HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR MULTIPLE INTEGRALS ON CONVEX BODIES

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Abstract. In this paper we establish some Hermite-Hadamard type inequalities for convex functions of n-variables defined on closed and bounded convex subsets of the Euclidian space $\mathbb{R}^n$, $n \geq 2$. Some examples for n-*hyper boxes* $R_n := [a_1, b_1] \times ... \times [a_n, b_n]$ and 3-dimensional balls are also provided.

1. Introduction

Let us consider a point $C = (a, b) \in \mathbb{R}^2$ and the disk

$$D(C, R) = \left\{ (x, y) \in \mathbb{R}^2 \middle| (x-a)^2 + (y-b)^2 \leq R^2 \right\}$$

centered at the point C and having the radius $R > 0$. The following inequality of Hermite-Hadamard's type holds [4].

Theorem 1. *If the mapping $f : D(C, R) \to \mathbb{R}$ is convex on $D(C, R)$, then one has the inequality:*

$$f(a, b) \leq \frac{1}{\pi R^2} \iint_{D(C,R)} f(x, y)\, dxdy \leq \frac{1}{2\pi R} \int_{\mathfrak{S}(C,R)} f(\gamma)\, dl(\gamma), \tag{1.1}$$

where $\mathfrak{S}(C, R)$ is the circle centered at the point C and having the radius R. The above inequalities are sharp.

Consider also the three dimensional ball $B(C, R)$ centered in $C = (a, b, c) \in \mathbb{R}^3$ and with radius $R > 0$, namely

$$B(C, R) := \left\{ (x, y, z) \in \mathbb{R}^3 \middle| (x-a)^2 + (y-b)^2 + (z-c)^2 \leq R^2 \right\}.$$

The following theorem holds [5]:

Theorem 2. *Let $f : B(C, R) \to \mathbb{R}$ be a convex function on the ball $B(C, R)$. Then we have the inequality:*

$$f(a, b, c) \leq \frac{1}{\nu(B(C, R))} \iiint_{B(C,R)} f(x, y, z)\, dxdydz \leq \frac{1}{\sigma(B(C, R))} \iint_{S(C,R)} f(x, y, z)\, ds, \tag{1.2}$$

where

$$S(C, R) := \left\{ (x, y, z) \in \mathbb{R}^3 \middle| (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \right\}$$

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is the sphere centered at C *and with radius* $R>0$ *and*

$$\nu\left(B\left(C,R\right)\right)=\frac{4\pi R^{3}}{3},\ \textit{the volume of the ball}$$

and

$$\sigma\left(S\left(C,R\right)\right)=4\pi R^{2},\ \textit{the area of the sphere.}$$

For other multivariate Hermite-Hadamard type inequalities, see [1]-[3] and [6]-[12].

In the following, consider G_n a closed and bounded convex subset of $\mathbb{R}^n$. Define

$$V_{G_n}:=\int\cdots\int_{G_n}dx_1...dx_n$$

the *n-volume* of G_n and $(\overline{x_{1G_n}},...,\overline{x_{nG_n}})$ the *n-centre of gravity* for G_n, where

$$\overline{x_{i,G_n}}:=\frac{1}{V_{G_n}}\int\cdots\int_{G_n}x_i dx_1...dx_n dxdy,\ \text{for } i\in\{1,...,n\}.$$

Consider the function of n variables $f=f(x_1,...,x_n)$ and denote by $\frac{\partial f}{\partial x_i}$ the partial derivative with respect to the variable x_i for $i\in\{1,...,n\}$.

As examples, we can consider the *n-hyper box* $R_n:=[a_1,b_1]\times...\times[a_n,b_n]\subset\mathbb{R}^n$, for which

$$V_{R_n}=\prod_{i=1}^{n}(b_i-a_i)\ \text{and}\ \overline{x_{i,R_n}}=\frac{b_i+a_i}{2}\ \text{for } i\in\{1,...,n\}.$$

Also, if we consider *n-hyper ball* centered in $C=(c_1,...,c_n)$ and with radius $R>0$ defined by

$$B_n\left(C,R\right):=\left\{(x_1,...,x_n)\mid\ \sum_{i=1}^{n}(x_i-c_i)^2\leq R^2\right\}\subset\mathbb{R}^n,$$

then the *n-volume of* $B_n(C,R)$ is

$$V_{B_n}=\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}R^n,\ n\geq 2$$

where Γ is Euler's *gamma function*.

Using explicit formulas for particular values of the gamma function at the integers and half integers gives formulas for the n-volume of the Euclidean ball as

$$V_{B_{2k}}=\frac{\pi^k}{k!}R^{2k}\ \text{and}\ V_{B_{2k+1}}=\frac{2\,(k!)\,(4\pi)^k}{(2k+1)!}R^{2k+1},\ k\geq 1.$$

We also have $\overline{x_{i,B_n}}=c_i$ for all $i\in\{1,...,n\}$.

