HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR DOUBLE INTEGRAL ON GENERAL DOMAINS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish some Hermite-Hadamard type inequalities for functions of two independent variables defined on closed and bounded convex subsets of the plane $\mathbb{R}^2$. Some examples for rectangles and disks are also provided.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \quad a,\ b \in \mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [11]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [11]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality. For a monograph devoted to this inequality see [10]. Related results can be also found in [8].

In 1990, [4] the author established the following refinement of Hermite-Hadamard inequality for double and triple integrals for the convex function $f:[a,b]\to\mathbb{R}$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \\ &\le \frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f((1-t)x+ty)\,dtdxdy \le \frac{1}{b-a}\int_a^b f(x)dx. \end{aligned} \tag{1.2}$$

More recently, [9] we obtained a different double integral inequality of Hermite-Hadamard type for the convex function $f:[a,b]\to\mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{(d-c)^2}\int_c^d\int_c^d f\left(\frac{\alpha a+\beta b}{\alpha+\beta}\right)d\beta d\alpha \le \frac{f(a)+f(b)}{2} \tag{1.3}$$

where $0<c<d$.

Let us consider a point $C=(a,b)\in\mathbb{R}^2$ and the disk $D(C,R)$ centered at the point C and having the radius $R>0$. In [5] we establish between others the

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Convex functions, Hermite-Hadamard inequality, Double integral inequalities.

RGMIA Res. Rep. Coll. 22 (2019), Art. 46, 10 pp. Received 25/04/19

following Hermite-Hadamard type inequality for a convex function $f : D(C,R) \to \mathbb{R}$,

$$f(C) \le \frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dxdy \le \frac{2}{3}\frac{1}{2\pi R}\int_{\mathcal{C}(C,R)} f(\gamma)\,d\ell(\gamma) + \frac{1}{3}f(C) \le \frac{1}{2\pi R}\int_{\mathcal{C}(C,R)} f(\gamma)\,d\ell(\gamma) \tag{1.4}$$

where $\mathcal{C}(C,R)$ is the circle centered at C and having the radius R and $\int_{\mathcal{C}(C,R)}$ is the path integral with respect to arc length.

Motivated by the above results, in this paper we establish some Hermite-Hadamard type inequalities for functions of two independent variables defined on closed and bounded convex subsets of the plane $\mathbb{R}^2$. Some examples for rectangles and disks are also provided.

2. Main Results

In the following, consider G a closed and bounded convex subset of $\mathbb{R}^2$. Define

$$A_G := \int\int_G dxdy$$

the *area* of G and $(\overline{x_G}, \overline{y_G})$ the *centre of mass* for G, where

$$\overline{x_G} := \frac{1}{A_G}\int\int_G xdxdy, \ \overline{y_G} := \frac{1}{A_G}\int\int_G ydxdy.$$

Consider the function of two variables $f = f(x,y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

Theorem 1. *Let $f : G \to \mathbb{R}$ be a differentiable convex function on G. Then for all $(u,v) \in G$ we have*

$$\begin{aligned} &\frac{\partial f}{\partial x}(u,v)(\overline{x_G} - u) + \frac{\partial f}{\partial y}(u,v)(\overline{y_G} - v) \\ &\le \frac{1}{A_G}\int\int_G f(x,y)\,dxdy - f(u,v) \\ &\le \frac{1}{A_G}\int\int_G \frac{\partial f}{\partial x}(x,y)(x-u)\,dxdy + \frac{1}{A_G}\int\int_G \frac{\partial f}{\partial y}(x,y)(y-v)\,dxdy. \end{aligned} \tag{2.1}$$

In particular,

$$\begin{aligned} 0 &\le \frac{1}{A_G}\int\int_G f(x,y)\,dxdy - f(\overline{x_G}, \overline{y_G}) \\ &\le \frac{1}{A_G}\int\int_G \frac{\partial f}{\partial x}(x,y)(x - \overline{x_G})\,dxdy + \frac{1}{A_G}\int\int_G \frac{\partial f}{\partial y}(x,y)(y - \overline{y_G})\,dxdy. \end{aligned} \tag{2.2}$$

Proof. Since $f : G \to \mathbb{R}$ is a differentiable convex function on G, then for all (x,y), $(u,v) \in D$ we have the gradient inequalities

$$\begin{aligned} \frac{\partial f}{\partial x}(u,v)(x-u) + \frac{\partial f}{\partial y}(u,v)(y-v) &\le f(x,y) - f(u,v) \\ &\le \frac{\partial f}{\partial x}(x,y)(x-u) + \frac{\partial f}{\partial y}(x,y)(y-v). \end{aligned} \tag{2.3}$$

Taking the integral mean $\frac{1}{A_G}\int\int_G$ in (2.3) over the variables (x,y) we deduce

$$\begin{aligned}(2.4)\qquad &\frac{1}{A_G}\int\int_G\left[\frac{\partial f}{\partial x}(u,v)(x-u)+\frac{\partial f}{\partial y}(u,v)(y-v)\right]dxdy\\ &\le\frac{1}{A_G}\int\int_G f(x,y)\,dxdy-f(u,v)\\ &\le\frac{1}{A_G}\int\int_G\left[\frac{\partial f}{\partial x}(x,y)(x-u)+\frac{\partial f}{\partial y}(x,y)(y-v)\right]dxdy.\end{aligned}$$

Since

$$\begin{aligned}&\frac{1}{A_G}\int\int_G\left[\frac{\partial f}{\partial x}(u,v)(x-u)+\frac{\partial f}{\partial y}(u,v)(y-v)\right]dxdy\\ &=\frac{1}{A_G}\int\int_G\frac{\partial f}{\partial x}(u,v)(x-u)\,dxdy+\frac{1}{A_G}\int\int_G\frac{\partial f}{\partial y}(u,v)(y-v)\,dxdy\\ &=\frac{\partial f}{\partial x}(u,v)(\overline{x_G}-u)+\frac{\partial f}{\partial y}(u,v)(\overline{y_G}-v)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{A_G}\int\int_G\left[\frac{\partial f}{\partial x}(x,y)(x-u)+\frac{\partial f}{\partial y}(x,y)(y-v)\right]dxdy\\ &=\frac{1}{A_G}\int\int_G\frac{\partial f}{\partial x}(x,y)(x-u)\,dxdy+\frac{1}{A_G}\int\int_G\frac{\partial f}{\partial y}(x,y)(y-v)\,dxdy,\end{aligned}$$

hence by (2.4) we get (2.1). □

Corollary 1. *Let $f:G\to\mathbb{R}$ be a differentiable convex function on G. Let*

$$x_S:=\frac{\int\int_G\frac{\partial f}{\partial x}(x,y)\,xdxdy}{\int\int_G\frac{\partial f}{\partial x}(x,y)\,dxdy},\ \ y_S:=\frac{\int\int_G\frac{\partial f}{\partial y}(x,y)\,ydxdy}{\int\int_G\frac{\partial f}{\partial y}(x,y)\,dxdy}.$$

If $(x_S,y_S)\in G$, then

$$\begin{aligned}(2.5)\qquad 0&\le f(x_S,y_S)-\frac{1}{A_G}\int\int_G f(x,y)\,dxdy\\ &\le\frac{\partial f}{\partial x}(x_S,y_S)(x_S-\overline{x_G})+\frac{\partial f}{\partial y}(x_S,y_S)(y_S-\overline{y_G}).\end{aligned}$$

Proof. If we take in (2.1) $(u,v)=(x_S,y_S)\in G$, then we get

$$\begin{aligned}&\frac{\partial f}{\partial x}(x_S,y_S)(\overline{x_G}-x_S)+\frac{\partial f}{\partial y}(x_S,y_S)(\overline{y_G}-y_S)\\ &\le\frac{1}{A_G}\int\int_G f(x,y)\,dxdy-f(x_S,y_S)\le 0,\end{aligned}$$

which is equivalent to (2.5). □

Corollary 2. *Let $f:G\to\mathbb{R}$ be a differentiable convex function on G. If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ satisfy the conditions*

$$(2.6)\qquad m_1\le\frac{\partial f}{\partial x}(x,y)\le M_1,\ \ m_2\le\frac{\partial f}{\partial y}(x,y)\le M_2\ \text{for any}\ (x,y)\in G$$

for some m_1, m_2, M_1 and M_2, then we have

$$0 \leq \frac{1}{A_G} \int \int_G f(x,y)\, dxdy - f(\overline{x_G}, \overline{y_G}) \tag{2.7}$$
$$\leq \frac{1}{2}(M_1 - m_1) \frac{1}{A_G} \int \int_G |x - \overline{x_G}|\, dxdy + \frac{1}{2}(M_2 - m_2) \frac{1}{A_G} \int \int_G |y - \overline{y_G}|\, dxdy.$$

Proof. Observe that for all α, β real numbers we have

$$\begin{aligned} &\frac{1}{A_G} \int \int_G \left[\frac{\partial f}{\partial x}(x,y) - \alpha\right](x - \overline{x_G})\, dxdy \\ &= \frac{1}{A_G} \int \int_G \frac{\partial f}{\partial x}(x,y)(x - \overline{x_G})\, dxdy + \alpha \frac{1}{A_G} \int \int_G (x - \overline{x_G})\, dxdy \\ &= \frac{1}{A_G} \int \int_G \frac{\partial f}{\partial x}(x,y)(x - \overline{x_G})\, dxdy \end{aligned}$$

and, similarly

$$\frac{1}{A_G} \int \int_G \left[\frac{\partial f}{\partial y}(x,y) - \beta\right](y - \overline{y_G})\, dxdy = \frac{1}{A_G} \int \int_G \frac{\partial f}{\partial y}(x,y)(y - \overline{y_G})\, dxdy.$$

If $f : G \to \mathbb{R}$ is a differentiable function on G, then for all α, β real numbers we have the following equality of interest in itself

$$\begin{aligned} &\frac{1}{A_G} \int \int_G \frac{\partial f}{\partial x}(x,y)(x - \overline{x_G})\, dxdy + \frac{1}{A_G} \int \int_G \frac{\partial f}{\partial y}(x,y)(y - \overline{y_G})\, dxdy \\ &\quad = \frac{1}{A_G} \int \int_G \left[\frac{\partial f}{\partial x}(x,y) - \alpha\right](x - \overline{x_G})\, dxdy \\ &\qquad + \frac{1}{A_G} \int \int_G \left[\frac{\partial f}{\partial y}(x,y) - \beta\right](y - \overline{y_G})\, dxdy. \end{aligned} \tag{2.8}$$

Now, if $f : G \to \mathbb{R}$ is a differentiable convex function on G and the condition (2.6) is satisfied, then

$$\begin{aligned} &0 \leq \frac{1}{A_G} \int \int_G \frac{\partial f}{\partial x}(x,y)(x - \overline{x_G})\, dxdy + \frac{1}{A_G} \int \int_G \frac{\partial f}{\partial y}(x,y)(y - \overline{y_G})\, dxdy \\ &= \left|\frac{1}{A_G} \int \int_G \frac{\partial f}{\partial x}(x,y)(x - \overline{x_G})\, dxdy + \frac{1}{A_G} \int \int_G \frac{\partial f}{\partial y}(x,y)(y - \overline{y_G})\, dxdy\right| \\ &\quad = \left|\frac{1}{A_G} \int \int_G \left[\frac{\partial f}{\partial x}(x,y) - \frac{m_1 + M_1}{2}\right](x - \overline{x_G})\, dxdy \right. \\ &\qquad \left. + \frac{1}{A_G} \int \int_G \left[\frac{\partial f}{\partial y}(x,y) - \frac{m_2 + M_2}{2}\right](y - \overline{y_G})\, dxdy\right| \end{aligned} \tag{2.9}$$

$$\leq \frac{1}{A_G}\left|\int\int_G \left[\frac{\partial f}{\partial x}(x,y) - \frac{m_1+M_1}{2}\right](y-\overline{y_G})\,dxdy\right|$$
$$+\frac{1}{A_G}\left|\int\int_G \left[\frac{\partial f}{\partial y}(x,y) - \frac{m_2+M_2}{2}\right](y-\overline{y_G})\,dxdy\right|$$
$$\leq \frac{1}{A_G}\int\int_G \left|\frac{\partial f}{\partial x}(x,y) - \frac{m_1+M_1}{2}\right||y-\overline{y_G}|\,dxdy$$
$$+\frac{1}{A_G}\int\int_G \left|\frac{\partial f}{\partial y}(x,y) - \frac{m_2+M_2}{2}\right||y-\overline{y_G}|\,dxdy$$
$$\leq \frac{1}{2}(M_1-m_1)\frac{1}{A_G}\int\int_G |x-\overline{x_G}|\,dxdy$$
$$+\frac{1}{2}(M_2-m_2)\frac{1}{A_G}\int\int_G |y-\overline{y_G}|\,dxdy.$$

By utilising the inequality (2.2) we deduce the desired result (2.7). □

Further, we assume that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on G and satisfy the following Lipschitz type conditions

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(u,v)\right| \leq L_1|x-u| + K_1|y-v| \text{ for all } (x,y),\ (u,v)\in G \tag{2.10}$$

and

$$\left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(u,v)\right| \leq L_2|x-u| + K_2|y-v| \text{ for all } (x,y),\ (u,v)\in G \tag{2.11}$$

where L_1, L_2, K_1 and K_2 are positive given numbers.

Corollary 3. *Let $f: G\to\mathbb{R}$ be a differentiable convex function on G. If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on G satisfy the conditions (2.10) and (2.11) where L_1, L_2, K_1 and K_2 are positive given numbers, then*

$$0 \leq \frac{1}{A_G}\int\int_G f(x,y)\,dxdy - f(\overline{x_G},\overline{y_G}) \tag{2.12}$$
$$\leq L_1\frac{1}{A_G}\int\int_G (x-\overline{x_G})^2\,dxdy + K_2\frac{1}{A_G}\int\int_G (y-\overline{y_G})^2\,dxdy$$
$$+(K_1+L_2)\frac{1}{A_G}\int\int_G |x-\overline{x_G}||y-\overline{y_G}|\,dxdy.$$

Proof. From (2.8) we get

$$\frac{1}{A_G}\int\int_G \frac{\partial f}{\partial x}(x,y)(x-\overline{x_G})\,dxdy + \frac{1}{A_G}\int\int_G \frac{\partial f}{\partial y}(x,y)(y-\overline{y_G})\,dxdy \tag{2.13}$$
$$=\frac{1}{A_G}\int\int_G\left[\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(\overline{x_G},\overline{y_G})\right](x-\overline{x_G})\,dxdy$$
$$+\frac{1}{A_G}\int\int_G\left[\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(\overline{x_G},\overline{y_G})\right](y-\overline{y_G})\,dxdy.$$

If $f : G \to \mathbb{R}$ is a differentiable convex function on G and if the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on G satisfy the conditions (2.10) and (2.11), then

$$\begin{aligned}
0 &\leq \frac{1}{A_G}\int\int_G \frac{\partial f}{\partial x}(x,y)(x-\overline{x_G})\,dxdy + \frac{1}{A_G}\int\int_G \frac{\partial f}{\partial y}(x,y)(y-\overline{y_G})\,dxdy \\
&\leq \frac{1}{A_G}\int\int_G \left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(\overline{x_G},\overline{y_G})\right| |x-\overline{x_G}|\,dxdy \\
&\quad + \frac{1}{A_G}\int\int_G \left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(\overline{x_G},\overline{y_G})\right| |y-\overline{y_G}|\,dxdy \\
&\leq \frac{1}{A_G}\int\int_G \left[L_1|x-\overline{x_G}| + K_1|y-\overline{y_G}|\right]|x-\overline{x_G}|\,dxdy \\
&\quad + \frac{1}{A_G}\int\int_G \left[L_2|x-\overline{x_G}| + K_2|y-\overline{y_G}|\right]|y-\overline{y_G}|\,dxdy \\
&= L_1\frac{1}{A_G}\int\int_G (x-\overline{x_G})^2\,dxdy + K_2\frac{1}{A_G}\int\int_G (y-\overline{y_G})^2\,dxdy \\
&\qquad + (K_1+L_2)\frac{1}{A_G}\int\int_G |x-\overline{x_G}|\,|y-\overline{y_G}|\,dxdy.
\end{aligned}$$

By utilising the inequality (2.2) we deduce the desired result (2.12). □

Theorem 2. *Assume that there exists the constants $m < M$ and $n < N$ such that $G \subset [m,M]\times[n,N]$ and f is convex on the box $[m,M]\times[n,N]$. Then we have*

$$\begin{aligned}
(2.14)\quad &\frac{1}{A_G}\int\int_G f(x,y)\,dxdy \\
&\leq \frac{1}{(M-m)(N-n)}\left[f(m,n)\frac{1}{A_G}\int\int_G (M-x)(N-y)\,dxdy\right. \\
&\quad + f(m,N)\frac{1}{A_G}\int\int_G (M-x)(y-n)\,dxdy \\
&\quad + f(M,n)\frac{1}{A_G}\int\int_G (x-m)(N-y)\,dxdy \\
&\quad \left. + f(M,N)\frac{1}{A_G}\int\int_G (x-m)(y-n)\,dxdy\right].
\end{aligned}$$

Proof. Observe that for $x \in [m,M]$ we have the convex combination

$$x = \frac{M-x}{M-m}m + \frac{x-m}{M-m}M$$

and by the convexity of f in the first variable we have

$$\begin{aligned}
(2.15)\quad f(x,y) &= f\left(\frac{M-x}{M-m}m + \frac{x-m}{M-M}M, y\right) \\
&\leq \frac{M-x}{M-m}f(m,y) + \frac{x-m}{M-m}f(M,y)
\end{aligned}$$

for all $(x,y) \in G$.

Also, for $y \in [n,N]$ we have the convex combination

$$y = \frac{N-y}{N-n}n + \frac{y-n}{N-n}N$$

and by the convexity of f in the second variable we have

$$f(m,y)=f\left(m,\frac{N-y}{N-n}n+\frac{y-n}{N-n}N\right)\leq\frac{N-y}{N-n}f(m,n)+\frac{y-n}{N-n}f(m,N) \tag{2.16}$$

and

$$\begin{aligned} f(M,y)&=f\left(M,\frac{N-y}{N-n}n+\frac{y-n}{N-n}N\right)\\ &\leq\frac{N-y}{N-n}f(M,n)+\frac{y-n}{N-n}f(M,N) \end{aligned} \tag{2.17}$$

for all $(x,y)\in G$.

Using (2.15)-(2.17) we get

$$\begin{aligned} f(x,y)&\leq\frac{M-x}{M-m}f(m,y)+\frac{x-m}{M-m}f(M,y)\\ &\leq\frac{M-x}{M-m}\left[\frac{N-y}{N-n}f(m,n)+\frac{y-n}{N-n}f(m,N)\right]\\ &\qquad+\frac{x-m}{M-m}\left[\frac{N-y}{N-n}f(M,n)+\frac{y-n}{N-n}f(M,N)\right] \end{aligned} \tag{2.18}$$

$$\begin{aligned} =\frac{1}{(M-m)(N-n)}&\left[(M-x)(N-y)f(m,n)+(M-x)(y-n)f(m,N)\right.\\ &\left.+(x-m)(N-y)f(M,n)+(x-m)(y-n)f(M,N)\right] \end{aligned}$$

for all $(x,y)\in G$.

Now, by the integral mean $\frac{1}{A_G}\int\int_G$ in (2.3) over the variables (x,y) we deduce the desired result (2.14). □

3. Examples for Rectangles

If $G=[a,b]\times[c,d]$ is a rectangle from $I\times I$, then

$$A_G=(b-a)(d-c),\ \overline{x_G}=\frac{a+b}{2}\text{ and }\overline{y_G}=\frac{c+d}{2}.$$

If $f:[a,b]\times[c,d]\to\mathbb{R}$ is differentiable convex, then from (2.2) we have

$$\begin{aligned} 0&\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\frac{\partial f}{\partial x}(x,y)\left(x-\frac{a+b}{2}\right)dxdy\\ &\qquad+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\frac{\partial f}{\partial y}(x,y)\left(y-\frac{c+d}{2}\right)dxdy. \end{aligned} \tag{3.1}$$

We also have

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\left|x-\frac{a+b}{2}\right|dxdy=\frac{1}{4}(b-a)$$

and

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\left|y-\frac{c+d}{2}\right|dxdy=\frac{1}{4}(d-c)$$

If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ satisfy the conditions (2.6) on $[a,b] \times [c,d]$ for some m_1, m_2, M_1 and M_2, then from (2.7) we have

$$0 \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dxdy - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{8}(M_1 - m_1)(b-a) + \frac{1}{8}(M_2 - m_2)(d-c). \tag{3.2}$$

We also have

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left(x - \frac{a+b}{2}\right)^2 dxdy = \frac{1}{12}(b-a)^2,$$

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left(y - \frac{c+d}{2}\right)^2 dxdy = \frac{1}{12}(d-c)^2$$

and

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left|x - \frac{a+b}{2}\right| \left|y - \frac{c+d}{2}\right| dxdy = \frac{1}{16}(b-a)(d-c).$$

If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on $[a,b] \times [c,d]$ satisfy the conditions (2.10) and (2.11) where L_1, L_2, K_1 and K_2 are positive given numbers, then from (2.12) we get

$$0 \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dxdy - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{12}(b-a)^2 L_1 + \frac{1}{12}K_2(d-c)^2 + \frac{1}{16}(b-a)(d-c)(K_1 + L_2). \tag{3.3}$$

If we take $[m,M] = [a,b]$ and $[n,N] = [c,d]$ and take into account that

$$\begin{aligned} \int_a^b \int_c^d (b-x)(d-y)\,dxd &= \int_a^b \int_c^d (b-x)(y-c)\,dxdy \\ &= \int_a^b \int_c^d (x-a)(y-c)\,dxdy \\ &= \int_a^b \int_c^d (x-a)(d-y)\,dxdy = \frac{1}{4}(b-a)^2(d-c)^2 \end{aligned}$$

then by (2.14) we get

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dxdy \leq \frac{1}{4}\left[f(a,c) + f(a,d) + f(b,c) + f(b,d)\right]. \tag{3.4}$$

4. Examples for Disks

Consider the disk centered in $C = (a,b)$ and of radius $R > 0$,

$$D(C,R) := \{(x,y) \mid x = r\cos\theta + a,\ y = r\sin\theta + b,\ r \in [0,R],\ \theta \in [0,2\pi]\}.$$

We have for $G = D(C,R)$ that

$$A_G = \pi R^2,\ \overline{x_G} = a \text{ and } \overline{y_G} = b.$$

We also have

$$\frac{1}{A_G}\int\int_G |x-\overline{x_G}|\,dxdy = \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} r^2|\cos\theta|\,drd\theta = \frac{4}{3\pi}R$$

and

$$\frac{1}{A_G}\int\int_G |y-\overline{y_G}|\,dxdy = \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} r^2|\sin\theta|\,drd\theta = \frac{4}{3\pi}R.$$

Let $f : D(C,R) \to \mathbb{R}$ be a differentiable convex function on $D(C,R)$. If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ satisfy the conditions

$$m_1 \le \frac{\partial f}{\partial x}(x,y) \le M_1,\ m_2 \le \frac{\partial f}{\partial y}(x,y) \le M_2 \text{ for any } (x,y)\in D(0,R)$$

then by (2.7) we get

$$0 \le \frac{1}{\pi R^2}\int\int_{D(0,R)} f(x,y)\,dxdy - f(a,b) \le \frac{2}{3\pi}R(M_1-m_1+M_2-m_2). \tag{4.1}$$

We also have for $G = D(C,R)$ that

$$\frac{1}{A_G}\int\int_G (x-\overline{x_G})^2\,dxdy = \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} r^3\cos^2\theta drd\theta = \frac{R^2}{4\pi}\int_0^{2\pi}\cos^2\theta d\theta = \frac{R^2}{4},$$

$$\frac{1}{A_G}\int\int_G (y-\overline{y_G})^2\,dxdy = \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} r^3\sin^2\theta drd\theta = \frac{R^2}{4\pi}\int_0^{2\pi}\sin^2\theta d\theta = \frac{R^2}{4}$$

and

$$\begin{aligned}\frac{1}{A_G}\int\int_G |x-\overline{x_G}|\,|y-\overline{y_G}|\,dxdy &= \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} r^3|\cos\theta\sin\theta|\,drd\theta\\ &= \frac{R^2}{4\pi}\int_0^{2\pi}|\cos\theta\sin\theta|\,d\theta = \frac{R^2}{8\pi}\int_0^{2\pi}|\sin 2\theta|\,d\theta\\ &= \frac{4R^2}{8\pi}\int_0^{\frac{\pi}{2}}\sin 2\theta d\theta = \frac{R^2}{2\pi}.\end{aligned}$$

By utilising (2.12), we then get

$$0 \le \frac{1}{\pi R^2}\int\int_{D(0,R)} f(x,y)\,dxdy - f(a,b) \le \left(\frac{L_1+K_2}{2}+\frac{K_1+L_2}{\pi}\right)\frac{R^2}{2} \tag{4.2}$$

provided that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on $D(C,R)$ and satisfy the conditions (2.10) and (2.11).

Observe that $D(C,R) \subset [a-R,a+R]\times[b-R,b+R]$. Now, if we take $[m,M] = [a-R,a+R]$ and $[n,N] = [b-R,b+R]$ then

$$\begin{aligned}&\frac{1}{A_G}\int\int_G (M-x)(N-y)\,dxdy\\ &= \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}(R-r\cos\theta)(R-r\sin\theta)\,rdrd\theta\\ &= \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}\left[R^2-Rr\cos\theta-Rr\sin\theta+r^2\sin\theta\cos\theta\right]rdrd\theta = R^2.\end{aligned}$$

Similarly,

$$\frac{1}{A_G}\int\int_G (M-x)(y-n)\,dxdy = \frac{1}{A_G}\int\int_G (x-m)(N-y)\,dxdy = \frac{1}{A_G}\int\int_G (x-m)(y-n)\,dxdy = R^2.$$

Now, if we assume that f is convex on the box $[a-R, a+R]\times[b-R, b+R]$ then by (2.14) we get

$$\frac{1}{\pi R^2}\int\int_{D(0,R)} f(x,y)\,dxdy \leq \frac{1}{4}\left[f(a-R,b-R)+f(a-R,b+R)+f(a+R,b-R)+f(a+R,b+R)\right]. \tag{4.3}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa