}\right) \tag{3.3}$$
$$\leq \frac{1}{A_G}\int\int_G f'\left(\frac{y}{x}\right)\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)dxdy.$$

The proof follows by taking $\frac{v}{u}=\frac{\overline{y_G}}{\overline{x_G}}$ in (3.1).

We define for $f:(0,\infty)\to\mathbb{R}$ a differentiable function on $(0,\infty)$ the quantity

$$\ell_G(f') := \frac{\int\int_G f'\left(\frac{y}{x}\right)ydxdy}{\int\int_G f'\left(\frac{y}{x}\right)xdxdy},$$

provided that the denominator is nonzero.

Corollary 3. *With the assumptions of Theorem 2 and if $\ell_G(f')>0$, then*

$$0\leq \overline{x_G}f(\ell_G(f'))-\frac{1}{A_G}\int\int_G P_f(x,y)\,dxdy \geq \left(\overline{x_G}\ell_G(f')-\overline{y_G}\right)f'(\ell_G(f')). \tag{3.4}$$

The proof follows by taking $\frac{v}{u}=\ell_G(f')$ in (3.1).

We observe that the condition f is strictly increasing on $(0,\infty)$ implies that $\ell_G(f')>0$.

In 2002, Cerone and Dragomir [2] obtained the following refinement of Grüss inequality for the general Lebesgue integral:

Lemma 2. *Let w, f, $g:\Omega\to\mathbb{R}$ be μ-measurable functions on Ω and $w\geq 0$ μ-almost everywhere on Ω. If there exists the constants δ, Δ such that*

$$-\infty<\delta\leq g\leq\Delta<\infty,$$

μ-almost everywhere on Ω, then

$$\left|\frac{\int_\Omega w(x)f(x)g(x)\,d\mu(x)}{\int_\Omega w(x)\,d\mu(x)} - \frac{\int_\Omega w(x)g(x)\,d\mu(x)}{\int_\Omega w(x)\,d\mu(x)}\frac{\int_\Omega w(x)f(x)\,d\mu(x)}{\int_\Omega w(x)\,d\mu(x)}\right| \tag{3.5}$$
$$\leq \frac{1}{2}\frac{\Delta-\delta}{\int_\Omega w(x)\,d\mu(x)}\int_\Omega\left|g(y)-\frac{1}{\int_\Omega w(x)\,d\mu(x)}\int_\Omega w(x)g(x)\,d\mu(x)\right|d\mu(y).$$

The constant $\frac{1}{2}$ is best possible.

We have:

Theorem 3. *If $f:(0,\infty)\to\mathbb{R}$ is differentiable convex on $(0,\infty)$ and G a closed and bounded subset of $(0,\infty)\times(0,\infty)$. Assume that there exists the constants γ, Γ such that*

$$-\infty<\gamma\leq f'\left(\frac{y}{x}\right)\leq\Gamma<\infty \tag{3.6}$$

for almost every $(x,y)\in G$, then

$$\begin{aligned}0&\leq\frac{1}{A_G}\int\int_G P_f(x,y)\,dxdy-\overline{x_G}f\left(\frac{\overline{y_G}}{\overline{x_G}}\right)\\&\leq\frac{1}{2}(\Gamma-\gamma)\frac{1}{A_G}\int\int_G\left|y-\frac{\overline{y_G}}{\overline{x_G}}x\right|dxdy\leq\frac{1}{2}(\Gamma-\gamma)I_G^{1/2},\end{aligned}\tag{3.7}$$

where

$$I_G:=\frac{1}{A_G}\int\int_G y^2dxdy-2\frac{\overline{y_G}}{\overline{x_G}}\frac{1}{A_G}\int\int_G xydxdy+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\frac{1}{A_G}\int\int_G x^2dxdy.$$

Proof. Observe that

$$\begin{aligned}\frac{1}{A_G}\int\int_G\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)dxdy&=\frac{1}{A_G}\left(\int\int_G ydxdy-\frac{\overline{y_G}}{\overline{x_G}}\int\int_G xdxdy\right)\\&=\overline{y_G}-\frac{\overline{y_G}}{\overline{x_G}}\overline{x_G}=0.\end{aligned}$$

Then by the inequality (3.5) for functions defined on G we get

$$\begin{aligned}&\left|\frac{1}{A_G}\int\int_G f'\left(\frac{y}{x}\right)\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)dxdy\right|=\left|\frac{1}{A_G}\int\int_G f'\left(\frac{y}{x}\right)\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)dxdy\right.\\&\qquad\left.-\frac{1}{A_G}\int\int_G f'\left(\frac{y}{x}\right)dxdy\frac{1}{A_G}\int\int_G\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)dxdy\right|\\&\leq\frac{1}{2}(\Gamma-\gamma)\frac{1}{A_G}\int\int_G\left|y-\frac{\overline{y_G}}{\overline{x_G}}x-\frac{1}{A_G}\int\int_G\left(u-\frac{\overline{y_G}}{\overline{x_G}}v\right)dudv\right|dxdy\\&\leq\frac{1}{2}(\Gamma-\gamma)\frac{1}{A_G}\int\int_G\left|y-\frac{\overline{y_G}}{\overline{x_G}}x\right|dxdy.\end{aligned}$$

By utilising (3.3) we get

$$\begin{aligned}0&\leq\frac{1}{A_G}\int\int_G P_f(x,y)\,dxdy-\overline{x_G}f\left(\frac{\overline{y_G}}{\overline{x_G}}\right)\\&\leq\frac{1}{2}(\Gamma-\gamma)\frac{1}{A_G}\int\int_G\left|y-\frac{\overline{y_G}}{\overline{x_G}}x\right|dxdy,\end{aligned}$$

which proves the second inequality in (3.7).

Using Cauchy-Schwarz inequality for the double integral, we have

$$\frac{1}{A_G}\int\int_G\left|y-\frac{\overline{y_G}}{\overline{x_G}}x\right|dxdy\leq\left(\frac{1}{A_G}\int\int_G\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)^2dxdy\right)^{1/2}. \tag{3.8}$$

Since

$$\begin{aligned}\int\int_G\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)^2dxdy&=\int\int_G\left(y^2-2\frac{\overline{y_G}}{\overline{x_G}}xy+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2x^2\right)dxdy\\&=\int\int_G y^2dxdy-2\frac{\overline{y_G}}{\overline{x_G}}\int\int_G xydxdy+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\int\int_G x^2dxdy,\end{aligned}$$

hence by (3.8) we get the last part of (3.7). □

Corollary 4. *With the assumptions of Theorem 3 and if there exists* $0<m<M<\infty$ *such that*

$$\frac{y}{x}\in[m,M]\ \text{for all}\ (x,y)\in G, \tag{3.9}$$

then

$$\begin{aligned}0&\le\frac{1}{A_G}\int\int_G P_f(x,y)\,dxdy-\overline{x_G}f\left(\frac{\overline{y_G}}{\overline{x_G}}\right)\\&\le\frac{1}{2}\left[f'(M)-f'(m)\right]\frac{1}{A_G}\int\int_G\left|y-\frac{\overline{y_G}}{\overline{x_G}}x\right|dxdy\le\frac{1}{2}\left[f'(M)-f'(m)\right]I_G^{1/2}.\end{aligned}\tag{3.10}$$

Proof. Since f' is increasing, then by (3.9) we have $f'(m)\le f'\left(\frac{y}{x}\right)\le f'(M)$, and by (3.7) we get the desired result. □

We have:

Theorem 4. *If* $f:(0,\infty)\to\mathbb{R}$ *is differentiable convex on* $(0,\infty)$ *and* G *a closed and bounded subset of* $(0,\infty)\times(0,\infty)$*. Assume that there exists the constants* γ, Γ *such that*

$$\left|f'\left(\frac{y}{x}\right)-f'\left(\frac{u}{v}\right)\right|\le\Lambda\left|\frac{y}{x}-\frac{u}{v}\right|<\infty \tag{3.11}$$

for almost every $(x,y)\in G$*, then*

$$0\le\frac{1}{A_G}\int\int_G P_f(x,y)\,dxdy-\overline{x_G}f\left(\frac{\overline{y_G}}{\overline{x_G}}\right)\le\Lambda J_G, \tag{3.12}$$

where

$$J_G:=\frac{1}{A_G}\int\int_G\frac{y^2}{x}dxdy-\frac{\left(\overline{y_G}\right)^2}{\overline{x_G}}.$$

Proof. Observe that

$$\begin{aligned}&\frac{1}{A_G}\int\int_G f'\left(\frac{y}{x}\right)\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)dxdy\\&\qquad=\frac{1}{A_G}\int\int_G\left[f'\left(\frac{y}{x}\right)-f'\left(\frac{\overline{y_G}}{\overline{x_G}}\right)\right]\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)dxdy.\end{aligned}$$

Therefore

$$\begin{aligned}(3.13)\quad & \frac{1}{A_G}\int\int_G f'\left(\frac{y}{x}\right)\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)dxdy \\ & \le \frac{1}{A_G}\int\int_G \left|\left[f'\left(\frac{y}{x}\right)-f'\left(\frac{\overline{y_G}}{\overline{x_G}}\right)\right]\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)\right|dxdy \\ & \le \frac{1}{A_G}\Lambda\int\int_G \left|\left(\frac{y}{x}-\frac{\overline{y_G}}{\overline{x_G}}\right)\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)\right|dxdy \\ & = \frac{1}{A_G}\Lambda\int\int_G \left|\left(\frac{y}{x}-\frac{\overline{y_G}}{\overline{x_G}}\right)\left(\frac{y}{x}-\frac{\overline{y_G}}{\overline{x_G}}\right)\right|xdxdy \\ & = \frac{1}{A_G}\Lambda\int\int_G \left(\frac{y}{x}-\frac{\overline{y_G}}{\overline{x_G}}\right)^2 xdxdy \\ & = \frac{1}{A_G}\Lambda\int\int_G \left[\frac{y^2}{x^2}-2\frac{\overline{y_G}}{\overline{x_G}}\frac{y}{x}+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\right]xdxdy.\end{aligned}$$

Since

$$\begin{aligned}&\int\int_G \left[\frac{y^2}{x^2}-2\frac{\overline{y_G}}{\overline{x_G}}\frac{y}{x}+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\right]xdxdy \\ &= \int\int_G \frac{y^2}{x}dxdy-2\frac{\overline{y_G}}{\overline{x_G}}\int\int_G \frac{y}{x}xdxdy+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\int\int_G xdxdy \\ &= \int\int_G \frac{y^2}{x}dxdy-2\frac{\overline{y_G}}{\overline{x_G}}\int\int_G ydxdy+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\int\int_G xdxdy \\ &= \int\int_G \frac{y^2}{x}dxdy-2A_G\frac{\overline{y_G}}{\overline{x_G}}\overline{y_G}+A_G\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\overline{x_G}=\int\int_G \frac{y^2}{x}dxdy-A_G\frac{(\overline{y_G})^2}{\overline{x_G}},\end{aligned}$$

hence

$$\frac{1}{A_G}\int\int_G \left[\frac{y^2}{x^2}-2\frac{\overline{y_G}}{\overline{x_G}}\frac{y}{x}+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\right]xdxdy=\frac{1}{A_G}\int\int_G \frac{y^2}{x}dxdy-\frac{(\overline{y_G})^2}{\overline{x_G}}$$

and by (3.13) we get (3.12). □

Corollary 5. *If $f:(0,\infty)\to\mathbb{R}$ is twice differentiable convex on $(0,\infty)$ and if there exists $0<m<M<\infty$ such that the condition (3.9) holds, then we have*

$$(3.14)\qquad 0\le\frac{1}{A_G}\int\int_G P_f(x,y)\,dxdy-\overline{x_G}f\left(\frac{\overline{y_G}}{\overline{x_G}}\right)\le\|f''\|_{[m,M],\infty}J_G,$$

where

$$\|f''\|_{[m,M],\infty}:=\sup_{t\in[m,M]}|f'(t)|<\infty.$$

4. Examples for Functions Defined on Rectangles

If $G=[a,b]\times[c,d]$ is a rectangle from $(0,\infty)\times(0,\infty)$, then

$$\int_a^b\int_c^d P_f(x,y)\,dxdy=\int_a^b x\left(\int_c^d f\left(\frac{y}{x}\right)dy\right)dx=\int_a^b x^2\left(\int_{\frac{c}{x}}^{\frac{d}{x}} f(u)\,du\right)dx,$$

and

$$A_G = (b-a)(d-c), \ \overline{x_G} = \frac{a+b}{2} \text{ and } \overline{y_G} = \frac{c+d}{2}.$$

If F is an antiderivative for f, namely $F'(x) = f(x)$, then integrating by parts we have the following identity that can be used in applications to calculate $\int_a^b \int_c^d P_f(x,y)\,dxdy$

$$\begin{aligned}
\int_a^b x^2 \left(\int_{\frac{c}{x}}^{\frac{d}{x}} f(u)\,du \right) dx &= \int_a^b x^2 \left(F\left(\frac{d}{x}\right) - F\left(\frac{c}{x}\right) \right) dx \\
&= \frac{1}{3} \int_a^b \left(F\left(\frac{d}{x}\right) - F\left(\frac{c}{x}\right) \right) d\left(x^3\right) \\
&= \frac{1}{3} \left[\left(F\left(\frac{d}{b}\right) - F\left(\frac{c}{b}\right) \right) b^3 - \left(F\left(\frac{d}{a}\right) - F\left(\frac{c}{a}\right) \right) a^3 \right] \\
&\quad - \frac{1}{3} \int_a^b x^3 \left(-F'\left(\frac{d}{x}\right)\left(\frac{d}{x^2}\right) + F'\left(\frac{c}{x}\right)\frac{c}{x^2} \right) dx \\
&= \frac{1}{3} \left[\left(F\left(\frac{d}{b}\right) - F\left(\frac{c}{b}\right) \right) b^3 - \left(F\left(\frac{d}{a}\right) - F\left(\frac{c}{a}\right) \right) a^3 \right] \\
&\quad - \frac{1}{3} \int_a^b x \left(-df\left(\frac{d}{x}\right) + cf\left(\frac{c}{x}\right) \right) dx \\
&= \frac{1}{3} \left[\left(F\left(\frac{d}{b}\right) - F\left(\frac{c}{b}\right) \right) b^3 - \left(F\left(\frac{d}{a}\right) - F\left(\frac{c}{a}\right) \right) a^3 \right] \\
&\quad + \frac{1}{3} d \int_a^b xf\left(\frac{d}{x}\right) dx - \frac{1}{3} c \int_a^b xf\left(\frac{c}{x}\right) dx.
\end{aligned}$$

We also have

$$\begin{aligned}
I_{[a,b]\times[c,d]} &= \frac{(b-a)\left(d^3-c^3\right)}{3(b-a)(d-c)} - 2\frac{c+d}{a+b}\frac{\left(b^2-a^2\right)\left(d^2-c^2\right)}{4(b-a)(d-c)} \\
&\quad + \left(\frac{c+d}{a+b}\right)^2 \frac{(d-c)\left(b^3-a^3\right)}{3(b-a)(d-c)} \\
&= \frac{\left(d^2+dc+c^2\right)}{3} - \frac{c+d}{a+b}\frac{(b+a)(d+c)}{2} + \left(\frac{c+d}{a+b}\right)^2 \frac{\left(b^2+ba+a^2\right)}{3}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6(a+b)^2} \\
&\quad \times \left[2\left(d^2+dc+c^2\right)(a+b)^2 - 3(b+a)^2(d+c)^2 + 2(d+c)^2\left(b^2+ba+a^2\right) \right] \\
&= \frac{1}{6(a+b)^2} \\
&\quad \times \left[2\left((d+c)^2 - dc\right)(a+b)^2 - 3(b+a)^2(d+c)^2 + 2(d+c)^2\left((b+a)^2 - ba\right) \right] \\
&= \frac{1}{6(a+b)^2} \left[(d+c)^2(a+b)^2 - 2dc(a+b)^2 - 2ba(d+c)^2 \right].
\end{aligned}$$

On the other hand,

$$J_{[a,b]\times[c,d]} := \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \frac{y^2}{x} dxdy - \frac{(c+d)^2}{2(a+b)}$$
$$= \frac{(\ln b - \ln a)\left(d^2 + dc + c^2\right)}{3(b-a)} - \frac{(c+d)^2}{2(a+b)}.$$

If $(x, y) \in [a, b] \times [c, d] \subset (0, \infty) \times (0, \infty)$, then

$$m = \frac{c}{b} \le \frac{y}{x} \le \frac{d}{a} = M$$

From the inequality (3.10) we have for a differentiable convex function $f : (0, \infty) \to \mathbb{R}$

$$(4.1) \quad 0 \le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d P_f(x, y)\, dxdy - \frac{a+b}{2} f\left(\frac{c+d}{a+b}\right)$$
$$\le \frac{1}{2}\left[f'\left(\frac{d}{a}\right) - f'\left(\frac{c}{b}\right)\right] \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left|y - \frac{c+d}{a+b}x\right| dxdy$$
$$\le \frac{1}{2\sqrt{6}(a+b)} \left[f'\left(\frac{d}{a}\right) - f'\left(\frac{c}{b}\right)\right]$$
$$\times \left[(d+c)^2 (a+b)^2 - 2dc(a+b)^2 - 2ba(d+c)^2\right]^{1/2}.$$

If $f : (0, \infty) \to \mathbb{R}$ is twice differentiable convex function, then by (3.14)

$$(4.2) \quad 0 \le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d P_f(x, y)\, dxdy - \frac{a+b}{2} f\left(\frac{c+d}{a+b}\right)$$
$$\le \|f''\|_{\left[\frac{c}{b}, \frac{d}{a}\right], \infty} \left[\frac{(\ln b - \ln a)\left(d^2 + dc + c^2\right)}{3(b-a)} - \frac{(c+d)^2}{2(a+b)}\right].$$

The case of squares $[a, b] \times [a, b]$ provides simpler forms as follows,

$$(4.3) \quad 0 \le \frac{1}{(b-a)^2} \int_a^b \int_a^b P_f(x, y)\, dxdy - \frac{a+b}{2} f(1)$$
$$\le \frac{1}{2}\left[f'\left(\frac{b}{a}\right) - f'\left(\frac{a}{b}\right)\right] \frac{1}{(b-a)^2} \int_a^b \int_a^b |y - x|\, dxdy$$
$$= \frac{1}{6}\left[f'\left(\frac{b}{a}\right) - f'\left(\frac{a}{b}\right)\right](b-a)$$

for a differentiable convex function $f : (0, \infty) \to \mathbb{R}$ and $[a, b] \subset (0, \infty)$, and

$$(4.4) \quad 0 \le \frac{1}{(b-a)^2} \int_a^b \int_a^b P_f(x, y)\, dxdy - \frac{a+b}{2} f(1)$$
$$\le \|f''\|_{\left[\frac{a}{b}, \frac{b}{a}\right], \infty} \left[\frac{(\ln b - \ln a)\left(a^2 + ab + b^2\right)}{3(b-a)} - \frac{a+b}{2}\right]$$

if $f : (0, \infty) \to \mathbb{R}$ is twice differentiable convex function and $[a, b] \subset (0, \infty)$.

5. Examples for Functions Defined on Circular Sectors

We consider the first quarter of the circle

$$Q(R) := \left\{ (x,y) \mid x = r\cos\theta,\ y = r\sin\theta \text{ with } r \in [0,R],\ \theta \in \left[0, \frac{\pi}{2}\right] \right\}.$$

Using the polar coordinates change of variable we have

$$\begin{aligned}\int\int_{Q(R)} P_f(x,y)\,dxdy &= \int\int_{Q(R)} xf\left(\frac{y}{x}\right) dxdy \\ &= \int_0^R \int_0^{\frac{\pi}{2}} r^2 \cos\theta f(\tan(\theta))\,drd\theta = \frac{R^3}{3}\int_0^{\frac{\pi}{2}} \cos\theta f(\tan(\theta))\,d\theta\end{aligned}$$

where $f : (0,\infty) \to \mathbb{R}$ is convex and the integral $\int_0^{\frac{\pi}{2}} \cos\theta f(\tan(\theta))\,d\theta$ is finite.

We have

$$A_{Q(R)} = \int\int_{Q(R)} dxdy = \int_0^R \int_0^{\frac{\pi}{2}} r dr d\theta = \frac{\pi R^2}{4}$$

$$\overline{x_{Q(R)}} := \frac{1}{A_{Q(R)}} \int\int_{Q(R)} x dxdy = \frac{1}{\frac{\pi R^2}{4}} \int_0^R \int_0^{\frac{\pi}{2}} r^2 \cos\theta dr d\theta = \frac{4}{3\pi} R$$

and

$$\overline{y_{Q(R)}} := \frac{1}{A_{Q(R)}} \int\int_{Q(R)} y dxdy = \frac{1}{\frac{\pi R^2}{4}} \int_0^R \int_0^{\frac{\pi}{2}} r^2 \sin\theta dr d\theta = \frac{4}{3\pi} R.$$

From the inequality (3.3) we have

$$0 \le \int_0^{\frac{\pi}{2}} \cos\theta f(\tan(\theta))\,d\theta - f(1) \le \int_0^{\frac{\pi}{2}} f'(\tan(\theta))(\sin\theta - \cos\theta)\,d\theta, \tag{5.1}$$

for $f : (0,\infty) \to \mathbb{R}$ convex and provided that the involved integral exist.

Consider

$$\ell_{Q(R)}(f') := \frac{\int_0^{\frac{\pi}{2}} f'(\tan(\theta))\cos\theta d\theta}{\int_0^{\frac{\pi}{2}} f'(\tan(\theta))\sin\theta d\theta}, \tag{5.2}$$

provided the involved integrals exists and assume that $\ell_{Q(R)}(f') > 0$, then by (3.4) we get

$$0 \le f\left(\ell_{Q(R)}(f')\right) - \int_0^{\frac{\pi}{2}} \cos\theta f(\tan(\theta))\,d\theta \le \left(\ell_{Q(R)}(f') - 1\right) f'\left(\ell_{Q(R)}(f')\right), \tag{5.3}$$

for $f : (0,\infty) \to \mathbb{R}$ convex and provided that the involved integral exist.

We can also consider the circular sector

$$Q(R,\theta_1,\theta_2) := \{(x,y) \mid x = r\cos\theta,\ y = r\sin\theta \text{ with } r \in [0,R],\ \theta \in [\theta_1,\theta_2]\},$$

where $[\theta_1,\theta_2] \subset \left[0,\frac{\pi}{2}\right]$.

Then

$$\int\int_{Q(R,\theta_1,\theta_2)} P_f(x,y)\,dxdy = \frac{R^3}{3}\int_{,\theta_1}^{\theta_2} \cos\theta f(\tan(\theta))\,d\theta$$

$$A_{Q(R,\theta_1,\theta_2)} = \frac{R^2}{2}(\theta_2 - \theta_1),$$

$$\overline{x_{Q(R,\theta_1,\theta_2)}} = \frac{2R}{3}\frac{\sin\theta_2 - \sin\theta_1}{\theta_2 - \theta_1}$$

and

$$\overline{y_{Q(R,\theta_1,\theta_2)}} = \frac{2R}{3}\frac{\cos\theta_1 - \cos\theta_2}{\theta_2 - \theta_1}.$$

We also have

$$J_{Q(R,\theta_1,\theta_2)} := \frac{2R}{3}\frac{1}{\theta_2 - \theta_1}\left[\int_{\theta_1}^{\theta_2}\frac{\sin^2\theta}{\cos\theta}d\theta - \frac{(\cos\theta_2 - \cos\theta_1)^2}{\sin\theta_2 - \sin\theta_1}\right].$$

Since

$$\int_{\theta_1}^{\theta_2}\frac{\sin^2\theta}{\cos\theta}d\theta = \ln\left(\frac{\tan\left(\frac{\theta_2}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\theta_1}{2}+\frac{\pi}{4}\right)}\right) - (\sin\theta_2 - \sin\theta_1),$$

hence

$$\begin{aligned}&\int_{\theta_1}^{\theta_2}\frac{\sin^2\theta}{\cos\theta}d\theta - \frac{(\cos\theta_2 - \cos\theta_1)^2}{\sin\theta_2 - \sin\theta_1}\\ &= \ln\left(\frac{\tan\left(\frac{\theta_2}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\theta_1}{2}+\frac{\pi}{4}\right)}\right) - \frac{(\sin\theta_2 - \sin\theta_1)^2 + (\cos\theta_2 - \cos\theta_1)^2}{\sin\theta_2 - \sin\theta_1}.\end{aligned}$$

Moreover,

$$\frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan(\theta) \in [\tan(\theta_1), \tan(\theta_2)]$$

and by (3.14) we get

$$\begin{aligned}(5.4)\quad 0 \le \frac{1}{\theta_2 - \theta_1}\int_{,\theta_1}^{\theta_2}\cos\theta f(\tan(\theta))\,d\theta - \frac{\sin\theta_2 - \sin\theta_1}{\theta_2 - \theta_1} f\left(\frac{\cos\theta_1 - \cos\theta_2}{\sin\theta_2 - \sin\theta_1}\right)\\ \le \|f''\|_{[\tan(\theta_1),\tan(\theta_2)],\infty}\\ \times\frac{1}{\theta_2 - \theta_1}\left[\ln\left(\frac{\tan\left(\frac{\theta_2}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\theta_1}{2}+\frac{\pi}{4}\right)}\right) - \frac{(\sin\theta_2 - \sin\theta_1)^2 + (\cos\theta_2 - \cos\theta_1)^2}{\sin\theta_2 - \sin\theta_1}\right]\end{aligned}$$

provided $f : (0,\infty) \to \mathbb{R}$ is twice differentiable convex on $(0,\infty)$ and $[\theta_1, \theta_2] \subset \left[0, \frac{\pi}{2}\right]$.

By utilising the above general results the interested reader may obtain other inequalities for the integral of perspective on the circular sectors. The details are not presented here.

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