Motivated by the above results, in this paper we establish some Hermite-Hadamard type inequalities for convex functions of n-variables defined on closed and bounded convex subsets of the Euclidian space $\mathbb{R}^n$, $n\geq 2$. Some examples for *n-hyper boxes* $R_n:=[a_1,b_1]\times...\times[a_n,b_n]$ and 3-dimensional balls are also provided.

2. The Main Results

The following double inequality holds:

Theorem 3. *Let $f:G_n\to\mathbb{R}$ be a differentiable convex function on G_n. Then for all $(y_1,...,y_n)\in G_n$ we have*

$$\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(y_1,...,y_n)\left(\overline{x_{i,G_n}}-y_i\right) \le \frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f(y_1,...,y_n) \le \sum_{i=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n} x_i\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n - \sum_{i=1}^{n} y_i\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n. \tag{2.1}$$

In particular, for $(y_1,...,y_n)=\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right)$ we obtain

$$0\le\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right) \le \sum_{i=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n} x_i\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n - \sum_{i=1}^{n}\overline{x_{i,G_n}}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n. \tag{2.2}$$

Proof. Since $f:G_n\to\mathbb{R}$ is a differentiable convex function on G_n, then for all $(x_1,...,x_n),\ (y_1,...,y_n)\in G_n$ we have the gradient inequalities

$$\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(y_1,...,y_n)(x_i-y_i)\le f(x_1,...,x_n)-f(y_1,...,y_n) \le\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x_1,...,x_n)(x_i-y_i). \tag{2.3}$$

Taking the integral mean $\frac{1}{V_{G_n}}\int\cdots\int_{G_n}$ in (2.3) over the variables $(x_1,...,x_n)$ we deduce

$$\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(y_1,...,y_n)(x_i-y_i)\,dx_1...dx_n \le\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f(y_1,...,y_n) \le\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x_1,...,x_n)(x_i-y_i)\,dx_1...dx_n. \tag{2.4}$$

Since

$$\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(y_1,...,y_n)(x_i-y_i)\,dx_1...dx_n$$
$$=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(y_1,...,y_n)\left(\overline{x_{i,G_n}}-y_i\right)$$

and

$$\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x_1,...,x_n)(x_i-y_i)\,dx_1...dx_n$$
$$=\sum_{i=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}x_i\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n$$
$$-\sum_{i=1}^{n}y_i\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n$$

hence by (2.4) we get (2.1). □

We have:

Corollary 1. *Let $f:G_n\to\mathbb{R}$ be a differentiable convex function on G_n and assume that*

$$x_{i,S}:=\frac{\int\cdots\int_{G_n}x_i\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n}{\int\cdots\int_{G_n}\frac{\partial f}{\partial x_i}(x_1,...,x_n)\,dx_1...dx_n},\ i\in\{1,...,n\} \tag{2.5}$$

exist.

If $(x_{1,S},...,x_{n,S})\in G_n$, then

$$0\le f(x_{1,S},...,x_{n,S})-\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f(x_1,...,x_n)\,dx_1...dx_n \le\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x_{1,S},...,x_{n,S})\left(x_{i,S}-\overline{x_{i,G_n}}\right). \tag{2.6}$$

Proof. If we take in (2.1) $(y_1,...,y_n)=(x_{1,S},...,x_{n,S})\in G_n$, then we get

$$\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x_{1,S},...,x_{n,S})\left(\overline{x_{i,G_n}}-x_{i,S}\right)$$
$$\le\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f(x_1,...,x_n)\,dx_1...dx_n-f(x_{1,S},...,x_{n,S})\le 0,$$

which is equivalent to (2.6). □

Remark 1. *We observe that from (2.6) we also have the inequality for the sup norm as follows*

$$0\le f(x_{1,S},...,x_{n,S})-\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f(x_1,...,x_n)\,dx_1...dx_n \le\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|_{G_n,\infty}\left|x_{i,S}-\overline{x_{i,G_n}}\right|. \tag{2.7}$$

Corollary 2. *Let $f : G_n \to \mathbb{R}$ be a differentiable convex function on G_n. If the partial derivatives $\frac{\partial f}{\partial x_i}$, $i \in \{1, ..., n\}$, satisfy the conditions*

$$m_i \leq \frac{\partial f}{\partial x_i}(x_1, ..., x_n) \leq M_i, \text{ for any } (x_1, ..., x_n) \in G_n \tag{2.8}$$

for some m_i and M_i, $i \in \{1, ..., n\}$, then we have

$$0 \leq \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n)\, dx_1 ... dx_n - f\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right) \leq \frac{1}{2} \sum_{i=1}^{n} (M_i - m_i) \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left|x_i - \overline{x_{i,G_n}}\right| dx_1 ... dx_n. \tag{2.9}$$

Proof. Observe that for all α_i, $i \in \{1, ..., n\}$, real numbers, we have

$$\frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left[\frac{\partial f}{\partial x_i}(x_1, ..., x_n) - \alpha_i\right] \left(x_i - \overline{x_{i,G_n}}\right) dx_1 ... dx_n$$
$$= \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \frac{\partial f}{\partial x_i}(x_1, ..., x_n) \left(x_i - \overline{x_{i,G_n}}\right) dx_1 ... dx_n$$
$$+ \alpha \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left(x_i - \overline{x_{i,G_n}}\right) dx_1 ... dx_n$$
$$= \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \frac{\partial f}{\partial x_i}(x_1, ..., x_n) \left(x_i - \overline{x_{i,G_n}}\right) dx_1 ... dx_n.$$

This implies that

$$\sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \frac{\partial f}{\partial x_i}(x_1, ..., x_n) \left(x_i - \overline{x_{i,G_n}}\right) dx_1 ... dx_n = \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left[\frac{\partial f}{\partial x_i}(x_1, ..., x_n) - \alpha_i\right] \left(x_i - \overline{x_{i,G_n}}\right) dx_1 ... dx_n. \tag{2.10}$$

If we take in this equality $\alpha_i = \frac{1}{2}(m_i + M_i)$, $i \in \{1, ..., n\}$, and use the condition (2.8) and the inequality (2.2), then we get

$$0 \leq \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n)\, dx_1 ... dx_n - f\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right)$$
$$\leq \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \frac{\partial f}{\partial x_i}(x_1, ..., x_n) \left(x_i - \overline{x_{i,G_n}}\right) dx_1 ... dx_n$$
$$= \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left[\frac{\partial f}{\partial x_i}(x_1, ..., x_n) - \frac{1}{2}(m_i + M_i)\right] \left(x_i - \overline{x_{i,G_n}}\right) dx_1 ... dx_n$$
$$= \left|\sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left[\frac{\partial f}{\partial x_i}(x_1, ..., x_n) - \frac{1}{2}(m_i + M_i)\right] \left(x_i - \overline{x_{i,G_n}}\right) dx_1 ... dx_n\right|$$
$$\leq \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left|\frac{\partial f}{\partial x_i}(x_1, ..., x_n) - \frac{1}{2}(m_i + M_i)\right| \left|x_i - \overline{x_{i,G_n}}\right| dx_1 ... dx_n$$
$$\leq \frac{1}{2}(M_i - m_i) \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left|x_i - \overline{x_{i,G_n}}\right| dx_1 ... dx_n$$

and the inequality (2.9) is proved. □

Remark 2. *If we use the discrete Cauchy-Bunyakovsky-Schwarz inequality we have*

$$\sum_{i=1}^{n}(M_i-m_i)\left|x_i-\overline{x_{i,G_n}}\right| \le \left[\sum_{i=1}^{n}(M_i-m_i)^2\right]^{1/2}\left[\sum_{i=1}^{n}\left(x_i-\overline{x_{i,G_n}}\right)^2\right]^{1/2}$$

and by the inequality (2.9) we get

$$(2.11)\quad 0 \le \frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right) \le \frac{1}{2}\left[\sum_{i=1}^{n}(M_i-m_i)^2\right]^{1/2}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left[\sum_{i=1}^{n}\left(x_i-\overline{x_{i,G_n}}\right)^2\right]^{1/2}dx_1...dx_n.$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we also have

$$\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|dx_1...dx_n \le \left[\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left(x_i-\overline{x_{i,G_n}}\right)^2dx_1...dx_n\right]^{1/2}$$

and by the inequality (2.9) we get

$$(2.12)\quad 0 \le \frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right) \le \frac{1}{2}\sum_{i=1}^{n}(M_i-m_i)\left[\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left(x_i-\overline{x_{i,G_n}}\right)^2dx_1...dx_n\right]^{1/2}.$$

We say that the partial derivatives $\frac{\partial f}{\partial x_i}$ are *Lipschitzian* with the positive constants $(L_{i,1},...,L_{i,n})$, $i\in\{1,...,n\}$ if

$$(2.13)\quad \left|\frac{\partial f}{\partial x_i}(x_1,...,x_n)-\frac{\partial f}{\partial x_i}(y_1,...,y_n)\right| \le \sum_{j=1}^{n}L_{i,j}\left|x_j-y_j\right|$$

for all $(x_1,...,x_n),(y_1,...,y_n)\in G_n$ and $i\in\{1,...,n\}$.

