

Complex Multivariate Taylor's formula

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Abstract

We derive here a Taylor's formula with integral remainder in the several complex variables and we estimate its remainder.

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1 Main Results

We need the following vector Taylor's formula:

Theorem 1 (*Shilov, [3], pp. 93-94*) Let $n \in \mathbb{N}$ and $f \in C^n([a, b], X)$, where $[a, b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Then

$$f(b) = f(a) + \sum_{i=1}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) + \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x) dx. \quad (1)$$

The remainder here is the Riemann X -valued integral (defined similar to numerical one) given by

$$Q_{n-1} = \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x) dx, \quad (2)$$

with the property:

$$\|Q_{n-1}\| \leq \max_{a \leq x \leq b} \|f^{(n)}(x)\| \frac{(b-a)^n}{n!}. \quad (3)$$

The derivatives above are defined similar to the numerical ones. We make

Remark 2 Here Q is an open convex subset of \mathbb{C}^k , $k \geq 2$; $z := (z_1, \dots, z_k)$, $x_0 := (x_{01}, \dots, x_{0k}) \in Q$. Let $f : Q \rightarrow \mathbb{C}$ be a coordinate-wise holomorphic

function. Then, by the famous Hartog's fundamental theorem ([1], [2]) f is jointly holomorphic and jointly continuous on Q . Let $n \in \mathbb{N}$. Each n^{th} order complex partial derivative is denoted by $f_\alpha := \frac{\partial^n f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, \dots, \alpha_k)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, \dots, k$ and $|\alpha| := \sum_{i=1}^k \alpha_i = n$.

Consider

$$g_z(t) := f(x_0 + t(z - x_0)), \quad 0 \leq t \leq 1. \quad (4)$$

Clearly it holds that $x_0 + t(z - x_0) \in Q$ and $g_z(t) \in \mathbb{C}$, $\forall t \in [0, 1]$.

Then we derive

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), \dots, x_{0k} + t(z_k - x_{0k})), \quad (5)$$

for all $j = 0, 1, \dots, n$.

Notice here that any mixed partials commute. We remind that $(\mathbb{C}, |\cdot|)$ is a Banach space. By Shilov's Theorem 1, about the Taylor's formula for Banach space valued functions, we obtain

Theorem 3 *It holds*

$$f(z_1, \dots, z_k) = g_z(1) = \sum_{j=0}^{n-1} \frac{g_z^{(j)}(0)}{j!} + R_n(z, 0), \quad (6)$$

where

$$R_n(z, 0) = \frac{1}{(n-1)!} \int_0^1 (1-\theta)^{n-1} g_z^{(n)}(\theta) d\theta, \quad (7)$$

and notice that $g_z(0) = f(x_0)$.

We make

Remark 4 *Notice that (by (7)) we get*

$$|R_n(z, 0)| \leq \left(\max_{0 \leq \theta \leq 1} |g_z^{(n)}(\theta)| \right) \frac{1}{n!}. \quad (8)$$

We also have for $j = 0, 1, \dots, n$:

$$g_z^{(j)}(0) = \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_k), \alpha_j \in \mathbb{Z}^+ \\ i=1, \dots, k; |\alpha| := \sum_{i=1}^k \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0). \quad (9)$$

Furthermore it holds

$$g_z^{(n)}(\theta) = \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_k), \alpha_j \in \mathbb{Z}^+ \\ i=1, \dots, k; |\alpha| := \sum_{i=1}^k \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0 + \theta(z - x_0)),$$

(10)

$0 \leq \theta \leq 1$.

Another version of (6) is

$$f(z_1, \dots, z_k) = g_z(1) = \sum_{j=0}^n \frac{g_z^{(j)}(0)}{j!} + \overline{R}_n(z, 0),$$

(11)

where

$$\overline{R}_n(z, 0) = \frac{1}{(n-1)!} \int_0^1 (1-\theta)^{n-1} \left(g_z^{(n)}(\theta) - g_z^{(n)}(0) \right) d\theta.$$

(12)

Identities (6) and (11) are the multivariate complex Taylor's formula with integral remainders.

We give

Example 5 Let $n = k = 2$. Then

$$g_z(t) = f(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})), \quad t \in [0, 1],$$

and

$$g'_z(t) = (z_1 - x_{01}) \frac{\partial f}{\partial x_1}(x_0 + t(z - x_0)) + (z_2 - x_{02}) \frac{\partial f}{\partial x_2}(x_0 + t(z - x_0)).$$

(13)

In addition,

$$\begin{aligned} g''_z(t) &= (z_1 - x_{01}) \left(\frac{\partial f}{\partial x_1}(x_0 + t(z - x_0)) \right)' + (z_2 - x_{02}) \left(\frac{\partial f}{\partial x_2}(x_0 + t(z - x_0)) \right)' \\ &= (z_1 - x_{01}) \left\{ (z_1 - x_{01}) \frac{\partial^2 f}{\partial x_1^2}(\ast) + (z_2 - x_{02}) \frac{\partial^2 f}{\partial x_2 \partial x_1}(\ast) \right\} + \\ &\quad (z_2 - x_{02}) \left\{ (z_1 - x_{01}) \frac{\partial^2 f}{\partial x_1 \partial x_2}(\ast) + (z_2 - x_{02}) \frac{\partial^2 f}{\partial x_2^2}(\ast) \right\}. \end{aligned}$$

(14)

Hence,

$$\begin{aligned} g''_z(t) &= (z_1 - x_{01})^2 \frac{\partial^2 f}{\partial x_1^2}(\ast) + (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial^2 f}{\partial x_2 \partial x_1}(\ast) + \\ &\quad (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial^2 f}{\partial x_1 \partial x_2}(\ast) + (z_2 - x_{02})^2 \frac{\partial^2 f}{\partial x_2^2}(\ast), \end{aligned}$$

(15)

where $\ast := x_0 + t(z - x_0)$.

Notice that $g_z(t), g'_z(t), g''_z(t) \in \mathbb{C}$.

We make

Remark 6 We define

$$\|f\|_{p, \overline{zx_0}} := \left(\int_0^1 |f(x_0 + \theta(z - x_0))|^p d\theta \right)^{\frac{1}{p}}, \quad p \geq 1, \quad (16)$$

where $\overline{zx_0}$ denotes the segment $\overline{zx_0} \subset Q$.

We also define

$$\|f\|_{\infty, \overline{zx_0}} := \max_{\theta \in [0,1]} |f(x_0 + \theta(z - x_0))|. \quad (17)$$

By (10) we obtain

$$\begin{aligned} |g_z^{(n)}(\theta)| \leq & \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_k), \alpha_j \in \mathbb{Z}^+ \\ i=1, \dots, k; |\alpha| := \sum_{i=1}^k \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k |z_i - x_{0i}|^{\alpha_i} \right) |f_\alpha(x_0 + \theta(z - x_0))|, \\ & \forall \theta \in [0, 1]. \end{aligned} \quad (18)$$

Therefore, by norm properties for $1 \leq p \leq \infty$, it holds

$$\begin{aligned} \|g_z^{(n)}\|_{p, \overline{zx_0}} & \leq \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_k), \alpha_j \in \mathbb{Z}^+ \\ i=1, \dots, k; |\alpha| := \sum_{i=1}^k \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k |z_i - x_{0i}|^{\alpha_i} \right) \|f_\alpha\|_{p, \overline{zx_0}} \\ & \leq \left(\sum_{i=1}^k |z_i - x_{0i}| \right)^n \|f_\alpha^*\|_{p, \overline{zx_0}}, \end{aligned} \quad (19)$$

where

$$\|f_\alpha^*\|_{p, \overline{zx_0}} := \max_{|\alpha|=n} \|f_\alpha\|_{p, \overline{zx_0}}, \quad (20)$$

for all $1 \leq p \leq \infty$.

That is

$$\|g_z^{(n)}\|_{p, \overline{zx_0}} \leq (\|z - x_0\|_{l_1})^n \|f_\alpha^*\|_{p, \overline{zx_0}}, \quad (21)$$

for all $1 \leq p \leq \infty$.

Therefore by (8) we obtain

$$|R_n(z, 0)| \leq \frac{(\|z - x_0\|_{l_1})^n \|f_\alpha^*\|_{\infty, \overline{zx_0}}}{n!}. \quad (22)$$

Next, we put things together and we further estimate $R_n(z, 0)$.

Theorem 7 Here $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. It holds

$$|R_n(z, 0)| \leq \min \left\{ \begin{array}{l} \frac{\|g_z^{(n)}\|_{\infty, \overline{z x_0}}}{n!}, \\ \frac{\|g_z^{(n)}\|_{1, \overline{z x_0}}}{(n-1)!}, \\ \frac{\|g_z^{(n)}\|_{p, \overline{z x_0}}}{(n-1)!(q(n-1)+1)^{\frac{1}{q}}} \end{array} \right\} \leq \quad (23)$$

$$(\|z - x_0\|_{l_1})^n \min \left\{ \begin{array}{l} \frac{\|f_\alpha^*\|_{\infty, \overline{z x_0}}}{n!}, \\ \frac{\|f_\alpha^*\|_{1, \overline{z x_0}}}{(n-1)!}, \\ \frac{\|f_\alpha^*\|_{p, \overline{z x_0}}}{(n-1)!(q(n-1)+1)^{\frac{1}{q}}}. \end{array} \right. \quad (24)$$

Proof. Based on (7), Hölder's inequality and (21). ■

References

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