HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR JENSEN'S DIVERGENCE

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let $f : I \to \mathbb{R}$ be a convex function on I. The associated two variables *Jensen's divergence* function $\mathcal{J}_f : I \times I \to \mathbb{R}_+$ is defined by

$$\mathcal{J}_f(t,s) := \frac{1}{2}[f(t)+f(s)] - f\left(\frac{t+s}{2}\right) \geq 0.$$

In this paper we establish some basic and double integral inequalities for the divergence function $\mathcal{J}_f$ defined above. Some double integral inequalities in the case of rectangles, squares and circular sectors are also given.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \quad a,\ b \in \mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [10]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [10]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality. For a monograph devoted to this inequality see [9]. Related results can be also found in [7].

In 1990, [4] the author established the following refinement of Hermite-Hadamard inequality for double and triple integrals for the convex function $f : [a,b] \to \mathbb{R}$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \\ &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f((1-t)x+ty)\,dtdxdy \leq \frac{1}{b-a}\int_a^b f(x)dx. \end{aligned} \tag{1.2}$$

More recently, [8] we obtained a different double integral inequality of Hermite-Hadamard type for the convex function $f : [a,b] \to \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(d-c)^2}\int_c^d\int_c^d f\left(\frac{\alpha a+\beta b}{\alpha+\beta}\right)d\beta d\alpha \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

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where $0 < c < d$.

For a function f defined on an interval I of the real line $\mathbb{R}$, by following the paper by Burbea & Rao [2], we consider the $\mathcal{J}$*-divergence* between the elements t, $s \in I$ given by

$$\mathcal{J}_f(t,s) := \frac{1}{2}[f(t)+f(s)] - f\left(\frac{t+s}{2}\right) \geq 0.$$

If f is convex on I, then $\mathcal{J}_f(t,s) \geq 0$ for all $(t,s) \in I \times I$.

As important examples of such divergences, we can consider for positive numbers (t,s) [2],

$$\mathcal{J}_\alpha(t,s) := \begin{cases} (\alpha-1)^{-1}\left[\frac{1}{2}(t^\alpha + s^\alpha) - \left(\frac{t+s}{2}\right)^\alpha\right], \ \alpha \neq 1, \\ \\ \left[t\ln(t) + s\ln(s) - (t+s)\ln\left(\frac{t+s}{2}\right)\right], \ \alpha = 1. \end{cases}$$

The following result concerning the joint convexity of $\mathcal{J}_f$ also holds:

Theorem 1 (Burbea-Rao, 1982 [2])**.** *Let f be a C^2 function on an interval I. Then $\mathcal{J}_f$ is convex (concave) on $I \times I$, if and only if f is convex (concave) and $\frac{1}{f''}$ is concave (convex) on I.*

Consider the power function $f_\alpha : [0,\infty) \to \mathbb{R}$, $f_\alpha(t) = (\alpha-1)^{-1} t^\alpha$ with $\alpha \in (1,2]$. This function is convex on $[0,\infty)$ and $\frac{1}{f''_\alpha}$ is concave on $(0,\infty)$ and therefore $\mathcal{J}_\alpha$ is jointly convex on $[0,\infty) \times [0,\infty)$. Also, the function $f_1 : (0,\infty) \to \mathbb{R}$, $f_1(t) = t\ln t$ is convex on $(0,\infty)$ and $\frac{1}{f''_1}$ is concave on $(0,\infty)$ showing that $\mathcal{J}_1$ is jointly convex on $(0,\infty) \times (0,\infty)$.

In this paper we establish some basic and double integral inequalities for the Jensen's divergence function $\mathcal{J}_f$ defined above. Some double integral inequalities in the case of rectangles, squares and circular sectors are also given.