Corollary 3. *Let $f:G_n\to\mathbb{R}$ be a differentiable convex function on G_n. If the partial derivatives $\frac{\partial f}{\partial x_i}$, $i\in\{1,...,n\}$, satisfy the conditions (2.13), then we have*

$$(2.14)\quad 0 \le \frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right) \le \sum_{i=1}^{n}\sum_{j=1}^{n}L_{i,j}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_j-\overline{x_{j,G_n}}\right|\left|x_i-\overline{x_{i,G_n}}\right|dx_1...dx_n.$$

Proof. By using the identity (2.10) for $\alpha_i = \frac{\partial f}{\partial x_i}\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right)$, $i \in \{1, ..., n\}$ and the fact that $\frac{\partial f}{\partial x_i}$ is Lipschitzian for each $i \in \{1, ..., n\}$, we get

$$
\begin{aligned}
0 &\le \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n)\, dx_1 ... dx_n - f\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right) \\
&\le \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \frac{\partial f}{\partial x_i}(x_1, ..., x_n)\left(x_i - \overline{x_{i,G_n}}\right) dx_1 ... dx_n \\
&= \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left[\frac{\partial f}{\partial x_i}(x_1, ..., x_n) - \frac{\partial f}{\partial x_i}\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right)\right] \\
&\qquad \times \left(x_i - \overline{x_{i,G_n}}\right) dx_1 ... dx_n \\
&\le \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left|\frac{\partial f}{\partial x_i}(x_1, ..., x_n) - \frac{\partial f}{\partial x_i}\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right)\right| \\
&\qquad \times \left|x_i - \overline{x_{i,G_n}}\right| dx_1 ... dx_n \\
&\le \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left(\sum_{j=1}^{n} L_{i,j}\left|x_j - \overline{x_{j,G_n}}\right|\right)\left|x_i - \overline{x_{i,G_n}}\right| dx_1 ... dx_n \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} L_{i,j} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left|x_j - \overline{x_{j,G_n}}\right|\left|x_i - \overline{x_{i,G_n}}\right| dx_1 ... dx_n,
\end{aligned}
$$

which proves the desired result (2.14). □

Remark 3. *We have*

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} L_{i,j} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left|x_j - \overline{x_{j,G_n}}\right|\left|x_i - \overline{x_{i,G_n}}\right| dx_1 ... dx_n \\
&\le \max_{i,j=1,...,n}\{L_{i,j}\} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left|x_j - \overline{x_{j,G_n}}\right|\left|x_i - \overline{x_{i,G_n}}\right| dx_1 ... dx_n \\
&= \max_{i,j=1,...,n}\{L_{i,j}\} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n}\sum_{j=1}^{n} \left|x_j - \overline{x_{j,G_n}}\right|\left|x_i - \overline{x_{i,G_n}}\right| dx_1 ... dx_n \\
&= \max_{i,j=1,...,n}\{L_{i,j}\} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{j=1}^{n} \left|x_j - \overline{x_{j,G_n}}\right| \sum_{i=1}^{n} \left|x_i - \overline{x_{i,G_n}}\right| dx_1 ... dx_n \\
&= \max_{i,j=1,...,n}\{L_{i,j}\} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left(\sum_{i=1}^{n} \left|x_i - \overline{x_{i,G_n}}\right|\right)^2 dx_1 ... dx_n
\end{aligned}
$$

and by (2.14) we get

$$
\begin{aligned}
(2.15)\quad 0 &\le \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n)\, dx_1 ... dx_n - f\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right) \\
&\le \max_{i,j=1,...,n}\{L_{i,j}\} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left(\sum_{i=1}^{n} \left|x_i - \overline{x_{i,G_n}}\right|\right)^2 dx_1 ... dx_n.
\end{aligned}
$$

We also have by Cauchy-Bunyakovsky-Schwarz integral inequality that

$$\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_j-\overline{x_{j,G_n}}\right|\left|x_i-\overline{x_{i,G_n}}\right|dx_1...dx_n \leq\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_j-\overline{x_{j,G_n}}\right|^2dx_1...dx_n\right)^{1/2} \times\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|^2dx_1...dx_n\right)^{1/2}$$

for all $i,\ j=1,...,n$.

Therefore

$$\sum_{i=1}^n\sum_{j=1}^n\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_j-\overline{x_{j,G_n}}\right|\left|x_i-\overline{x_{i,G_n}}\right|dx_1...dx_n$$
$$\leq\sum_{i=1}^n\sum_{j=1}^n\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_j-\overline{x_{j,G_n}}\right|^2dx_1...dx_n\right)^{1/2}$$
$$\times\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|^2dx_1...dx_n\right)^{1/2}$$
$$=\left[\sum_{i=1}^n\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|^2dx_1...dx_n\right)^{1/2}\right]^2.$$

Consequently, by (2.14) we get

$$0\leq\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f(x_1,...,x_n)\,dx_1...dx_n-f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right) \leq\max_{i,j=1,...,n}\{L_{i,j}\}\left[\sum_{i=1}^n\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|^2dx_1...dx_n\right)^{1/2}\right]^2. \tag{2.16}$$

Also, by Cauchy-Schwarz discrete inequality we have

$$\left[\sum_{i=1}^n\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|^2dx_1...dx_n\right)^{1/2}\right]^2$$
$$\leq n\sum_{i=1}^n\left[\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|^2dx_1...dx_n\right)^{1/2}\right]^2$$
$$=n\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^n\left|x_i-\overline{x_{i,G_n}}\right|^2dx_1...dx_n$$

and by (2.16) we get

$$0\leq\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f(x_1,...,x_n)\,dx_1...dx_n-f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right) \leq n\max_{i,j=1,...,n}\{L_{i,j}\}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^n\left|x_i-\overline{x_{i,G_n}}\right|^2dx_1...dx_n. \tag{2.17}$$

Corollary 4. *Let $f : G_n \to \mathbb{R}$ be a twice differentiable convex function on G_n. If the partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $i,\ j \in \{1, ..., n\}$, are bounded on G_n, namely*

$$\left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{G_n,\infty} := \sup_{(x_1,...,x_n)\in G_n} \left| \frac{\partial^2 f}{\partial x_i \partial x_j} (x_1, ..., x_n) \right| < \infty,$$

then we have

$$0 \le \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n)\, dx_1...dx_n - f\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right) \le \sum_{i=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{G_n,\infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left| x_j - \overline{x_{j,G_n}} \right| \left| x_i - \overline{x_{i,G_n}} \right| dx_1...dx_n. \tag{2.18}$$

For a continuous function $g : G_n \to \mathbb{C}$ we consider the usual p-norms as follows

$$\|g\|_{G_n,p} := \begin{cases} \sup_{(x_1,...,x_n)\in G_n} |g(x_1, ..., x_n)| \\ \left(\int \cdots \int_{G_n} |g(x_1, ..., x_n)|^p\, dx_1...dx_n \right)^{1/p}, \ p \ge 1. \end{cases}$$

We also have:

Theorem 4. *Let $f : G_n \to \mathbb{R}$ be a differentiable convex function on G_n. Then*

$$0 \le \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n)\, dx_1...dx_n - f\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right) \le \begin{cases} \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{G_n,\infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left| x_i - \overline{x_{i,G_n}} \right| dx_1...dx_n, \\ \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{G_n,p} \left(\frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left| x_i - \overline{x_{i,G_n}} \right|^q dx_1...dx_n \right)^{1/q} \\ \text{for } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{G_n,1} \sup_{(x_1,...,x_n)\in G_n} \left| x_i - \overline{x_{i,G_n}} \right| \end{cases} \tag{2.19}$$

for any $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$.

In particular, for $(\alpha_1, ..., \alpha_n) = (0, ..., 0)$ we get

$$0 \le \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n)\, dx_1...dx_n - f\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right) \le \begin{cases} \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{G_n,\infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left| x_i - \overline{x_{i,G_n}} \right| dx_1...dx_n, \\ \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{G_n,p} \left(\frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left| x_i - \overline{x_{i,G_n}} \right|^q dx_1...dx_n \right)^{1/q} \\ \text{for } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{G_n,1} \sup_{(x_1,...,x_n)\in G_n} \left| x_i - \overline{x_{i,G_n}} \right|. \end{cases} \tag{2.20}$$

Proof. We have the equality

$$\sum_{i=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\frac{\partial f}{\partial x_i}(x_1,...,x_n)(x_i-\overline{x_{i,G_n}})\,dx_1...dx_n = \sum_{i=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left[\frac{\partial f}{\partial x_i}(x_1,...,x_n)-\alpha_i\right](x_i-\overline{x_{i,G_n}})\,dx_1...dx_n. \tag{2.21}$$

Using Hölder's integral inequality we have

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left[\frac{\partial f}{\partial x_i}(x_1,...,x_n)-\alpha_i\right](x_i-\overline{x_{i,G_n}})\,dx_1...dx_n\right|$$
$$\leq\begin{cases}\sup_{(x_1,...,x_n)\in G_n}\left|\frac{\partial f}{\partial x_i}(x_1,...,x_n)-\alpha_i\right|\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|dx_1...dx_n\\[2ex] \left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|\frac{\partial f}{\partial x_i}(x_1,...,x_n)-\alpha_i\right|^p\right)^{1/p}\\ \times\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|^q dx_1...dx_n\right)^{1/q}\\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \sup_{(x_1,...,x_n)\in G_n}\left|x_i-\overline{x_{i,G_n}}\right|\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|\frac{\partial f}{\partial x_i}(x_1,...,x_n)-\alpha_i\right|dx_1...dx_n\end{cases}$$
$$=\begin{cases}\left\|\frac{\partial f}{\partial x_i}-\alpha_i\right\|_{G_n,\infty}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|dx_1...dx_n\\[2ex] \left\|\frac{\partial f}{\partial x_i}-\alpha_i\right\|_{G_n,p}\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|^q dx_1...dx_n\right)^{1/q}\\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1\\[2ex] \left\|\frac{\partial f}{\partial x_i}-\alpha_i\right\|_{G_n,1}\sup_{(x_1,...,x_n)\in G_n}\left|x_i-\overline{x_{i,G_n}}\right|.\end{cases}$$

Therefore by (2.10) and (2.21) we get

$$\begin{aligned}0&\leq\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f(x_1,...,x_n)\,dx_1...dx_n-f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right)\\&\leq\left|\sum_{i=1}^{n}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left[\frac{\partial f}{\partial x_i}(x_1,...,x_n)-\alpha_i\right](x_i-\overline{x_{i,G_n}})\,dx_1...dx_n\right|\\&\leq\sum_{i=1}^{n}\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left[\frac{\partial f}{\partial x_i}(x_1,...,x_n)-\alpha_i\right](x_i-\overline{x_{i,G_n}})\,dx_1...dx_n\right|\\&\leq\begin{cases}\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}-\alpha_i\right\|_{G_n,\infty}\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|dx_1...dx_n,\\[2ex] \sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}-\alpha_i\right\|_{G_n,p}\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|^q dx_1...dx_n\right)^{1/q}\\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}-\alpha_i\right\|_{G_n,1}\sup_{(x_1,...,x_n)\in G_n}\left|x_i-\overline{x_{i,G_n}}\right|\end{cases}\end{aligned} \tag{2.22}$$

for any $(\alpha_1,...,\alpha_n)\in\mathbb{R}^n$. □

Remark 4. *If we put*

$$B_{\infty}(\alpha_1,...,\alpha_n) := \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{G_n,\infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - \overline{x_{i,G_n}}| \, dx_1...dx_n,$$

then we have

$$B_{\infty}(\alpha_1,...,\alpha_n) \le \max_{i\in\{1,...,n\}} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{G_n,\infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left(\sum_{i=1}^{n} |x_i - \overline{x_{i,G_n}}| \right) dx_1...dx_n. \tag{2.23}$$

Also, if we put

$$B_p(\alpha_1,...,\alpha_n) := \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{G_n,p} \left(\frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - \overline{x_{i,G_n}}|^q \, dx_1...dx_n \right)^{1/q},$$

then by Hölder's inequality we have

$$B_p(\alpha_1,...,\alpha_n) \le \left(\sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{G_n,p}^{p} \right)^{1/p} \left(\frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} |x_i - \overline{x_{i,G_n}}|^q \, dx_1...dx_n \right)^{1/q} \tag{2.24}$$

for $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

If we denote

$$B_1(\alpha_1,...,\alpha_n) := \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{G_n,1} \sup_{(x_1,...,x_n)\in G_n} |x_i - \overline{x_{i,G_n}}|,$$

then we also have

$$B_1(\alpha_1,...,\alpha_n) \le \max_{i\in\{1,...,n\}} \left\{ \sup_{(x_1,...,x_n)\in G_n} |x_i - \overline{x_{i,G_n}}| \right\} \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{G_n,1}. \tag{2.25}$$

3. Examples for n-Dimensional Boxes

We can consider the *n-hyper box* $R_n := [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$, $n \ge 2$ and assume that $f : R_n \to \mathbb{R}$ is a differentiable convex function on R_n. We have

$$V_{R_n} = \prod_{i=1}^{n} (b_i - a_i) \text{ and } \overline{x_{i,R_n}} = \frac{b_i + a_i}{2} \text{ for } i \in \{1,...,n\}.$$

Also for $i \in \{1,...,n\}$ we have

$$\begin{aligned} &\frac{1}{V_{R_n}} \int \cdots \int_{R_n} |x_i - \overline{x_{i,G_n}}| \, dx_1...dx_n \\ &= \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_i}^{b_i} ... \int_{a_n}^{b_n} \left| x_i - \frac{b_i + a_i}{2} \right| dx_1...dx_i...dx_n \\ &= \frac{1}{b_i - a_i} \int_{a_i}^{b_i} \left| x_i - \frac{b_i + a_i}{2} \right| dx_i = \frac{1}{4}(b_i - a_i). \end{aligned}$$

If the partial derivatives $\frac{\partial f}{\partial x_i}$, $i \in \{1,...,n\}$, satisfy the conditions

$$m_i \le \frac{\partial f}{\partial x_i}(x_1,...,x_n) \le M_i, \text{ for any } (x_1,...,x_n) \in R_n \tag{3.1}$$

for some m_i and M_i, $i \in \{1, ..., n\}$, then by (2.9) we have

$$0 \leq \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(x_1, ..., x_n)\, dx_1 ... dx_n - f\left(\frac{b_1 + a_1}{2}, ..., \frac{b_n + a_n}{2}\right) \leq \frac{1}{8} \sum_{i=1}^{n} (M_i - m_i)(b_i - a_i). \tag{3.2}$$

Also, observe that

$$\begin{aligned} &\frac{1}{V_{R_n}} \int \cdots \int_{R_n} \left|x_i - \overline{x_{i,R_n}}\right|^q dx_1 ... dx_n \\ &= \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_i}^{b_i} ... \int_{a_n}^{b_n} \left|x_i - \frac{b_i + a_i}{2}\right|^q dx_1 ... dx_i ... dx_n \\ &= \frac{1}{b_i - a_i} \int_{a_i}^{b_i} \left|x_i - \frac{b_i + a_i}{2}\right|^q dx_i = \frac{(b_i - a_i)^{q+1}}{2^q (q+1)} \end{aligned}$$

and

$$\sup_{(x_1,...,x_n) \in G_n} \left|x_i - \overline{x_{i,R_n}}\right| = \sup_{(x_1,...,x_n) \in G_n} \left|x_i - \frac{b_i + a_i}{2}\right| = \frac{1}{2}(b_i - a_i)$$

and by (2.19) we get

$$0 \leq \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(x_1, ..., x_n)\, dx_1 ... dx_n - f\left(\frac{b_1 + a_1}{2}, ..., \frac{b_n + a_n}{2}\right) \leq \begin{cases} \frac{1}{4} \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{R_n,\infty} (b_i - a_i), \\ \\ \frac{1}{2^q(q+1)} \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{R_n,p} (b_i - a_i)^{q+1} \\ \text{for } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{2} \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} - \alpha_i \right\|_{R_n,1} (b_i - a_i) \end{cases} \tag{3.3}$$

for all $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$.

In particular, we have

$$0 \leq \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(x_1, ..., x_n)\, dx_1 ... dx_n - f\left(\frac{b_1 + a_1}{2}, ..., \frac{b_n + a_n}{2}\right) \leq \begin{cases} \frac{1}{4} \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{R_n,\infty} (b_i - a_i), \\ \\ \frac{1}{2^q(q+1)} \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{R_n,p} (b_i - a_i)^{q+1} \\ \text{for } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{2} \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{R_n,1} (b_i - a_i). \end{cases} \tag{3.4}$$

For $i,\, j \in \{1, ..., n\}$ with $i \neq j$ we have

$$\frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} \left| x_j - \frac{a_j + b_j}{2} \right| \left| x_i - \frac{a_i + b_i}{2} \right| dx_1 ... dx_n = \frac{1}{16}(b_j - a_j)(b_i - a_i)$$

and for $i \in \{1, ..., n\}$

$$\frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} \left| x_i - \frac{a_i + b_i}{2} \right|^2 dx_1 ... dx_n = \frac{1}{12}(b_i - a_i)^2 .$$

Now, since

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{R_n,\infty} \frac{1}{V_{R_n}} \int \cdots \int_{R_n} |x_j - \overline{x_{j,R_n}}|\, |x_i - \overline{x_{i,R_n}}|\, dx_1 ... dx_n \\
&= \sum_{i=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_{R_n,\infty} \frac{1}{V_{R_n}} \int \cdots \int_{R_n} |x_i - \overline{x_{i,R_n}}|^2\, dx_1 ... dx_n \\
&+ 2 \sum_{1 \leq i < j \leq n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{R_n,\infty} \frac{1}{V_{R_n}} \int \cdots \int_{R_n} |x_j - \overline{x_{j,R_n}}|\, |x_i - \overline{x_{i,R_n}}|\, dx_1 ... dx_n \\
&= \frac{1}{12} \sum_{i=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_{R_n,\infty} (b_i - a_i)^2 + \frac{1}{8} \sum_{1 \leq i < j \leq n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{R_n,\infty} (b_j - a_j)(b_i - a_i)
\end{aligned}$$

then by (2.18) we have

$$\begin{aligned}
(3.5) \quad 0 &\leq \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(x_1, ..., x_n)\, dx_1 ... dx_n \\
&\qquad - f\left(\frac{b_1 + a_1}{2}, ..., \frac{b_n + a_n}{2} \right) \\
&\leq \frac{1}{12} \sum_{i=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_{R_n,\infty} (b_i - a_i)^2 + \frac{1}{8} \sum_{1 \leq i < j \leq n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{R_n,\infty} (b_j - a_j)(b_i - a_i)
\end{aligned}$$

provided that $f : G_n \to \mathbb{R}$ is a twice differentiable convex function on R_n and the partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $i,\, j \in \{1, ..., n\}$, are bounded on R_n.

4. Examples for 3-Dimensional Balls

In this section we will point out some inequalities of Hermite-Hadamard's type for convex functions defined on a ball $B(C, R)$ where $C = (a, b, c) \in \mathbb{R}^3$, $R > 0$ and

$$B(C, R) := \left\{ (x, y, z) \in \mathbb{R}^3 \middle| (x - a)^2 + (y - b)^2 + (z - c)^2 \leq R^2 \right\}.$$

Let us consider the transformation $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$T_2(r, \psi, \varphi) := (r \cos\psi \cos\varphi + a, r \cos\psi \sin\varphi + b, r \sin\psi + c).$$

It is well known that the Jacobian of T_2 is $J(T_2) = r^2 \cos\psi$ and T_2 is a one-to-one mapping defined on the interval of $\mathbb{R}^3$, $[0, R] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, 2\pi]$, with values in the ball $B(C,R)$ from $\mathbb{R}^3$. Thus we have the change of variable:

$$\iiint_{B(C,R)} f(x,y,z)\,dxdydz$$
$$=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c)\,r^2\cos\psi drd\psi d\varphi.$$

We have

$$\overline{x_{B(C,R)}}=a,\ \overline{y_{B(C,R)}}=b \text{ and } \overline{z_{B(C,R)}}=c$$

and

$$\iiint_{B(C,R)}\left|x-\overline{x_{B(C,R)}}\right|dxdydz$$
$$=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}|r\cos\psi\cos\varphi+a-a|\,r^2\cos\psi drd\psi d\varphi$$
$$=\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}|\cos\varphi|\,r^3\cos^2\psi drd\psi d\varphi=\frac{R^4}{2}\pi.$$

Similarly

$$\iiint_{B(C,R)}\left|y-\overline{y_{B(C,R)}}\right|dxdydz=\iiint_{B(C,R)}\left|z-\overline{z_{B(C,R)}}\right|dxdydz=\frac{R^4}{2}\pi.$$

Let $f:B(C,R)\to\mathbb{R}$ be a differentiable convex function on $B(C,R)$. If the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ satisfy the conditions

$$m \le \frac{\partial f}{\partial x}(x,y,z)\le M \text{ for any } (x,y,z)\in B(C,R),$$
$$s \le \frac{\partial f}{\partial y}(x,y,z)\le S \text{ for any } (x,y,z)\in B(C,R),$$
$$t \le \frac{\partial f}{\partial y}(x,y,z)\le T \text{ for any } (x,y,z)\in B(C,R),$$

then we have

$$(4.1)\quad 0\le\frac{1}{\frac{4\pi R^3}{3}}\iiint_{B(C,R)} f(x,y,z)\,dxdydz-f(a,b,c)$$
$$\le\frac{3}{16}R(M+S+T-m-s-t).$$

From the inequality (2.19) we also have

$$(4.2)\quad 0\le\frac{1}{\frac{4\pi R^3}{3}}\iiint_{B(C,R)} f(x,y,z)\,dxdydz-f(a,b,c)$$
$$\le\frac{3}{8}R\left[\left\|\frac{\partial f}{\partial x}-\alpha_1\right\|_{B(C,R),\infty}+\left\|\frac{\partial f}{\partial y}-\alpha_2\right\|_{B(C,R),\infty}+\left\|\frac{\partial f}{\partial z}-\alpha_3\right\|_{B(C,R),\infty}\right]$$

for all $(\alpha_1, ..., \alpha_3) \in \mathbb{R}^3$ and, in particular,

$$0 \le \frac{1}{\frac{4\pi R^3}{3}} \iiint_{B(C,R)} f(x,y,z)\, dxdydz - f(a,b,c) \le \frac{3}{8} R \left[\left\| \frac{\partial f}{\partial x} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial z} \right\|_{B(C,R),\infty} \right]. \tag{4.3}$$

We also have

$$\begin{aligned} &\iiint_{B(C,R)} \left| z - \overline{z_{B(C,R)}} \right|^2 dxdydz \\ &= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^2 \sin^2 \psi r^2 \cos \psi drd\psi d\varphi \\ &= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^4 \sin^2 \psi \cos \psi drd\psi d\varphi = \frac{4}{15}\pi R^5 \end{aligned}$$

and, similarly

$$\iiint_{B(C,R)} \left| x - \overline{x_{B(C,R)}} \right|^2 dxdydz = \iiint_{B(C,R)} \left| y - \overline{y_{B(C,R)}} \right|^2 dxdydz = \frac{4}{15}\pi R^5.$$

Also

$$\begin{aligned} &\iiint_{B(C,R)} \left| x - \overline{x_{B(C,R)}} \right| \left| y - \overline{y_{B(C,R)}} \right| dxdydz \\ &= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \left| r \cos\psi \cos\varphi \right| \left| r \cos\psi \sin\varphi \right| r^2 \cos \psi drd\psi d\varphi \\ &= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^4 \cos^3 \psi \left| \sin\varphi \cos\varphi \right| drd\psi d\varphi = \frac{8}{15} R^5 \end{aligned}$$

and, similarly

$$\begin{aligned} &\iiint_{B(C,R)} \left| x - \overline{x_{B(C,R)}} \right| \left| z - \overline{z_{B(C,R)}} \right| dxdydz \\ &= \iiint_{B(C,R)} \left| y - \overline{y_{B(C,R)}} \right| \left| z - \overline{z_{B(C,R)}} \right| dxdydz = \frac{8}{15} R^5. \end{aligned}$$

By the inequality (2.18) we have

$$\begin{aligned} 0 &\le \frac{1}{\frac{4\pi R^3}{3}} \iiint_{B(C,R)} f(x,y,z)\, dxdydz - f(a,b,c) \\ &\le \frac{1}{5} R^2 \left(\left\| \frac{\partial^2 f}{\partial x^2} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial z^2} \right\|_{B(C,R),\infty} \right) \\ &+ \frac{4}{5\pi} R^2 \left(\left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial x \partial z} \right\|_{B(C,R),\infty} + \left\| \frac{\partial^2 f}{\partial y \partial z} \right\|_{B(C,R),\infty} \right). \end{aligned} \tag{4.4}$$

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