OSTROWSKI TYPE INTEGRAL INEQUALITIES FOR DOUBLE INTEGRAL OF FUNCTIONS WITH LIPSCHITZIAN PARTIAL DERIVATIVES

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Abstract. In this paper we establish some Ostrowski type inequalities for functions of two independent variables defined on closed and bounded convex subsets of the plane $\mathbb{R}^2$ and whose partial derivatives are Lipschitzian. Some examples for rectangles and disks are also provided.

1. Introduction

In paper [1], the authors obtained among others the following results concerning the difference between the double integral on the disk and the values in the center or the path integral on the circle:

Theorem 1. *If $f : D(C, R) \to \mathbb{R}$ has continuous partial derivatives on $D(C, R)$, the disk centered in the point $C = (a, b)$ with the radius $R > 0$, and*

$$\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial x} \right| < \infty,$$
$$\left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial y} \right| < \infty;$$

then

$$\left| f(C) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dx\, dy \right| \leq \frac{4}{3\pi} R \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right]. \tag{1.1}$$

The constant $\frac{4}{3\pi}$ is sharp.

We also have

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dx\, dy - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\, dl(\gamma) \right| \leq \frac{2R}{3\pi} \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right], \tag{1.2}$$

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Ostrowski inequality, Hermite-Hadamard inequality, Double integral inequalities.

RGMIA Res. Rep. Coll. 22 (2019), Art. 48, 12 pp. Received 30/04/19

where $\sigma(C,R)$ *is the circle centered in* $C=(a,b)$ *with the radius* $R>0$ *and*

$$\left|f(C)-\frac{1}{2\pi R}\int_{\sigma(C,R)}f(\gamma)\,dl(\gamma)\right| \leq \frac{2R}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty}+\left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right]. \tag{1.3}$$

In the same paper [1] the authors also established the following Ostrowski type inequality:

Theorem 2. *If* f *has bounded partial derivatives on* $D(0,1)$*, then*

$$\begin{aligned}\left|f(u,v)-\frac{1}{\pi}\iint_{D(0,1)}f(x,y)\,dx\,dy\right| &\leq \frac{2}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(0,1),\infty}\left(u\arcsin u+\frac{1}{3}\sqrt{1-u^2}\,(2+u^2)\right)\right.\\ &\quad\left.+\left\|\frac{\partial f}{\partial y}\right\|_{D(0,1),\infty}\left(v\arcsin v+\frac{1}{3}\sqrt{1-v^2}\,(2+v^2)\right)\right]\end{aligned} \tag{1.4}$$

for any $(u,v)\in D(0,1)$.

For other Ostrowski type integral inequalities for double integrals see [2]-[13].

In the following, consider G a closed and bounded convex subset of $\mathbb{R}^2$. Define

$$A_G:=\int\int_G dxdy$$

the *area* of G and $(\overline{x_G},\overline{y_G})$ the *centre of mass* for G, where

$$\overline{x_G}:=\frac{1}{A_G}\int\int_G xdxdy,\ \overline{y_G}:=\frac{1}{A_G}\int\int_G ydxdy.$$

Consider the function of two variables $f=f(x,y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

Motivated by the above results, in this paper we establish some bounds for the absolute value of Ostrowski difference

$$\frac{1}{A_G}\iint_G f(x,y)\,dxdy-f(u,v)$$

and, in particular, for centre of mass difference

$$\frac{1}{A_G}\iint_G f(x,y)\,dxdy-f(\overline{x_G},\overline{y_G})$$

in the general case of closed and bounded convex subset of $\mathbb{R}^2$ and differentiable functions f defined on G with complex values whose partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ satisfy some Lipschitz type conditions. Some examples for rectangles and disks are also provided.

2. The Main Results

We have:

Lemma 1. *If $f : G \to \mathbb{C}$ is differentiable on G, then for all (x, y), $(u, v) \in G$ and λ, $\mu \in \mathbb{C}$ we have the equality*

$$
\begin{aligned}
f(x,y) &= f(u,v) + (x-u)\lambda + (y-v)\mu \\
&\quad + (x-u)\int_0^1 \left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)] - \lambda\right) dt \\
&\quad + (y-v)\int_0^1 \left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)] - \mu\right) dt.
\end{aligned} \tag{2.1}
$$

Proof. By Taylor's multivariate theorem with integral remainder, we have

$$
\begin{aligned}
f(x,y) &= f(u,v) + (x-u)\int_0^1 \frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]\, dt \\
&\quad + (y-v)\int_0^1 \frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]\, dt
\end{aligned} \tag{2.2}
$$

for all (x, y), $(u, v) \in G$.

If λ, $\mu \in \mathbb{C}$, then

$$
\begin{aligned}
&(x-u)\int_0^1 \left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)] - \lambda\right) dt \\
&= (x-u)\int_0^1 \frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)]\, dt - (x-u)\lambda
\end{aligned}
$$

and

$$
\begin{aligned}
&(y-v)\int_0^1 \left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)] - \mu\right) dt \\
&= (y-v)\int_0^1 \frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)]\, dt - (y-v)\mu
\end{aligned}
$$

and by (2.2) we get the desired result (2.1). □

We assume that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions

$$
\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(u,v)\right| \le L_1|x-u| + K_1|y-v| \tag{2.3}
$$

and

$$
\left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(u,v)\right| \le L_2|x-u| + K_2|y-v| \tag{2.4}
$$

for any (x, y), $(u, v) \in G$, where L_1, K_1, L_2 and K_2 are given positive constants.

Theorem 3. *If $f : G \to \mathbb{C}$ is differentiable on G and the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions (2.3) and (2.4), then for all $(u, v) \in G$ we* *have*

$$\left|\frac{1}{A_G}\iint_G f(x,y)\,dxdy - f(u,v) - (\overline{x_G}-u)\frac{\partial f}{\partial x}(u,v) - (\overline{y_G}-v)\frac{\partial f}{\partial y}(u,v)\right| \tag{2.5}$$
$$\leq \frac{1}{2}\left[L_1\frac{1}{A_G}\iint_G (x-u)^2\,dxdy + K_2\frac{1}{A_G}\iint_G (y-v)^2\,dxdy\right]$$
$$+\frac{K_1+L_2}{2}\frac{1}{A_G}\iint_G |x-u|\,|y-v|\,dxdy.$$

In particular,

$$\left|\frac{1}{A_G}\iint_G f(x,y)\,dxdy - f(\overline{x_G},\overline{y_G})\right| \tag{2.6}$$
$$\leq \frac{1}{2}\left[L_1\frac{1}{A_G}\iint_G (x-\overline{x_G})^2\,dxdy + K_2\frac{1}{A_G}\iint_G (y-\overline{y_G})^2\,dxdy\right]$$
$$+\frac{K_1+L_2}{2}\frac{1}{A_G}\iint_G |x-\overline{x_G}|\,|y-\overline{y_G}|\,dxdy.$$

Proof. From Lemma 1 we have for $\lambda = \frac{\partial f}{\partial x}(u,v)$ and $\mu = \frac{\partial f}{\partial y}(u,v)$

$$f(x,y) = f(u,v) + (x-u)\frac{\partial f}{\partial x}(u,v) + (y-v)\frac{\partial f}{\partial y}(u,v) \tag{2.7}$$
$$+(x-u)\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial x}(u,v)\right)dt$$
$$+(y-v)\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial y}(u,v)\right)dt$$

for any (x,y), $(u,v)\in G$.

By taking the integral mean $\frac{1}{A_G}\iint_G$ over (x,y) we get

$$\frac{1}{A_G}\iint_G f(x,y)\,dxdy = f(u,v) + \left(\frac{1}{A_G}\iint_G x\,dxdy - u\right)\frac{\partial f}{\partial x}(u,v) \tag{2.8}$$
$$+\left(\frac{1}{A_G}\iint_G y\,dxdy - v\right)\frac{\partial f}{\partial y}(u,v)$$
$$+\frac{1}{A_G}\iint_G (x-u)\left(\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial x}(u,v)\right)dt\right)dxdy$$
$$+\frac{1}{A_G}\iint_G (y-v)\left(\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial y}(u,v)\right)dt\right)dxdy$$

for all $(u,v)\in G$.

By using the equality (2.8) we get

$$\left|\frac{1}{A_G}\iint_G f(x,y)\,dxdy - f(u,v) - (\overline{x_G}-u)\frac{\partial f}{\partial x}(u,v) - (\overline{y_G}-v)\frac{\partial f}{\partial y}(u,v)\right| \tag{2.9}$$
$$\leq \frac{1}{A_G}\left|\iint_G (x-u)\left(\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial x}(u,v)\right)dt\right)dxdy\right|$$
$$+\frac{1}{A_G}\left|\iint_G (y-v)\left(\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial y}(u,v)\right)dt\right)dxdy\right|$$

$$
\begin{aligned}
&\le \frac{1}{A_G}\iint_G \left|(x-u)\left(\int_0^1 \left(\frac{\partial f}{\partial x}\left[t(x,y)+(1-t)(u,v)\right]-\frac{\partial f}{\partial x}(u,v)\right)dt\right)\right| dxdy \\
&+\frac{1}{A_G}\iint_G \left|(y-v)\left(\int_0^1 \left(\frac{\partial f}{\partial y}\left[t(x,y)+(1-t)(u,v)\right]-\frac{\partial f}{\partial y}(u,v)\right)dt\right)\right| dxdy \\
&\le \frac{1}{A_G}\iint_G |x-u|\int_0^1 \left|\frac{\partial f}{\partial x}\left[t(x,y)+(1-t)(u,v)\right]-\frac{\partial f}{\partial x}(u,v)\right| dtdxdy \\
&+\frac{1}{A_G}\iint_G |y-v|\left(\int_0^1 \left|\frac{\partial f}{\partial y}\left[t(x,y)+(1-t)(u,v)\right]-\frac{\partial f}{\partial y}(u,v)\right| dt\right) dxdy \\
&\le \frac{1}{2}\frac{1}{A_G}\iint_G |x-u|\left[L_1|x-u|+K_1|y-v|\right]dxdy \\
&+\frac{1}{2}\frac{1}{A_G}\iint_G |y-v|\left[L_2|x-u|+K_2|y-v|\right]dxdy \\
&=\frac{1}{2}\left[L_1\frac{1}{A_G}\iint_G (x-u)^2 dxdy+K_2\frac{1}{A_G}\iint_G (y-v)^2 dxdy\right] \\
&\qquad+\frac{K_1+L_2}{2}\frac{1}{A_G}\iint_G |x-u||y-v|\,dxdy,
\end{aligned}
$$

which proves the desired result (2.5). □

Remark 1. *We observe that if we put* $L=\max\{L_1,K_1,L_2,K_2\}$, *then from (2.5) we get*

$$
\begin{aligned}
(2.10)\quad &\left|\frac{1}{A_G}\iint_G f(x,y)\,dxdy-f(u,v)\right. \\
&\qquad\left.-(\overline{x_G}-u)\frac{\partial f}{\partial x}(u,v)-(\overline{y_G}-v)\frac{\partial f}{\partial y}(u,v)\right| \\
&\qquad\qquad\le \frac{1}{2}L\frac{1}{A_G}\iint_G (|x-u|+|y-v|)^2 dxdy,
\end{aligned}
$$

while from (2.6) we get

$$
\begin{aligned}
(2.11)\quad &\left|f(\overline{x_G},\overline{y_G})-\frac{1}{A_G}\iint_G f(x,y)\,dxdy\right| \\
&\qquad\le \frac{1}{2}L\frac{1}{A_G}\iint_G (|x-\overline{x_G}|+|y-\overline{y_G}|)^2 dxdy
\end{aligned}
$$

for all $(u,v)\in G$.

From Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$
\begin{aligned}
&\frac{1}{A_G}\iint_G |x-u||y-v|\,dxdy \\
&\le \left(\frac{1}{A_G}\iint_G (x-u)^2 dxdy\right)^{1/2}\left(\frac{1}{A_G}\iint_G (y-v)^2 dxdy\right)^{1/2}
\end{aligned}
$$

which gives by (2.5), that

$$\left|\frac{1}{A_G}\iint_G f(x,y)\,dxdy - f(u,v) - (\overline{x_G}-u)\frac{\partial f}{\partial x}(u,v) - (\overline{y_G}-v)\frac{\partial f}{\partial y}(u,v)\right| \leq \frac{1}{2}L\left[\left(\frac{1}{A_G}\iint_G (x-u)^2\,dxdy\right)^{1/2} + \left(\frac{1}{A_G}\iint_G (y-v)^2\,dxdy\right)^{1/2}\right]^2 \tag{2.12}$$

for all $(u,v)\in G$.

In particular,

$$\left|f(\overline{x_G},\overline{y_G}) - \frac{1}{A_G}\iint_G f(x,y)\,dxdy\right| \leq \frac{1}{2}L\left[\left(\frac{1}{A_G}\iint_G (x-\overline{x_G})^2\,dxdy\right)^{1/2} + \left(\frac{1}{A_G}\iint_G (y-\overline{y_G})^2\,dxdy\right)^{1/2}\right]^2. \tag{2.13}$$

Corollary 1. *Assume that* $f: G\to\mathbb{C}$ *is twice differentiable on* G *and the second partial derivatives* $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ *and* $\frac{\partial^2 f}{\partial x\partial y}$ *are bounded on* G. *Put*

$$\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{G,\infty} := \sup_{(x,y)\in G}\left|\frac{\partial^2 f}{\partial x^2}(x,y)\right|,\quad \left\|\frac{\partial^2 f}{\partial y^2}\right\|_{G,\infty} := \sup_{(x,y)\in G}\left|\frac{\partial^2 f}{\partial y^2}(x,y)\right|$$

and

$$\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{G,\infty} := \sup_{(x,y)\in G}\left|\frac{\partial^2 f}{\partial x\partial y}(x,y)\right|,$$

then

$$\begin{aligned}\left|\frac{1}{A_G}\iint_G f(x,y)\,dxdy - f(u,v) - (\overline{x_G}-u)\frac{\partial f}{\partial x}(u,v) - (\overline{y_G}-v)\frac{\partial f}{\partial y}(u,v)\right| \\ \leq \frac{1}{2}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{G,\infty}\frac{1}{A_G}\iint_G (x-u)^2\,dxdy + \left\|\frac{\partial^2 f}{\partial y^2}\right\|_{G,\infty}\frac{1}{A_G}\iint_G (y-v)^2\,dxdy\right] \\ + \left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{G,\infty}\frac{1}{A_G}\iint_G |x-u|\,|y-v|\,dxdy\end{aligned} \tag{2.14}$$

for all $(u,v)\in G$.

In particular,

$$\begin{aligned}\left|f(\overline{x_G},\overline{y_G}) - \frac{1}{A_G}\iint_G f(x,y)\,dxdy\right| \\ \leq \frac{1}{2}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{G,\infty}\frac{1}{A_G}\iint_G (x-\overline{x_G})^2\,dxdy + \left\|\frac{\partial^2 f}{\partial y^2}\right\|_{G,\infty}\frac{1}{A_G}\iint_G (y-\overline{y_G})^2\,dxdy\right] \\ + \left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{G,\infty}\frac{1}{A_G}\iint_G |x-\overline{x_G}|\,|y-\overline{y_G}|\,dxdy.\end{aligned} \tag{2.15}$$

We also have:

Theorem 4. *If $f : G \to \mathbb{C}$ is differentiable on G and the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions (2.3) and (2.4), then for all $(u,v) \in G$ we have*

$$\begin{aligned}(2.16)\quad &\left|\frac{1}{A_G}\iint_G f(x,y)\,dxdy - f(u,v)\right.\\ &\left.-\frac{1}{A_G}\iint_G (x-u)\frac{\partial f}{\partial x}(x,y)\,dxdy - \frac{1}{A_G}\iint_G (y-v)\frac{\partial f}{\partial y}(x,y)\,dxdy\right|\\ &\leq \frac{1}{2}\left[L_1\frac{1}{A_G}\iint_G (x-u)^2\,dxdy + K_2\frac{1}{A_G}\iint_G (y-v)^2\,dxdy\right]\\ &\qquad + \frac{K_1+L_2}{2}\frac{1}{A_G}\iint_G |x-u|\,|y-v|\,dxdy.\end{aligned}$$

Proof. From Lemma 1 we have for $\lambda = \frac{\partial f}{\partial x}(x,y)$ and $\mu = \frac{\partial f}{\partial y}(x,y)$

$$\begin{aligned}(2.17)\quad f(x,y) &= f(u,v) + (x-u)\frac{\partial f}{\partial x}(x,y) + (y-v)\frac{\partial f}{\partial y}(x,y)\\ &+ (x-u)\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial x}(x,y)\right)dt\\ &+ (y-v)\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial y}(x,y)\right)dt\end{aligned}$$

for any (x,y), $(u,v) \in G$.

By taking the integral mean $\frac{1}{A_G}\iint_G$ over (x,y) we get

$$\begin{aligned}(2.18)\quad &\frac{1}{A_G}\iint_G f(x,y)\,dxdy = f(u,v)\\ &+ \frac{1}{A_G}\iint_G (x-u)\frac{\partial f}{\partial x}(x,y)\,dxdy + \frac{1}{A_G}\iint_G (y-v)\frac{\partial f}{\partial y}(x,y)\,dxdy\\ &+ \frac{1}{A_G}\iint_G (x-u)\left(\int_0^1\left(\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial x}(x,y)\right)dt\right)dxdy\\ &+ \frac{1}{A_G}\iint_G (y-v)\left(\int_0^1\left(\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial y}(x,y)\right)dt\right)dxdy\end{aligned}$$

for all $(u,v) \in G$.

The equality (2.18) implies that

$$\begin{aligned}(2.19)\quad &\left|\frac{1}{A_G}\iint_G f(x,y)\,dxdy - f(u,v)\right.\\ &\left.-\frac{1}{A_G}\iint_G (x-u)\frac{\partial f}{\partial x}(x,y)\,dxdy - \frac{1}{A_G}\iint_G (y-v)\frac{\partial f}{\partial y}(x,y)\,dxdy\right|\\ &\leq \frac{1}{A_G}\iint_G |x-u|\left(\int_0^1\left|\frac{\partial f}{\partial x}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial x}(x,y)\right|dt\right)dxdy\\ &+ \frac{1}{A_G}\iint_G |y-v|\left(\int_0^1\left|\frac{\partial f}{\partial y}[t(x,y)+(1-t)(u,v)] - \frac{\partial f}{\partial y}(x,y)\right|dt\right)dxdy\end{aligned}$$

$$\leq \frac{1}{2}\frac{1}{A_G}\iint_G |x-u|\left[L_1|x-u|+K_1|y-v|\right]dxdy$$
$$+\frac{1}{2}\frac{1}{A_G}\iint_G |y-v|\left[L_2|x-u|+K_2|y-v|\right]dxdy$$
$$=\frac{1}{2}\left[L_1\frac{1}{A_G}\iint_G (x-u)^2\,dxdy+K_2\frac{1}{A_G}\iint_G (y-v)^2\,dxdy\right]$$
$$+\frac{K_1+L_2}{2}\frac{1}{A_G}\iint_G |x-u|\,|y-v|\,dxdy,$$

which proves the desired result (2.16). □

We define

$$u_S:=\frac{\iint_G x\frac{\partial f}{\partial x}(x,y)\,dxdy}{\iint_G \frac{\partial f}{\partial x}(x,y)\,dxdy} \text{ and } v_S:=\frac{\iint_G y\frac{\partial f}{\partial y}(x,y)\,dxdy}{\iint_G \frac{\partial f}{\partial y}(x,y)\,dxdy}, \tag{2.20}$$

provided that the integrals from the denominators are not zero.

Corollary 2. *With the assumptions of Theorem 4 and if* $(u_S,v_S)\in G$*, then*

$$\left|f(u_S,v_S)-\frac{1}{A_G}\iint_G f(x,y)\,dxdy\right| \tag{2.21}$$
$$\leq \frac{1}{2}\left[L_1\frac{1}{A_G}\iint_G (x-u_S)^2\,dxdy+K_2\frac{1}{A_G}\iint_G (y-v_S)^2\,dxdy\right]$$
$$+\frac{K_1+L_2}{2}\frac{1}{A_G}\iint_G |x-u_S|\,|y-v_S|\,dxdy.$$

Remark 2. *Assume that* $f:G\to\mathbb{C}$ *is twice differentiable on* G *and the second partial derivatives* $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ *and* $\frac{\partial^2 f}{\partial x\partial y}$ *are bounded on* G*. Then*

$$\left|f(u_S,v_S)-\frac{1}{A_G}\iint_G f(x,y)\,dxdy\right| \tag{2.22}$$
$$\leq \frac{1}{2}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{G,\infty}\frac{1}{A_G}\iint_G (x-u_S)^2\,dxdy+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{G,\infty}\frac{1}{A_G}\iint_G (y-v_S)^2\,dxdy\right]$$
$$+\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{G,\infty}\frac{1}{A_G}\iint_G |x-u_S|\,|y-v_S|\,dxdy.$$

3. Examples for Rectangles

If $G=[a,b]\times[c,d]$ is a rectangle from $\mathbb{R}^2$, then

$$A_{[a,b]\times[c,d]}=(b-a)(d-c),\ \overline{x_{[a,b]\times[c,d]}}=\frac{a+b}{2} \text{ and } \overline{y_{[a,b]\times[c,d]}}=\frac{c+d}{2}.$$

If $f:[a,b]\times[c,d]\to\mathbb{C}$ is differentiable on $[a,b]\times[c,d]$ and the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions (2.3) and (2.4), then for all $(u,v)\in$

$[a,b]\times[c,d]$ we have

$$\begin{aligned}(3.1)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - f(u,v)\right.\\ &\left.-\left(\frac{a+b}{2}-u\right)\frac{\partial f}{\partial x}(u,v)-\left(\frac{c+d}{2}-v\right)\frac{\partial f}{\partial y}(u,v)\right|\\ &\le \frac{1}{2}\left[L_1\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x-u)^2\,dxdy\right.\\ &\left.+K_2\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y-v)^2\,dxdy\right]\\ &+\frac{K_1+L_2}{2}\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |x-u|\,|y-v|\,dxdy\end{aligned}$$

for all $(u,v)\in[a,b]\times[c,d]$.

Since

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x-u)^2\,dxdy=\frac{(b-u)^3+(u-a)^3}{3(b-a)},$$
$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (y-v)^2\,dxdy=\frac{(d-v)^3+(v-c)^3}{3(d-c)}$$

and

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |x-u|\,|y-v|\,dxdy\\ &=\frac{\left[(b-u)^2+(u-a)^2\right]\left[(d-v)^2+(v-c)^2\right]}{4(b-a)(d-c)},\end{aligned}$$

hence by (3.1) we get

$$\begin{aligned}(3.2)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - f(u,v)\right.\\ &\left.-\left(\frac{a+b}{2}-u\right)\frac{\partial f}{\partial x}(u,v)-\left(\frac{c+d}{2}-v\right)\frac{\partial f}{\partial y}(u,v)\right|\\ &\le \frac{1}{2}\left[L_1\frac{(b-u)^3+(u-a)^3}{3(b-a)}+K_2\frac{(d-v)^3+(v-c)^3}{3(d-c)}\right]\\ &+\frac{K_1+L_2}{2}\frac{\left[(b-u)^2+(u-a)^2\right]\left[(d-v)^2+(v-c)^2\right]}{4(b-a)(d-c)}\end{aligned}$$

for all $(u,v)\in[a,b]\times[c,d]$.

In particular, we have

$$\begin{aligned}(3.3)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\\ &\le\frac{1}{24}\left[L_1(b-a)^2+K_2(d-c)^2\right]+\frac{K_1+L_2}{32}(b-a)(d-c).\end{aligned}$$

Assume that $f:[a,b]\times[c,d]\rightarrow\mathbb{C}$ is twice differentiable on $[a,b]\times[c,d]$ and the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x\partial y}$ are bounded on $[a,b]\times[c,d]$. Then

$$\begin{aligned}(3.4)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy-f(u,v)\right.\\&\left.-\left(\frac{a+b}{2}-u\right)\frac{\partial f}{\partial x}(u,v)-\left(\frac{c+d}{2}-v\right)\frac{\partial f}{\partial y}(u,v)\right|\\&\leq\frac{1}{2}\left[\frac{(b-u)^3+(u-a)^3}{3(b-a)}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{[a,b]\times[c,d],\infty}\right.\\&\left.+\frac{(d-v)^3+(v-c)^3}{3(d-c)}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{[a,b]\times[c,d],\infty}\right]\\&+\frac{\left[(b-u)^2+(u-a)^2\right]\left[(d-v)^2+(v-c)^2\right]}{4(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{[a,b]\times[c,d],\infty}\end{aligned}$$

for all $(u,v)\in[a,b]\times[c,d]$.

In particular,

$$\begin{aligned}(3.5)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\\&\leq\frac{1}{24}\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{[a,b]\times[c,d],\infty}(b-a)^2+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{[a,b]\times[c,d],\infty}(d-c)^2\right]\\&+\frac{1}{16}(b-a)(d-c)\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{[a,b]\times[c,d],\infty}.\end{aligned}$$

4. Examples for Disks

Consider the disk $D(C,R)$ centered in $C=(a,b)$ and with radius $R>0$. Observe that

$$\begin{aligned}(4.1)\quad &\frac{1}{D(C,R)}\iint_{D(C,R)}(x-u)^2\,dxdy\\&=\frac{1}{D(C,R)}\iint_{D(C,R)}\left(x^2-2xu+u^2\right)dxdy\\&=\frac{1}{D(C,R)}\iint_{D(C,R)}x^2dxdy-2u\frac{1}{D(C,R)}\iint_{D(C,R)}xdxdy+u^2.\end{aligned}$$

Since, by the polar change of variables,

$$\begin{aligned}\frac{1}{D(C,R)}\iint_{D(C,R)}x^2dxdy&=\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}(r\cos\theta+a)^2\,rdrd\theta\\&=\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}\left[r^2\cos^2(\theta)+2r\cos\theta+a^2\right]rdrd\theta\end{aligned}$$

$$= \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} \left(r^3 \cos^2(\theta) + 2r^2 \cos\theta + a^2 r\right) dr d\theta$$

$$= \frac{1}{\pi R^2} \left[\int_0^R \int_0^{2\pi} r^3 \cos^2(\theta)\, dr d\theta + \int_0^R \int_0^{2\pi} 2r^2 \cos\theta dr d\theta + \int_0^R \int_0^{2\pi} a^2 r dr d\theta\right]$$

$$= \frac{1}{\pi R^2} \left(\frac{R^4}{4}\pi + \pi a^2 R^2\right) = \frac{R^2}{4} + a^2,$$

and

$$\frac{1}{D(C,R)} \iint_{D(C,R)} x dx dy = \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} (r\cos\theta + a)\, r dr d\theta$$

$$= \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} \left(r^2 \cos\theta + ra\right) dr d\theta = \frac{a}{\pi R^2} 2\pi \frac{R^2}{2} = a$$

hence

$$\frac{1}{D(C,R)} \iint_{D(C,R)} (x-u)^2 dx dy = \frac{R^2}{4} + a^2 - 2ua + u^2 = \frac{R^2}{4} + (a-u)^2.$$

Similarly,

$$\frac{1}{D(C,R)} \iint_{D(C,R)} (y-v)^2 dx dy = \frac{R^2}{4} + (b-v)^2.$$

If $f : D(C,R) \to \mathbb{C}$ is differentiable on $D(C,R)$ and the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions (2.3) and (2.4), then for all $(u,v) \in D(C,R)$ we have by (2.12) that

$$\begin{aligned}(4.2) \quad & \left|\frac{1}{D(C,R)} \iint_{D(C,R)} f(x,y)\, dx dy - f(u,v)\right. \\ & \left. - (a-u) \frac{\partial f}{\partial x}(u,v) - (b-v) \frac{\partial f}{\partial y}(u,v)\right| \\ & \leq \frac{1}{2} L \left(\left[\frac{R^2}{4} + (a-u)^2\right]^{1/2} + \left[\frac{R^2}{4} + (b-v)^2\right]^{1/2}\right)^2,\end{aligned}$$

where $L = \max\{L_1, K_1, L_2, K_2\}$.

In particular,

$$(4.3) \qquad \left|\frac{1}{D(C,R)} \iint_{D(C,R)} f(x,y)\, dx dy - f(a,b)\right| \leq \frac{1}{2} L R^2.$$

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