2. General Results

Consider G a closed and bounded subset of $I \times I$. Define

$$A_G := \int\int_G dxdy$$

the *area* of G and $(\overline{x_G}, \overline{y_G})$ the *centre of mass* for G, where

$$\overline{x_G} := \frac{1}{A_G}\int\int_G xdxdy, \ \overline{y_G} := \frac{1}{A_G}\int\int_G ydxdy.$$

Observe that if $f : I \to \mathbb{R}$ is convex and G a closed and bounded subset of $I \times I$, then the double integral

$$\int\int_G \mathcal{J}_f(x,y)\,dxdy = \frac{1}{2}\left[\int\int_G f(x)\,dx + \int\int_G f(y)\,dy\right] - \int\int_G f\left(\frac{x+y}{2}\right)dxdy \geq 0 \tag{2.1}$$

exists.

We have the following general result:

Theorem 2. *Let f be a $C^1(I)$ function on an interval I. If f is convex on I, then*

$$0 \leq \int\int_G \mathcal{J}_f(x,y)\,dxdy \leq \frac{1}{4}\Phi_G(f'), \tag{2.2}$$

where

$$\Phi_G(f') := \int\int_G [f'(y) - f'(x)](y-x)\,dxdy \tag{2.3}$$
$$= \int\int_G f'(y)\,ydxdy + \int\int_G f'(x)\,xdxdy - \int\int_G xf'(y)\,dxdy - \int\int_G f'(x)\,ydxdy.$$

Proof. We use the following inequality for differentiable convex functions obtained in [6]

$$0 \le \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \le \frac{1}{4}[f'(y) - f'(x)](y-x)$$

for any $x,\, y \in \mathring{I}$ with the constant $\frac{1}{4}$ as best possible. □

Corollary 1. *With the assumptions of Theorem 2 and if* $\gamma = \inf_{t\in\mathring{I}} f'(t)$ *and* $\Gamma = \sup_{t\in\mathring{I}} f'(t)$ *are finite, then*

$$0 \le \int\int_G \mathcal{J}_f(x,y)\,dxdy \le \frac{1}{4}(\Gamma - \gamma)\int\int_G |y-x|\,dxdy. \tag{2.4}$$

Moreover, if $G \subset [a,b]\times[a,b] \subset I\times I$, *then*

$$0 \le \int\int_G \mathcal{J}_f(x,y)\,dxdy \le \frac{1}{4}[f'(b) - f'(a)]\int\int_G |y-x|\,dxdy. \tag{2.5}$$

Proof. We have

$$\begin{aligned} 0 \le \Phi_G(f') &= \int\int_G [f'(y) - f'(x)](y-x)\,dxdy \\ &\le \int\int_G |[f'(y) - f'(x)](y-x)|\,dxdy \le \int\int_G |f'(y) - f'(x)|\,|y-x|\,dxdy \\ &\le (\Gamma-\gamma)\int\int_G |y-x|\,dxdy, \end{aligned}$$

which together with (2.2) gives (2.4). □

Corollary 2. *With the assumptions of Theorem 2 and if the derivative* f' *is Lipschitzian with the constant* $K > 0$, *namely*

$$|f'(t) - f'(s)| \le K\,|t-s| \text{ for all } t,\ s \in \mathring{I},$$

where $\mathring{I}$ *is the interior of* I, *then we have the inequality*

$$0 \le \int\int_G \mathcal{J}_f(x,y)\,dxdy \le \frac{1}{4}K\int\int_G (y-x)^2\,dxdy. \tag{2.6}$$

Moreover, if f *is a* $C^2(I)$ *function on an interval* I *and* $\|f''\|_{I,\infty} := \sup_{t\in I}|f''(t)| < \infty$, *then*

$$0 \le \int\int_G \mathcal{J}_f(x,y)\,dxdy \le \frac{1}{4}\|f''\|_{I,\infty}\int\int_G (y-x)^2\,dxdy. \tag{2.7}$$

Proof. We have

$$
\begin{aligned}
0 &\leq \Phi_G(f') = \int\int_G [f'(y) - f'(x)](y-x)\,dxdy \\
&\leq \int\int_G |[f'(y) - f'(x)](y-x)|\,dxdy \leq \int\int_G |f'(y) - f'(x)|\,|y-x|\,dxdy \\
&\leq K \int\int_G (y-x)^2\,dxdy,
\end{aligned}
$$

which together with (2.2) gives (2.4). □

We need the following lemma that is of interest in itself:

Lemma 1. *Let f be a $C^2(I)$ function on an interval I. If f is convex on I and $\frac{1}{f''}$ is concave on I, then for all $(t,s), (u,v) \in \mathring{I} \times \mathring{I}$ we have the double inequality*

$$
\begin{aligned}
&\frac{1}{2}\left[f'(t) - f'\left(\frac{t+s}{2}\right)\right](t-u) + \frac{1}{2}\left[f'(s) - f'\left(\frac{t+s}{2}\right)\right](s-v) \\
&\qquad\qquad \geq \mathcal{J}_f(t,s) - \mathcal{J}_f(u,v) \\
&\qquad \geq \frac{1}{2}\left[f'(u) - f'\left(\frac{u+v}{2}\right)\right](t-u) + \frac{1}{2}\left[f'(v) - f'\left(\frac{u+v}{2}\right)\right](s-v).
\end{aligned} \tag{2.8}
$$

Proof. It is well known that if the function of two independent variables $F : D \subset \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is convex on the convex domain D and has partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ on D then for all $(t,s), (u,v) \in D$ we have the gradient inequalities

$$
\begin{aligned}
&\frac{\partial F(t,s)}{\partial x}(t-u) + \frac{\partial F(t,s)}{\partial y}(s-v) \\
&\geq F(t,s) - F(u,v) \\
&\geq \frac{\partial F(u,v)}{\partial x}(t-u) + \frac{\partial F(u,v)}{\partial y}(s-v).
\end{aligned} \tag{2.9}
$$

Now, if we take $F : I \times I \to \mathbb{R}$ given by

$$F(t,s) = \frac{1}{2}[f(t) + f(s)] - f\left(\frac{t+s}{2}\right)$$

and observe that

$$\frac{\partial F(t,s)}{\partial x} = \frac{1}{2}\left[f'(t) - f'\left(\frac{t+s}{2}\right)\right]$$

and

$$\frac{\partial F(t,s)}{\partial y} = \frac{1}{2}\left[f'(s) - f'\left(\frac{t+s}{2}\right)\right]$$

and since F is convex on $I \times I$, then by (2.9) we get (2.8). □

We have the following double integral inequality:

Theorem 3. *Let f be a $C^2(I)$ function on an interval I. If f is convex on I and $\frac{1}{f''}$ is concave on I, then for all $(u,v) \in \mathring{I} \times \mathring{I}$, we have the double integral inequality*

$$\begin{aligned}(2.10)\quad & \frac{1}{2}\frac{1}{A_G}\int\int_G \left[f'(x) - f'\left(\frac{x+y}{2}\right)\right](x-u)\,dxdy \\ &\quad + \frac{1}{2}\frac{1}{A_G}\int\int_G \left[f'(y) - f'\left(\frac{x+y}{2}\right)\right](y-v)\,dxdy \\ &\quad \geq \frac{1}{A_G}\int\int_G \mathcal{J}_f(x,y)\,dxdy - \mathcal{J}_f(u,v) \\ &\geq \frac{1}{2}\left[f'(u) - f'\left(\frac{u+v}{2}\right)\right](\overline{x_G} - u) + \frac{1}{2}\left[f'(v) - f'\left(\frac{u+v}{2}\right)\right](\overline{y_G} - v).\end{aligned}$$

In particular,

$$\begin{aligned}(2.11)\quad 0 &\leq \frac{1}{A_G}\int\int_G \mathcal{J}_f(x,y)\,dxdy - \mathcal{J}_f(\overline{x_G},\overline{y_G}) \\ &\leq \frac{1}{2}\frac{1}{A_G}\left[\int\int_G \left[f'(x) - f'\left(\frac{x+y}{2}\right)\right](x-\overline{x_G})\,dxdy\right. \\ &\qquad \left. + \int\int_G \left[f'(y) - f'\left(\frac{x+y}{2}\right)\right](y-\overline{y_G})\,dxdy\right].\end{aligned}$$

Proof. From (2.8) we have

$$\begin{aligned}(2.12)\quad & \frac{1}{2}\left[f'(x) - f'\left(\frac{x+y}{2}\right)\right](x-u) + \frac{1}{2}\left[f'(y) - f'\left(\frac{x+y}{2}\right)\right](y-v) \\ &\quad \geq \mathcal{J}_f(x,y) - \mathcal{J}_f(u,v) \\ &\geq \frac{1}{2}\left[f'(u) - f'\left(\frac{u+v}{2}\right)\right](x-u) + \frac{1}{2}\left[f'(v) - f'\left(\frac{u+v}{2}\right)\right](y-v)\end{aligned}$$

for all (x,y), $(u,v) \in \mathring{I} \times \mathring{I}$.

If we take the integral mean $\frac{1}{A_G}\int\int_G$ over $(x,y) \in G$ in (2.12) we get the desired result (2.10). □

Corollary 3. *With the assumptions of Theorem 3 and if $\gamma = \inf_{t\in\mathring{I}} f'(t)$ and $\Gamma = \sup_{t\in\mathring{I}} f'(t)$ are finite, then*

$$\begin{aligned}(2.13)\quad 0 &\leq \frac{1}{A_G}\int\int_G \mathcal{J}_f(x,y)\,dxdy - \mathcal{J}_f(\overline{x_G},\overline{y_G}) \\ &\leq \frac{1}{2}(\Gamma - \gamma)\frac{1}{A_G}\int\int_G (|x-\overline{x_G}| + |y-\overline{y_G}|)\,dxdy.\end{aligned}$$

Moreover, if $G \subset [a,b] \times [a,b] \subset I \times I$, then

$$\begin{aligned}(2.14)\quad 0 &\leq \frac{1}{A_G}\int\int_G \mathcal{J}_f(x,y)\,dxdy - \mathcal{J}_f(\overline{x_G},\overline{y_G}) \\ &\leq \frac{1}{2}[f'(b) - f'(a)]\frac{1}{A_G}\int\int_G (|x-\overline{x_G}| + |y-\overline{y_G}|)\,dxdy.\end{aligned}$$

Proof. We have

$$\begin{aligned}
0 &\leq \int\int_G \left[f'(x) - f'\left(\frac{x+y}{2}\right)\right](x - \overline{x_G})\,dxdy \\
&\qquad + \int\int_G \left[f'(y) - f'\left(\frac{x+y}{2}\right)\right](y - \overline{y_G})\,dxdy\Bigg] \\
&\leq \int\int_G \left|f'(x) - f'\left(\frac{x+y}{2}\right)\right| |x - \overline{x_G}|\,dxdy \\
&\qquad + \int\int_G \left|f'(y) - f'\left(\frac{x+y}{2}\right)\right| |y - \overline{y_G}|\,dxdy \\
&\leq (\Gamma - \gamma)\int\int_G |x - \overline{x_G}|\,dxdy + (\Gamma - \gamma)\int\int_G |y - \overline{y_G}|\,dxdy \\
&= (\Gamma - \gamma)\left[\int\int_G |x - \overline{x_G}|\,dxdy + \int\int_G |y - \overline{y_G}|\,dxdy\right]
\end{aligned}$$

and by (2.11) we get the desired result (2.13). □

Corollary 4. *With the assumptions of Theorem 3 and if the derivative f' is Lipschitzian with the constant K, then*

$$\begin{aligned}
(2.15)\quad 0 &\leq \frac{1}{A_G}\int\int_G \mathcal{J}_f(x,y)\,dxdy - \mathcal{J}_f(\overline{x_G}, \overline{y_G}) \\
&\leq \frac{1}{4}K\frac{1}{A_G}\int\int_G |x-y|\left[|x - \overline{x_G}| + |y - \overline{y_G}|\right]dxdy.
\end{aligned}$$

Moreover, if f is a $C^2(I)$ function on an interval I and $\|f''\|_{I,\infty} := \sup_{t\in I}|f''(t)| < \infty$, then

$$\begin{aligned}
(2.16)\quad 0 &\leq \frac{1}{A_G}\int\int_G \mathcal{J}_f(x,y)\,dxdy - \mathcal{J}_f(\overline{x_G}, \overline{y_G}) \\
&\leq \frac{1}{4}\|f''\|_{I,\infty}\frac{1}{A_G}\int\int_G |x-y|\left(|x - \overline{x_G}| + |y - \overline{y_G}|\right)dxdy.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
0 &\leq \int\int_G \left[f'(x) - f'\left(\frac{x+y}{2}\right)\right](x - \overline{x_G})\,dxdy \\
&\qquad + \int\int_G \left[f'(y) - f'\left(\frac{x+y}{2}\right)\right](y - \overline{y_G})\,dxdy\Bigg] \\
&\leq \int\int_G \left|f'(x) - f'\left(\frac{x+y}{2}\right)\right| |x - \overline{x_G}|\,dxdy \\
&\qquad + \int\int_G \left|f'(y) - f'\left(\frac{x+y}{2}\right)\right| |y - \overline{y_G}|\,dxdy \\
&\leq K\int\int_G \left|x - \frac{x+y}{2}\right| |x - \overline{x_G}|\,dxdy + K\int\int_G \left|y - \frac{x+y}{2}\right| |x - \overline{x_G}|\,dxdy \\
&= \frac{1}{2}K\int\int_G |x-y|\left(|x - \overline{x_G}| + |y - \overline{y_G}|\right)dxdy
\end{aligned}$$

and by (2.11) we get the desired result (2.16). □

3. Examples for Functions Defined on Squares

If $G = [a,b]^2 := [a,b] \times [a,b] \subset I \times I$ then

$$\begin{aligned} &\frac{1}{(b-a)^2} \int_a^b \int_a^b \mathcal{J}_f (x,y)\, dxdy \\ &= \frac{1}{b-a} \int_a^b f(x)\, dx - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \Phi_{[a,b]^2}(f') &= \int_a^b \int_a^b \left[f'(y) - f'(x)\right](y-x)\, dxdy \\ &= 2\left[(b-a)\int_a^b f'(x)\, xdx - \int_a^b f'(x)\, dx \int_a^b xdx\right] \\ &= 2(b-a)\int_a^b f'(x)\left(x - \frac{a+b}{2}\right) dx \\ &= 2(b-a)^2 \left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\, dx\right]. \end{aligned} \tag{3.2}$$

From (2.2) we then have for a differentiable convex function f on I that

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f(x)\, dx - \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right) dxdy \\ &\le \frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\, dx\right] \\ &\le \frac{1}{16}\left[f'(b) - f'(a)\right](b-a), \text{ conform with [5].} \end{aligned} \tag{3.3}$$

If f is twice differentiable and $\|f''\|_{(a,b),\infty} := \sup_{t\in(a,b)} |f''(t)| < \infty$ and since

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\, dx\right| \le \frac{1}{12}\|f''\|_{(a,b),\infty}(b-a)^2,$$

then

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f(x)\, dx - \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right) dxdy \\ &\le \frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\, dx\right] \le \frac{1}{24}\|f''\|_{(a,b),\infty}(b-a)^2, \end{aligned} \tag{3.4}$$

provided that f is twice differentiable convex on (a,b) with $\|f''\|_{(a,b),\infty} < \infty$.

4. Example for Functions Defined on Rectangle

If $G = [a,b] \times [c,d]$ is a rectangle from $I \times I$, then

$$A_{[a,b]\times[c,d]} = (b-a)(d-c), \ \overline{x_{[a,b]\times[c,d]}} = \frac{a+b}{2} \text{ and } \overline{y_{[a,b]\times[c,d]}} = \frac{c+d}{2}.$$

Also

$$\begin{aligned}
&\frac{1}{A_{[a,b]\times[c,d]}}\int_a^b\int_c^d \mathcal{J}_f(x,y)\,dxdy \\
&=\frac{1}{2}\left(\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{1}{d-c}\int_a^b f(y)\,dy\right) \\
&-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\frac{x+y}{2}\right)dxdy
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{J}_f\left(\overline{x_{[a,b]\times[c,d]}},\overline{y_{[a,b]\times[c,d]}}\right) \\
&=\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+f\left(\frac{c+d}{2}\right)\right]-f\left(\frac{a+b+c+d}{4}\right).
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_a^b\int_c^d\left(\left|x-\frac{a+b}{2}\right|+\left|y-\frac{c+d}{2}\right|\right)dxdy \\
&=\frac{1}{4}(d-c)(b-a)^2+\frac{1}{4}(b-a)(d-c)^2 \\
&=\frac{1}{4}(b-a)(d-c)(b-a+d-c).
\end{aligned}$$

Assume that $[a,b],[c,d]\subset[m,M]\subset I$ and f is twice differentiable convex on I with $\frac{1}{f''}$ concave, then from (2.14) we get

$$\begin{aligned}
0 &\le \frac{1}{2}\left(\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{1}{d-c}\int_a^b f(y)\,dy\right) \\
&\quad-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\frac{x+y}{2}\right)dxdy \\
&\quad-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+f\left(\frac{c+d}{2}\right)\right]+f\left(\frac{a+b+c+d}{4}\right) \\
&\le\frac{1}{8}\left[f'(M)-f'(m)\right](b-a+d-c).
\end{aligned}\tag{4.1}$$

5. Example for Functions Defined on Disks

Consider the disk centered in zero and of radius $R>0$,

$$D(0,R):=\{(x,y)\mid x=r\cos\theta,\ y=r\sin\theta,\ r\in[0,R],\ \theta\in[0,2\pi]\}.$$

Using the polar change of variable we have for a function $f : I \to R$ with $D(0,R) \subset I \times I$

$$\begin{aligned}\int\int_{D(0,R)} \mathcal{J}_f(x,y)\,dxdy &= \int_0^R \int_0^{2\pi} \mathcal{J}_f(r\cos\theta, r\sin\theta)\, rdrd\theta \\ &= \int_0^R \int_0^{2\pi} \left[\frac{f(r\cos\theta)+f(r\sin\theta)}{2} - f\left(\frac{r\cos\theta + r\sin\theta}{2}\right)\right] rdrd\theta \\ &= \frac{1}{2}\left[\int_0^R \int_0^{2\pi} f(r\cos\theta)\, rdrd\theta + \int_0^R \int_0^{2\pi} f(r\sin\theta)\, rdrd\theta\right] \\ &\qquad - \int_0^R \int_0^{2\pi} f\left(\frac{r\cos\theta + r\sin\theta}{2}\right) rdrd\theta\end{aligned}$$

and

$$\begin{aligned}\Phi_{D(0,R)}(f') &:= \int\int_{D(0,R)} [f'(y) - f'(x)](y-x)\,dxdy \\ &= \int_0^R \int_0^{2\pi} [f'(r\sin\theta) - f'(r\cos\theta)](\sin\theta - \cos\theta)\, r^2 drd\theta.\end{aligned}$$

Assume that f is twice differentiable convex on I with $\|f''\|_{I,\infty} := \sup_{t\in I} |f''(t)| < \infty$, then

$$\begin{aligned}\Phi_{D(0,R)}(f') &\le \int_0^R \int_0^{2\pi} |f'(r\sin\theta) - f'(r\cos\theta)|\, |\sin\theta - \cos\theta|\, r^2 drd\theta \\ &\le \|f''\|_{I,\infty} \int_0^R \int_0^{2\pi} (\sin\theta - \cos\theta)^2 r^2 drd\theta \\ &= \|f''\|_{I,\infty} \int_0^R \int_0^{2\pi} \left(\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta\right) r^2 drd\theta.\end{aligned}$$

Observe that

$$\begin{aligned}&\int_0^R \int_0^{2\pi} \left(\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta\right) r^2 drd\theta \\ &= \int_0^R \int_0^{2\pi} (1 - \sin 2\theta)\, r^2 drd\theta = \frac{R^3}{3}\int_0^{2\pi} (1-\sin 2\theta)\, d\theta = \frac{2\pi R^3}{3}\end{aligned}$$

and by the inequality (2.2) we get

$$\begin{aligned}(5.1)\quad 0 \le \frac{1}{2}&\left[\frac{1}{\pi R^2}\int_0^R \int_0^{2\pi} f(r\cos\theta)\, rdrd\theta + \frac{1}{\pi R^2}\int_0^R \int_0^{2\pi} f(r\sin\theta)\, rdrd\theta\right] \\ &- \frac{1}{\pi R^2}\int_0^R \int_0^{2\pi} f\left(\frac{r\cos\theta + r\sin\theta}{2}\right) rdrd\theta \le \frac{1}{6} R \|f''\|_{I,\infty}.\end{aligned}$$

Consider the disk centered in the point (a,b) and of radius R,

$$D((a,b),R) := \{(x,y) \mid x = r\cos\theta + a,\ y = r\sin\theta + b,\ r \in [0,R],\ \theta \in [0,2\pi]\}.$$

We have

$$\overline{x_{D((a,b),R)}} = a,\ \overline{y_{D((a,b),R)}} = b,$$

$$\frac{1}{A_{D((a,b),R)}}\int\int_{D((a,b),R)}\mathcal{J}_f(x,y)\,dxdy=$$
$$=\frac{1}{2}\left[\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}f(r\cos\theta+a)\,rdrd\theta+\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}f(r\sin\theta+b)\,rdrd\theta\right]$$
$$-\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}f\left(\frac{r\cos\theta+r\sin\theta}{2}+\frac{a+b}{2}\right)rdrd\theta.$$

Assume that $D((a,b),R)\subset[m,M]^2\subset I\times I$ and f is twice differentiable convex on I and with $\frac{1}{f''}$ concave on I, then by (2.13) we get

$$0\le\frac{1}{2}\left[\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}f(r\cos\theta+a)\,rdrd\theta+\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}f(r\sin\theta+b)\,rdrd\theta\right]$$
$$-\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}f\left(\frac{r\cos\theta+r\sin\theta}{2}+\frac{a+b}{2}\right)rdrd\theta$$
$$-\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)$$
$$\le\frac{1}{2}\left(f'(M)-f'(m)\right)\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}r^2\left(|\cos\theta|+|\sin\theta|\right)drd\theta$$
$$=\frac{1}{6}\left(f'(M)-f'(m)\right)\frac{R}{\pi}\int_0^{2\pi}\left(|\cos\theta|+|\sin\theta|\right)d\theta.$$

Since

$$\int_0^{2\pi}\left(|\cos\theta|+|\sin\theta|\right)d\theta=8,$$

hence we obtain the inequalities

(5.2)
$$0\le\frac{1}{2}\left[\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}f(r\cos\theta+a)\,rdrd\theta+\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}f(r\sin\theta+b)\,rdrd\theta\right]$$
$$-\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}f\left(\frac{r\cos\theta+r\sin\theta}{2}+\frac{a+b}{2}\right)rdrd\theta$$
$$-\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\le\frac{4}{3\pi}\left[f'(M)-f'(m)\right]R.$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa