OSTROWSKI TYPE INTEGRAL INEQUALITIES FOR DOUBLE INTEGRAL ON GENERAL DOMAINS

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ABSTRACT. In this paper we establish some Ostrowski type inequalities for functions of two independent variables defined on closed and bounded convex subsets of the plane $\mathbb{R}^2$. Some examples for rectangles and disks are also provided.

1. INTRODUCTION

In paper [1], the authors obtained among others the following results concerning the difference between the double integral on the disk and the values in the center or the path integral on the circle:

Theorem 1. *If $f : D(C, R) \to \mathbb{R}$ has continuous partial derivatives on $D(C, R)$, the disk centered in the point $C = (a, b)$ with the radius $R > 0$, and*

$$\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial x} \right| < \infty,$$

$$\left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial y} \right| < \infty;$$

then

$$\left| f(C) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dx\, dy \right| \leq \frac{4}{3\pi} R \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right]. \tag{1.1}$$

The constant $\frac{4}{3\pi}$ is sharp.

We also have

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dx\, dy - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\, dl(\gamma) \right| \leq \frac{2R}{3\pi} \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right], \tag{1.2}$$

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where $\sigma(C,R)$ *is the circle centered in* $C=(a,b)$ *with the radius* $R>0$ *and*

$$\left| f(C) - \frac{1}{2\pi R}\int_{\sigma(C,R)} f(\gamma)\,dl(\gamma)\right| \le \frac{2R}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty} + \left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right]. \tag{1.3}$$

In the same paper [1] the authors also established the following Ostrowski type inequality:

Theorem 2. *If* f *has bounded partial derivatives on* $D(0,1)$*, then*

$$\begin{aligned}\left| f(u,v) - \frac{1}{\pi}\iint_{D(0,1)} f(x,y)\,dx\,dy\right| &\le \frac{2}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(0,1),\infty}\left(u\arcsin u + \frac{1}{3}\sqrt{1-u^2}\,(2+u^2)\right)\right.\\ &\left.\quad + \left\|\frac{\partial f}{\partial y}\right\|_{D(0,1),\infty}\left(v\arcsin v + \frac{1}{3}\sqrt{1-v^2}\,(2+v^2)\right)\right]\end{aligned} \tag{1.4}$$

for any $(u,v)\in D(0,1)$.

For other Ostrowski type integral inequalities for double integrals see [2]-[13].

In the following, consider G a closed and bounded convex subset of $\mathbb{R}^2$. Define

$$A_G := \int\int_G dxdy$$

the *area* of G and $(\overline{x_G},\overline{y_G})$ the *centre of mass* for G, where

$$\overline{x_G} := \frac{1}{A_G}\int\int_G xdxdy,\ \overline{y_G} := \frac{1}{A_G}\int\int_G ydxdy.$$

Consider the function of two variables $f=f(x,y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

Motivated by the above results, in this paper we establish some bounds for the absolute value of Ostrowski difference

$$\frac{1}{A_G}\iint_G f(x,y)\,dxdy - f(u,v)$$

and, in particular, for centre of mass difference

$$\frac{1}{A_G}\iint_G f(x,y)\,dxdy - f(\overline{x_G},\overline{y_G})$$

in the general case of closed and bounded convex subset of $\mathbb{R}^2$ and differentiable functions f defined on G with complex values whose partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ satisfy some natural conditions. Some examples for rectangles and disks are also provided.

2. The Main Results

We have:

Lemma 1. *If $f:G\rightarrow\mathbb{C}$ is differentiable on G, then for all $(x,y)\,,\,(u,v)\in G$ and $\lambda,\ \mu\in\mathbb{C}$ we have the equality*

$$f(x,y)=f(u,v)+(x-u)\lambda+(y-v)\mu + (x-u)\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]-\lambda\right)dt + (y-v)\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]-\mu\right)dt. \tag{2.1}$$

Proof. By Taylor's multivariate theorem with integral remainder, we have

$$f(x,y)=f(u,v)+(x-u)\int_0^1\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]\,dt + (y-v)\int_0^1\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]\,dt \tag{2.2}$$

for all $(x,y)\,,\,(u,v)\in G$.

If $\lambda,\ \mu\in\mathbb{C}$, then

$$(x-u)\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]-\lambda\right)dt = (x-u)\int_0^1\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]\,dt-(x-u)\lambda$$

and

$$(y-v)\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]-\mu\right)dt = (y-v)\int_0^1\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]\,dt-(y-v)\mu$$

and by (2.2) we get the desired result (2.1). □

Suppose that $G\subset\mathbb{R}^2$ is a convex subset in $\mathbb{R}^2$. Now, for $\phi,\ \Phi\in\mathbb{C}$, define the sets of complex-valued functions

$$\bar{U}_G(\phi,\Phi) := \left\{f:G\rightarrow\mathbb{C}|\operatorname{Re}\left[(\Phi-f(x,y))\left(\overline{f(x,y)}-\overline{\phi}\right)\right]\geq 0 \text{ for each } (x,y)\in G\right\}$$

and

$$\bar{\Delta}_G(\phi,\Phi) := \left\{f:G\rightarrow\mathbb{C}|\ \left|f(x,y)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi| \text{ for each } (x,y)\in G\right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\phi,\ \Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_G(\phi,\Phi)$ and $\bar{\Delta}_G(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_G(\phi,\Phi)=\bar{\Delta}_G(\phi,\Phi). \tag{2.3}$$

Proof. We observe that for any $w \in \mathbb{C}$ we have the equivalence

$$\left| w - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} \left| \Phi - \phi \right|$$

if and only if

$$\operatorname{Re}\left[(\Phi - w) \left(\overline{w} - \overline{\phi} \right) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4} \left| \Phi - \phi \right|^2 - \left| w - \frac{\phi + \Phi}{2} \right|^2 = \operatorname{Re}\left[(\Phi - w) \left(\overline{w} - \overline{\phi} \right) \right]$$

that holds for any $w \in \mathbb{C}$.

The equality (2.3) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

(2.4)
$$\begin{aligned} \bar{U}_G(\phi, \Phi) = \{ f : G \to \mathbb{C} \mid \ & (\operatorname{Re}\Phi - \operatorname{Re} f(x,y)) (\operatorname{Re} f(x,y) - \operatorname{Re}\phi) \\ & + (\operatorname{Im}\Phi - \operatorname{Im} f(x,y)) (\operatorname{Im} f(x,y) - \operatorname{Im}\phi) \geq 0 \text{ for each } (x,y) \in G \}. \end{aligned}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_G(\phi, \Phi) := \{ f : G \to \mathbb{C} \mid \ & \operatorname{Re}(\Phi) \geq \operatorname{Re} f(x,y) \geq \operatorname{Re}(\phi) \\ & \text{and } \operatorname{Im}(\Phi) \geq \operatorname{Im} f(x,y) \geq \operatorname{Im}(\phi) \text{ for each } (x,y) \in G \}. \end{aligned} \tag{2.5}$$

One can easily observe that $\bar{S}_G(\phi, \Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_G(\phi, \Phi) \subseteq \bar{U}_G(\phi, \Phi). \tag{2.6}$$

We have:

Theorem 3. *Assume that $f : G \to \mathbb{C}$ is differentiable on G and $(u, v) \in G$. Let $(\phi_1, \Phi_1), (\phi_2, \Phi_2) \in \mathbb{C}$ and assume that $\frac{\partial f}{\partial x} \in \bar{\Delta}_G(\phi_1, \Phi_1)$ and $\frac{\partial f}{\partial y} \in \bar{\Delta}_G(\phi_2, \Phi_2)$, then*

$$\begin{aligned} & \left| \frac{1}{A_G} \iint_G f(x,y)\, dxdy - f(u,v) - (\overline{x_G} - u) \frac{\phi_1 + \Phi_1}{2} - (\overline{y_G} - v) \frac{\phi_2 + \Phi_2}{2} \right| \\ & \leq \frac{1}{2} \left| \Phi_1 - \phi_1 \right| \frac{1}{A_G} \iint_G |x - u|\, dxdy + \frac{1}{2} \left| \Phi_2 - \phi_2 \right| \frac{1}{A_G} \iint_G |y - v|\, dxdy. \end{aligned} \tag{2.7}$$

In particular,

$$\begin{aligned} & \left| f(\overline{x_G}, \overline{y_G}) - \frac{1}{A_G} \iint_G f(x,y)\, dxdy \right| \\ & \leq \frac{1}{2} \left| \Phi_1 - \phi_1 \right| \frac{1}{A_G} \iint_G |x - \overline{x_G}|\, dxdy + \frac{1}{2} \left| \Phi_2 - \phi_2 \right| \frac{1}{A_G} \iint_G |y - \overline{y_G}|\, dxdy. \end{aligned} \tag{2.8}$$

Proof. From Lemma 1 we have

$$f(x,y)=f(u,v)+(x-u)\frac{\phi_1+\Phi_1}{2}+(y-v)\frac{\phi_2+\Phi_2}{2}+(x-u)\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]-\frac{\phi_1+\Phi_1}{2}\right)dt+(y-v)\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]-\frac{\phi_2+\Phi_2}{2}\right)dt \tag{2.9}$$

for all (x,y), $(u,v)\in G$.

By taking the integral mean $\frac{1}{A_G}\iint_G$ over (x,y) we get

$$\frac{1}{A_G}\iint_G f(x,y)\,dxdy=f(u,v)+\left(\frac{1}{A_G}\iint_G xdxdy-u\right)\frac{\phi_1+\Phi_1}{2}+\left(\frac{1}{A_G}\iint_G ydxdy-v\right)\frac{\phi_2+\Phi_2}{2}+\frac{1}{A_G}\iint_G(x-u)\left(\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]-\frac{\phi_1+\Phi_1}{2}\right)dt\right)dxdy+\frac{1}{A_G}\iint_G(y-v)\left(\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]-\frac{\phi_2+\Phi_2}{2}\right)dt\right)dxdy \tag{2.10}$$

for all $(u,v)\in G$.

By using the equality (2.10) we get

$$\begin{aligned}&\left|\frac{1}{A_G}\iint_G f(x,y)\,dxdy-f(u,v)-(\overline{x_G}-u)\frac{\phi_1+\Phi_1}{2}-(\overline{y_G}-v)\frac{\phi_2+\Phi_2}{2}\right| \\ &\le\frac{1}{A_G}\left|\iint_G(x-u)\left(\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]-\frac{\phi_1+\Phi_1}{2}\right)dt\right)dxdy\right| \\ &+\frac{1}{A_G}\left|\iint_G(y-v)\left(\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]-\frac{\phi_2+\Phi_2}{2}\right)dt\right)dxdy\right| \\ &\le\frac{1}{A_G}\iint_G|x-u|\left|\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]-\frac{\phi_1+\Phi_1}{2}\right)dt\right|dxdy \\ &+\frac{1}{A_G}\iint_G|y-v|\left|\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]-\frac{\phi_2+\Phi_2}{2}\right)dt\right|dxdy \\ &\le\frac{1}{A_G}\iint_G|x-u|\left(\int_0^1\left|\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]-\frac{\phi_1+\Phi_1}{2}\right|dt\right)dxdy \\ &+\frac{1}{A_G}\iint_G|y-v|\left(\int_0^1\left|\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]-\frac{\phi_2+\Phi_2}{2}\right|dt\right)dxdy=:I\end{aligned} \tag{2.11}$$

By the fact that $\frac{\partial f}{\partial x}\in\bar{\Delta}_G(\phi_1,\Phi_1)$ and $\frac{\partial f}{\partial y}\in\bar{\Delta}_G(\phi_2,\Phi_2)$, it follows that

$$\int_0^1\left|\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]-\frac{\phi_1+\Phi_1}{2}\right|dt\le\frac{1}{2}|\Phi_1-\phi_1|$$

and

$$\int_0^1\left|\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]-\frac{\phi_2+\Phi_2}{2}\right|dt\le\frac{1}{2}|\Phi_2-\phi_2|.$$

Therefore

$$I \leq \frac{1}{2}\left|\Phi_1-\phi_1\right| \frac{1}{A_G} \iint_G |x-u|\, dxdy + \frac{1}{2}\left|\Phi_2-\phi_2\right| \frac{1}{A_G} \iint_G |y-v|\, dxdy$$

and by (2.11) we obtain the desired result (2.7). □

We also have:

Theorem 4. *Assume that $f : G \to \mathbb{C}$ is differentiable on G and $(u,v) \in G$. Then*

$$\begin{aligned} &\left| f(u,v) - \frac{1}{A_G}\iint_G f(x,y)\, dxdy \right| \\ &\leq \frac{1}{A_G}\iint_G |x-u| \left(\int_0^1 \left| \frac{\partial f}{\partial x}\left[t(x,y)+(1-t)(u,v)\right] \right| dt \right) dxdy \\ &\quad + \frac{1}{A_G}\iint_G |y-v| \left(\int_0^1 \left| \frac{\partial f}{\partial y}\left[t(x,y)+(1-t)(u,v)\right] \right| dt \right) dxdy \\ &\leq \left\| \frac{\partial f}{\partial x} \right\|_{G,\infty} \frac{1}{A_G}\iint_G |x-u|\, dxdy + \left\| \frac{\partial f}{\partial y} \right\|_{G,\infty} \frac{1}{A_G}\iint_G |y-v|\, dxdy, \end{aligned} \tag{2.12}$$

provided

$$\left\| \frac{\partial f}{\partial x} \right\|_{G,\infty} := \sup_{(z,w)\in G} \left| \frac{\partial f}{\partial x}(z,w) \right| < \infty \text{ and } \left\| \frac{\partial f}{\partial y} \right\|_{G,\infty} := \sup_{(z,w)\in G} \left| \frac{\partial f}{\partial y}(z,w) \right| < \infty.$$

In particular,

$$\begin{aligned} &\left| f(\overline{x_G},\overline{y_G}) - \frac{1}{A_G}\iint_G f(x,y)\, dxdy \right| \\ &\leq \frac{1}{A_G}\iint_G |x-\overline{x_G}| \left(\int_0^1 \left| \frac{\partial f}{\partial x}\left[t(x,y)+(1-t)(\overline{x_G},\overline{y_G})\right] \right| dt \right) dxdy \\ &\quad + \frac{1}{A_G}\iint_G |y-\overline{y_G}| \left(\int_0^1 \left| \frac{\partial f}{\partial y}\left[t(x,y)+(1-t)(\overline{x_G},\overline{y_G})\right] \right| dt \right) dxdy \\ &\leq \left\| \frac{\partial f}{\partial x} \right\|_{G,\infty} \frac{1}{A_G}\iint_G |x-\overline{x_G}|\, dxdy + \left\| \frac{\partial f}{\partial y} \right\|_{G,\infty} \frac{1}{A_G}\iint_G |y-\overline{y_G}|\, dxdy. \end{aligned} \tag{2.13}$$

Proof. We have from (2.1) that

$$\begin{aligned} f(x,y) = f(u,v) &+ (x-u)\int_0^1 \frac{\partial f}{\partial x}\left[t(x,y)+(1-t)(u,v)\right] dt \\ &+ (y-v)\int_0^1 \frac{\partial f}{\partial y}\left[t(x,y)+(1-t)(u,v)\right] dt. \end{aligned} \tag{2.14}$$

for all (x,y), $(u,v) \in G$.

By taking the integral mean $\frac{1}{A_G}\iint_G$ over (x,y) in (2.14) we get the following identity of interest

$$\begin{aligned}(2.15)\quad \frac{1}{A_G}\iint_G f(x,y)\,dxdy &= f(u,v)\\ &+\frac{1}{A_G}\iint_G (x-u)\left(\int_0^1 \frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]\,dt\right)dxdy\\ &+\frac{1}{A_G}\iint_G (y-v)\left(\int_0^1 \frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]\,dt\right)dxdy.\end{aligned}$$

for all $(u,v)\in G$.

From (2.15) we get

$$\begin{aligned}(2.16)\quad &\left|\frac{1}{A_G}\iint_G f(x,y)\,dxdy - f(u,v)\right|\\ &\le \frac{1}{A_G}\left|\iint_G (x-u)\left(\int_0^1 \frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]\,dt\right)dxdy\right|\\ &+\frac{1}{A_G}\left|\iint_G (y-v)\left(\int_0^1 \frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]\,dt\right)dxdy\right|\\ &\le \frac{1}{A_G}\iint_G |x-u|\left|\int_0^1 \frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]\,dt\right|dxdy\\ &+\frac{1}{A_G}\iint_G |y-v|\left|\int_0^1 \frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]\,dt\right|dxdy\\ &\le \frac{1}{A_G}\iint_G |x-u|\left(\int_0^1 \left|\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]\right|dt\right)dxdy\\ &+\frac{1}{A_G}\iint_G |y-v|\left(\int_0^1 \left|\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]\right|dt\right)dxdy =: J,\end{aligned}$$

which proves the first part of (2.12).

We also have that

$$\left|\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]\right| \le \sup_{(z,w)\in G}\left|\frac{\partial f}{\partial x}(z,w)\right| = \left\|\frac{\partial f}{\partial x}\right\|_{G,\infty}$$

and

$$\left|\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]\right| \le \sup_{(z,w)\in G}\left|\frac{\partial f}{\partial y}(z,w)\right| = \left\|\frac{\partial f}{\partial y}\right\|_{G,\infty}$$

for all $(x,y),(u,v)\in G$ and for all $t\in[0,1]$.

Therefore

$$J \le \left\|\frac{\partial f}{\partial x}\right\|_{G,\infty}\frac{1}{A_G}\iint_G |x-u|\,dxdy + \left\|\frac{\partial f}{\partial y}\right\|_{G,\infty}\frac{1}{A_G}\iint_G |y-v|\,dxdy,$$

which proves the last part of (2.12). □

Remark 1. *If we denote*

$$B_1(u,v) := \frac{1}{A_G}\iint_G |x-u|\left(\int_0^1 \left|\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]\right|dt\right)dxdy$$

and

$$B_2(u,v) := \frac{1}{A_G} \iint_G |y-v| \left(\int_0^1 \left| \frac{\partial f}{\partial y} [t(x,y) + (1-t)(u,v)] \right| dt \right) dxdy$$

then by Hölder's integral inequalities for the double integral we have

$$\begin{aligned} &B_1(u,v) \\ &\leq \frac{1}{A_G} \begin{cases} \sup_{(x,y)\in G} |x-u| \iint_G \left(\int_0^1 \left| \frac{\partial f}{\partial x} [t(x,y)+(1-t)(u,v)] \right| dt \right) dxdy, \\ \\ \left(\iint_G |x-u|^p \, dxdy \right)^{1/p} \left[\iint_G \left(\int_0^1 \left| \frac{\partial f}{\partial x} [t(x,y)+(1-t)(u,v)] \right| dt \right)^q dxdy \right]^{1/q} \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \sup_{(x,y)\in G} \left(\int_0^1 \left| \frac{\partial f}{\partial x} [t(x,y)+(1-t)(u,v)] \right| dt \right) \iint_G |x-u| \, dxdy \end{cases} \\ &= M_1(u,v) \end{aligned}$$

and

$$\begin{aligned} &B_2(u,v) \\ &\leq \frac{1}{A_G} \begin{cases} \sup_{(x,y)\in G} |y-v| \iint_G \left(\int_0^1 \left| \frac{\partial f}{\partial y} [t(x,y)+(1-t)(u,v)] \right| dt \right) dxdy \\ \\ \left(\iint_G |y-v|^p \, dxdy \right)^{1/p} \left[\iint_G \left(\int_0^1 \left| \frac{\partial f}{\partial y} [t(x,y)+(1-t)(u,v)] \right| dt \right)^q dxdy \right]^{1/q} \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \sup_{(x,y)\in G} \left(\int_0^1 \left| \frac{\partial f}{\partial y} [t(x,y)+(1-t)(u,v)] \right| dt \right) \iint_G |y-v| \, dxdy \end{cases} \\ &=: M_2(u,v) \end{aligned}$$

for all $(u,v) \in G$.

Therefore, by the first inequality in (2.12) we obtain

$$\left| f(u,v) - \frac{1}{A_G} \iint_G f(x,y) \, dxdy \right| \leq M_1(u,v) + M_2(u,v) \tag{2.17}$$

for all $(u,v) \in G$.

In particular, we have

$$\left| f(\overline{x_G}, \overline{y_G}) - \frac{1}{A_G} \iint_G f(x,y) \, dxdy \right| \leq M_1(\overline{x_G}, \overline{y_G}) + M_2(\overline{x_G}, \overline{y_G}). \tag{2.18}$$

When the partial derivatives are convex in absolute value, we have:

Corollary 2. *With the assumptions of Theorem 4 and if* $\left|\frac{\partial f}{\partial x}\right|$ *and* $\left|\frac{\partial f}{\partial y}\right|$ *are convex on* G*, then we have*

$$\left|f(u,v)-\frac{1}{A_G}\iint_G f(x,y)\,dxdy\right| \tag{2.19}$$

$$\leq \frac{1}{2}\frac{1}{A_G}\begin{cases} \sup_{(x,y)\in G}|x-u|\iint_G\left[\left|\frac{\partial f}{\partial x}\right|(x,y)+\left|\frac{\partial f}{\partial x}\right|(u,v)\right]dxdy, \\ \left(\iint_G|x-u|^p\,dxdy\right)^{1/p}\left[\iint_G\left[\left|\frac{\partial f}{\partial x}\right|(x,y)+\left|\frac{\partial f}{\partial x}\right|(u,v)\right]^q dxdy\right]^{1/q} \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \sup_{(x,y)\in G}\left(\left[\left|\frac{\partial f}{\partial x}\right|(x,y)+\left|\frac{\partial f}{\partial x}\right|(u,v)\right]\right)\iint_G|x-u|\,dxdy \end{cases}$$

$$+\frac{1}{2}\frac{1}{A_G}\begin{cases} \sup_{(x,y)\in G}|y-v|\iint_G\left[\left|\frac{\partial f}{\partial y}\right|(x,y)+\left|\frac{\partial f}{\partial y}\right|(u,v)\right]dxdy, \\ \left(\iint_G|y-v|^p\,dxdy\right)^{1/p}\left[\iint_G\left[\left|\frac{\partial f}{\partial y}\right|(x,y)+\left|\frac{\partial f}{\partial y}\right|(u,v)\right]^q dxdy\right]^{1/q} \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \sup_{(x,y)\in G}\left[\left|\frac{\partial f}{\partial y}\right|(x,y)+\left|\frac{\partial f}{\partial y}\right|(u,v)\right]\iint_G|y-v|\,dxdy \end{cases}$$

for all $(u,v)\in G$.

In particular,

$$\left|f(\overline{x_G},\overline{y_G})-\frac{1}{A_G}\iint_G f(x,y)\,dxdy\right| \tag{2.20}$$

$$\leq \frac{1}{2}\frac{1}{A_G}\begin{cases} \sup_{(x,y)\in G}|x-\overline{x_G}|\iint_G\left[\left|\frac{\partial f}{\partial x}\right|(x,y)+\left|\frac{\partial f}{\partial x}\right|(\overline{x_G},\overline{y_G})\right]dxdy, \\ \left(\iint_G|x-\overline{x_G}|^p\,dxdy\right)^{1/p}\left[\iint_G\left[\left|\frac{\partial f}{\partial x}\right|(x,y)+\left|\frac{\partial f}{\partial x}\right|(\overline{x_G},\overline{y_G})\right]^q dxdy\right]^{1/q} \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \sup_{(x,y)\in G}\left(\left[\left|\frac{\partial f}{\partial x}\right|(x,y)+\left|\frac{\partial f}{\partial x}\right|(\overline{x_G},\overline{y_G})\right]\right)\iint_G|x-\overline{x_G}|\,dxdy \end{cases}$$

$$+\frac{1}{2}\frac{1}{A_G}\begin{cases} \sup_{(x,y)\in G}|y-\overline{y_G}|\iint_G\left[\left|\frac{\partial f}{\partial y}\right|(x,y)+\left|\frac{\partial f}{\partial y}\right|(\overline{x_G},\overline{y_G})\right]dxdy, \\ \left(\iint_G|y-\overline{y_G}|^p\,dxdy\right)^{1/p}\left[\iint_G\left[\left|\frac{\partial f}{\partial y}\right|(x,y)+\left|\frac{\partial f}{\partial y}\right|(\overline{x_G},\overline{y_G})\right]^q dxdy\right]^{1/q} \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \sup_{(x,y)\in G}\left[\left|\frac{\partial f}{\partial y}\right|(x,y)+\left|\frac{\partial f}{\partial y}\right|(\overline{x_G},\overline{y_G})\right]\iint_G|y-\overline{y_G}|\,dxdy. \end{cases}$$

Remark 2. *From (2.20) we have*

$$\left|f\left(\overline{x_G},\overline{y_G}\right)-\frac{1}{A_G}\iint_G f\left(x,y\right)dxdy\right| \tag{2.21}$$
$$\leq \frac{1}{2}\sup_{(x,y)\in G}\left|x-\overline{x_G}\right|\left[\frac{1}{A_G}\iint_G\left|\frac{\partial f}{\partial x}\right|\left(x,y\right)dxdy+\left|\frac{\partial f}{\partial x}\right|\left(\overline{x_G},\overline{y_G}\right)\right],$$
$$+\frac{1}{2}\sup_{(x,y)\in G}\left|y-\overline{y_G}\right|\left[\frac{1}{A_G}\iint_G\left|\frac{\partial f}{\partial y}\right|\left(x,y\right)dxdy+\left|\frac{\partial f}{\partial y}\right|\left(\overline{x_G},\overline{y_G}\right)\right]$$

and

$$\left|f\left(\overline{x_G},\overline{y_G}\right)-\frac{1}{A_G}\iint_G f\left(x,y\right)dxdy\right| \tag{2.22}$$
$$\leq \frac{1}{2}\left[\sup_{(x,y)\in G}\left|\frac{\partial f}{\partial x}\right|\left(x,y\right)+\left|\frac{\partial f}{\partial x}\right|\left(\overline{x_G},\overline{y_G}\right)\right]\frac{1}{A_G}\iint_G\left|x-\overline{x_G}\right|dxdy$$
$$+\frac{1}{2}\left[\sup_{(x,y)\in G}\left|\frac{\partial f}{\partial y}\right|\left(x,y\right)+\left|\frac{\partial f}{\partial y}\right|\left(\overline{x_G},\overline{y_G}\right)\right]\frac{1}{A_G}\iint_G\left|y-\overline{y_G}\right|dxdy.$$

3. Examples for Rectangles

If $G=[a,b]\times[c,d]$ is a rectangle from $\mathbb{R}^2$, then

$$A_{[a,b]\times[c,d]}=(b-a)(d-c),\ \overline{x_{[a,b]\times[c,d]}}=\frac{a+b}{2}\text{ and }\overline{y_{[a,b]\times[c,d]}}=\frac{c+d}{2}.$$

Assume that $f:[a,b]\times[c,d]\to\mathbb{C}$ is differentiable on $[a,b]\times[c,d]$ and $(u,v)\in[a,b]\times[c,d]$. Let $(\phi_1,\Phi_1),(\phi_2,\Phi_2)\in\mathbb{C}$ and assume that $\frac{\partial f}{\partial x}\in\bar{\Delta}_{[a,b]\times[c,d]}(\phi_1,\Phi_1)$ and $\frac{\partial f}{\partial y}\in\bar{\Delta}_{[a,b]\times[c,d]}(\phi_2,\Phi_2)$, then by Theorem 3

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy-f(u,v)\right. \tag{3.1}$$
$$\left.-\left(\frac{a+b}{2}-u\right)\frac{\phi_1+\Phi_1}{2}-\left(\frac{c+d}{2}-v\right)\frac{\phi_2+\Phi_2}{2}\right|$$
$$\leq\frac{1}{2}\left|\Phi_1-\phi_1\right|\left[\frac{1}{4}+\left(\frac{u-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)$$
$$+\frac{1}{2}\left|\Phi_2-\phi_2\right|\left[\frac{1}{4}+\left(\frac{v-\frac{c+d}{2}}{d-c}\right)^2\right](d-c).$$

In particular,

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy\right| \tag{3.2}$$
$$\leq\frac{1}{8}\left[\left|\Phi_1-\phi_1\right|(b-a)+\left|\Phi_2-\phi_2\right|(d-c)\right].$$

Assume that $f : [a,b] \times [c,d] \to \mathbb{C}$ is differentiable on G and $(u,v) \in [a,b] \times [c,d]$. Then by Theorem 4

$$\left| f(u,v) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dxdy \right| \leq \left\| \frac{\partial f}{\partial x} \right\|_{[a,b]\times[c,d],\infty} \left[\frac{1}{4} + \left(\frac{u - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) + \left\| \frac{\partial f}{\partial y} \right\|_{[a,b]\times[c,d],\infty} \left[\frac{1}{4} + \left(\frac{v - \frac{c+d}{2}}{d-c} \right)^2 \right] (d-c), \tag{3.3}$$

provided

$$\left\| \frac{\partial f}{\partial x} \right\|_{[a,b]\times[c,d],\infty} := \sup_{(z,w)\in[a,b]\times[c,d]} \left| \frac{\partial f}{\partial x}(z,w) \right| < \infty \text{ and}$$
$$\left\| \frac{\partial f}{\partial y} \right\|_{[a,b]\times[c,d],\infty} := \sup_{(z,w)\in[a,b]\times[c,d]} \left| \frac{\partial f}{\partial y}(z,w) \right| < \infty.$$

In particular,

$$\left| f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dxdy \right| \leq \frac{1}{4} \left[\left\| \frac{\partial f}{\partial x} \right\|_{[a,b]\times[c,d],\infty} (b-a) + \left\| \frac{\partial f}{\partial y} \right\|_{[a,b]\times[c,d],\infty} (d-c) \right]. \tag{3.4}$$

4. Examples for Disks

We start with the following identity that was stated in [1] without a proof:

Lemma 2. *For any $u \in [-1,1]$ we have*

$$\iint_{D(0,1)} |x-u|\, dxdy = 2\left[u \arcsin(u) + \frac{2+u^2}{3}\sqrt{1-u^2} \right]. \tag{4.1}$$

Proof. We have for any continuous function f defined on the disk $D(0,1)$ that

$$\iint_{D(0,1)} f(x,y)\, dxdy = \int_{-1}^1 \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\, dy \right) dx.$$

Therefore

$$\begin{aligned} \iint_{D(0,1)} |x-u|\, dxdy &= \int_{-1}^1 \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} |x-u|\, dy \right) dx \\ &= 2\int_{-1}^1 |x-u| \sqrt{1-x^2} dx \\ &= 2\left[\int_{-1}^u (u-x)\sqrt{1-x^2} dx + \int_u^1 (x-u)\sqrt{1-x^2} dx \right] \end{aligned}$$

for any $u \in [-1,1]$.

Put

$$I_1(u) := \int_{-1}^{u} (u-x)\sqrt{1-x^2}dx = u\int_{-1}^{u}\sqrt{1-x^2}dx - \int_{-1}^{u} x\sqrt{1-x^2}dx.$$

Using the change of variable $x = \sin s$, $s \in \left[-\frac{\pi}{2}, \arcsin(u)\right]$ we have

$$\begin{aligned}
&\int_{-1}^{u}\sqrt{1-x^2}dx \\
&= \int_{-\frac{\pi}{2}}^{\arcsin(u)} \cos^2 s ds = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\arcsin(u)} [1+\cos(2s)]\, ds \\
&= \frac{1}{2}\left(s + \frac{1}{2}\sin(2s)\right)\Big|_{-\frac{\pi}{2}}^{\arcsin(u)} = \frac{1}{2}\left(s + \sin s\cos s\right)\Big|_{-\frac{\pi}{2}}^{\arcsin(u)} \\
&= \frac{1}{2}\left(\arcsin(u) + \sin[\arcsin(u)]\cos[\arcsin(u)]\right) + \frac{\pi}{4} \\
&= \frac{1}{2}\left(\arcsin(u) + u\sqrt{1-u^2} + \frac{\pi}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\int_{-1}^{u} x\sqrt{1-x^2}dx &= \int_{-\frac{\pi}{2}}^{\arcsin(u)} \cos^2 s\sin s ds = -\int_{-\frac{\pi}{2}}^{\arcsin(u)} \cos^2 s d(\cos s) \\
&= -\left(\frac{\cos^3 s}{3}\right)\Big|_{-\frac{\pi}{2}}^{\arcsin(u)} = -\frac{1}{3}\cos^3[\arcsin(u)] \\
&= -\frac{1}{3}\left(1-u^2\right)\sqrt{1-u^2}
\end{aligned}$$

for all $u \in [-1,1]$.

Therefore

$$\begin{aligned}
I_1(u) &= \frac{1}{2}u\left(\arcsin(u) + u\sqrt{1-u^2} + \frac{\pi}{2}\right) + \frac{1}{3}\left(1-u^2\right)\sqrt{1-u^2} \\
&= \frac{1}{2}u\arcsin(u) + \frac{1}{2}u^2\sqrt{1-u^2} + \frac{1}{2}u\frac{\pi}{2} + \frac{1}{3}\left(1-u^2\right)\sqrt{1-u^2} \\
&= \frac{\pi}{4}u + \frac{1}{2}u\arcsin(u) + \frac{1}{3}\left(1+\frac{1}{2}u^2\right)\sqrt{1-u^2},
\end{aligned}$$

for all $u \in [-1,1]$.

Further, put

$$I_2(u) := \int_{u}^{1} (x-u)\sqrt{1-x^2}dx = \int_{u}^{1} x\sqrt{1-x^2}dx - u\int_{u}^{1}\sqrt{1-x^2}dx$$

for $u \in [-1,1]$.

As above, we have

$$\int_{u}^{1} x\sqrt{1-x^2}dx = -\left(\frac{\cos^3 s}{3}\right)\Big|_{\arcsin(u)}^{\frac{\pi}{2}} = \frac{1}{3}\left(1-u^2\right)\sqrt{1-u^2}$$

and

$$\int_u^1 \sqrt{1-x^2}dx = \frac{1}{2}\left(s+\sin s\cos s\right)\Big|_{\arcsin(u)}^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2}-\arcsin(u)-u\sqrt{1-u^2}\right)$$

for all $u \in [-1,1]$.

Therefore

$$\begin{aligned} I_2(u) &= \frac{1}{3}\left(1-u^2\right)\sqrt{1-u^2}-\frac{1}{2}u\left(\frac{\pi}{2}-\arcsin(u)-u\sqrt{1-u^2}\right) \\ &= \frac{1}{3}\left(1-u^2\right)\sqrt{1-u^2}-\frac{\pi}{4}u+\frac{1}{2}u\arcsin(u)+\frac{1}{2}u^2\sqrt{1-u^2} \\ &= -\frac{\pi}{4}u+\frac{1}{2}u\arcsin(u)+\frac{1}{3}\left(1+\frac{1}{2}u^2\right)\sqrt{1-u^2} \end{aligned}$$

for all $u \in [-1,1]$.

We then have

$$\begin{aligned} I_1(u)+I_2(u) &= \frac{\pi}{4}u+\frac{1}{2}u\arcsin(u)+\frac{1}{3}\left(1+\frac{1}{2}u^2\right)\sqrt{1-u^2} \\ &\quad -\frac{\pi}{4}u+\frac{1}{2}u\arcsin(u)+\frac{1}{3}\left(1+\frac{1}{2}u^2\right)\sqrt{1-u^2} \\ &= u\arcsin(u)+\frac{2}{3}\left(1+\frac{1}{2}u^2\right)\sqrt{1-u^2} \\ &= u\arcsin(u)+\frac{1}{3}\left(2+u^2\right)\sqrt{1-u^2}, \end{aligned}$$

which gives the desired result (4.1). □

Corollary 3. *For any $u \in [-R,R]$, $R>0$ we have*

$$\int\int_{D(0,R)} |x-u|\,dxdy = 2R^2\left[u\arcsin\left(\frac{u}{R}\right)+\frac{2R^2+u^2}{3R^2}\sqrt{R^2-u^2}\right]. \tag{4.2}$$

Proof. We have

$$\begin{aligned} \int\int_{D(0,R)} |x-u|\,dxdy &= \int_{-R}^{R}\left(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} |x-u|\,dy\right)dx \\ &= 2\int_{-R}^{R}|x-u|\sqrt{R^2-x^2}dx. \end{aligned}$$

If we change the variable as $x = Rt$, $t \in [-1, 1]$ we have $dx = Rdt$ and

$$\begin{aligned}
&\int_{-R}^{R} |x-u| \sqrt{R^2 - x^2} dx \\
&= \int_{-1}^{1} |Rt - u| \sqrt{R^2 - R^2 t^2} R dt \\
&= R^3 \int_{-1}^{1} \left| t - \frac{u}{R} \right| \sqrt{1-t^2} dt \text{ (and by Lemma 2)} \\
&= R^3 \left[\frac{u}{R} \arcsin\left(\frac{u}{R}\right) + \frac{1}{3}\left(2 + \left(\frac{u}{R}\right)^2\right) \sqrt{1 - \left(\frac{u}{R}\right)^2} \right] \\
&= R^3 \left[\frac{u}{R} \arcsin\left(\frac{u}{R}\right) + \frac{1}{3R}\left(\frac{2R^2 + u^2}{R^2}\right) \sqrt{R^2 - u^2} \right] \\
&= R^2 \left[u \arcsin\left(\frac{u}{R}\right) + \frac{1}{3}\left(\frac{2R^2 + u^2}{R^2}\right) \sqrt{R^2 - u^2} \right],
\end{aligned}$$

which proves (4.2). □

Consider now the disk $D(C, R)$ centered in $C = (a, b) \in \mathbb{R}^2$ and having the radius $R > 0$. We have:

Corollary 4. *For any* $u \in [a - R, a + R]$, $R > 0$ *we have*

$$\begin{aligned}
&\iint_{D(C,R)} |x-u| \, dxdy \\
&= 2R^2 \left[(u-a) \arcsin\left(\frac{u-a}{R}\right) + \frac{2R^2 + (u-a)^2}{3R^2} \sqrt{R^2 - (u-a)^2} \right].
\end{aligned} \tag{4.3}$$

Proof. We have

$$\begin{aligned}
\iint_{D(C,R)} |x-u| \, dxdy &= \int_{a-R}^{a+R} \left(\int_{b-\sqrt{R^2-(x-a)^2}}^{b+\sqrt{R^2-(x-a)^2}} |x-u| \, dy \right) dx \\
&= 2 \int_{a-R}^{a+R} |x-u| \sqrt{R^2 - (x-a)^2} dx.
\end{aligned}$$

Using the change of variable $p = x - a$, $x \in [a - R, a + R]$ we have

$$\begin{aligned}
&\int_{a-R}^{a+R} |x-u| \sqrt{R^2 - (x-a)^2} dx \\
&= \int_{-R}^{R} |p + a - u| \sqrt{R^2 - p^2} dp \\
&= \int_{-R}^{R} |p - (u-a)| \sqrt{R^2 - p^2} dp \text{ (and by Corollary 3)} \\
&= R^2 \left[(u-a) \arcsin\left(\frac{u-a}{R}\right) + \frac{2R^2 + (u-a)^2}{3R^2} \sqrt{R^2 - (u-a)^2} \right],
\end{aligned}$$

which proves the desired result (4.3). □

Assume that $f : D(C,R) \to \mathbb{C}$ is differentiable on $D(C,R)$ and $(u,v) \in D(C,R)$. Let (ϕ_1, Φ_1), $(\phi_2, \Phi_2) \in \mathbb{C}$ and assume that $\frac{\partial f}{\partial x} \in \bar{\Delta}_{D(C,R)}(\phi_1, \Phi_1)$ and $\frac{\partial f}{\partial y} \in \bar{\Delta}_{D(C,R)}(\phi_2, \Phi_2)$, then $\overline{x_{D(C,R)}} = a$, $\overline{y_G} = b$ and by the inequality (2.7) we have

$$\begin{aligned}
&\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dxdy - f(u,v) \right. \\
&\qquad \left. - (a-u)\frac{\phi_1 + \Phi_1}{2} - (b-v)\frac{\phi_2 + \Phi_2}{2} \right| \\
&\le \frac{1}{\pi}|\Phi_1 - \phi_1| \left[(u-a)\arcsin\left(\frac{u-a}{R}\right) + \frac{2R^2 + (u-a)^2}{3R^2}\sqrt{R^2 - (u-a)^2} \right] \\
&\quad + \frac{1}{\pi}|\Phi_2 - \phi_2| \left[(v-b)\arcsin\left(\frac{v-b}{R}\right) + \frac{2R^2 + (v-b)^2}{3R^2}\sqrt{R^2 - (v-b)^2} \right].
\end{aligned} \tag{4.4}$$

In particular, we have

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dxdy - f(a,b) \right| \le \frac{2}{3\pi} R\left[|\Phi_1 - \phi_1| + |\Phi_2 - \phi_2| \right]. \tag{4.5}$$

Assume that $f : D(C,R) \to \mathbb{C}$ is differentiable on $D(C,R)$ and $(u,v) \in D(C,R)$. Then by (2.12) we have

$$\begin{aligned}
&\left| f(u,v) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dxdy \right| \\
&\le \frac{1}{\pi R^2} \iint_{D(C,R)} |x-u| \left(\int_0^1 \left| \frac{\partial f}{\partial x}[t(x,y) + (1-t)(u,v)] \right| dt \right) dxdy \\
&\quad + \frac{1}{\pi R^2} \iint_{D(C,R)} |y-v| \left(\int_0^1 \left| \frac{\partial f}{\partial y}[t(x,y) + (1-t)(u,v)] \right| dt \right) dxdy \\
&\le \frac{2}{\pi} \left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} \left[(u-a)\arcsin\left(\frac{u-a}{R}\right) + \frac{2R^2 + (u-a)^2}{3R^2}\sqrt{R^2 - (u-a)^2} \right] \\
&\quad + \frac{2}{\pi} \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \left[(v-b)\arcsin\left(\frac{v-b}{R}\right) + \frac{2R^2 + (v-b)^2}{3R^2}\sqrt{R^2 - (v-b)^2} \right],
\end{aligned} \tag{4.6}$$

provided

$$\begin{aligned}
\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} &:= \sup_{(z,w)\in D(C,R)} \left| \frac{\partial f}{\partial x}(z,w) \right| < \infty \text{ and} \\
\left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} &:= \sup_{(z,w)\in D(C,R)} \left| \frac{\partial f}{\partial y}(z,w) \right| < \infty.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
(4.7)\quad & \left| f(a,b) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dxdy \right| \\
&\leq \frac{1}{\pi R^2} \iint_{D(C,R)} |x-a| \left(\int_0^1 \left| \frac{\partial f}{\partial x}\left[t(x,y)+(1-t)(a,b)\right] \right| dt \right) dxdy \\
&+ \frac{1}{\pi R^2} \iint_{D(C,R)} |y-b| \left(\int_0^1 \left| \frac{\partial f}{\partial y}\left[t(x,y)+(1-t)(a,b)\right] \right| dt \right) dxdy \\
&\leq \frac{4}{3\pi} R \left(\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right).
\end{aligned}$$

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