OSTROWSKI TYPE INTEGRAL INEQUALITIES FOR MULTIPLE INTEGRAL ON GENERAL CONVEX BODIES

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Abstract. In this paper we establish some Ostrowski type inequalities for functions of n-variables defined on closed and bounded convex bodies of the Euclidean space $\mathbb{R}^n$. Some examples for *n-hyper boxes* $R_n := [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$, $n \geq 2$ and 3-dimensional balls are also provided.

1. Introduction

In paper [1], the authors obtained among others the following results concerning the difference between the double integral on the disk and the values in the center or the path integral on the circle:

Theorem 1. *If $f : D(C, R) \to \mathbb{R}$ has continuous partial derivatives on $D(C, R)$, the disk centered in the point $C = (a, b)$ with the radius $R > 0$, and*

$$\begin{aligned} \left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} &: = \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial x} \right| < \infty, \\ \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} &: = \sup_{(x,y)\in D(C,R)} \left| \frac{\partial f(x,y)}{\partial y} \right| < \infty; \end{aligned}$$

then

$$\left| f(C) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy \right| \leq \frac{4}{3\pi} R \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right]. \tag{1.1}$$

The constant $\frac{4}{3\pi}$ is sharp.

We also have

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\,dx\,dy - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\,dl(\gamma) \right| \leq \frac{2R}{3\pi} \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right], \tag{1.2}$$

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where $\sigma(C,R)$ is the circle centered in $C=(a,b)$ with the radius $R>0$ and

$$\left|f(C)-\frac{1}{2\pi R}\int_{\sigma(C,R)}f(\gamma)\,dl(\gamma)\right| \leq \frac{2R}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty}+\left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right]. \tag{1.3}$$

In the same paper [1] the authors also established the following Ostrowski type inequality:

Theorem 2. *If f has bounded partial derivatives on $D(0,1)$, the unity disk, then*

$$\begin{aligned}\left|f(u,v)-\frac{1}{\pi}\iint_{D(0,1)}f(x,y)\,dx\,dy\right| &\leq \frac{2}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(0,1),\infty}\left(u\arcsin u+\frac{1}{3}\sqrt{1-u^2}\left(2+u^2\right)\right)\right.\\ &\quad\left.+\left\|\frac{\partial f}{\partial y}\right\|_{D(0,1),\infty}\left(v\arcsin v+\frac{1}{3}\sqrt{1-v^2}\left(2+v^2\right)\right)\right]\end{aligned} \tag{1.4}$$

for any $(u,v)\in D(0,1)$.

For other Ostrowski type integral inequalities for multiple integrals see [2]-[13].

In the following, consider G_n a closed and bounded convex subset of $\mathbb{R}^n$. Define

$$V_{G_n}:=\int\cdots\int_{G_n}dx_1...dx_n$$

the *n-volume* of G_n and $(\overline{x_{1G_n}},...,\overline{x_{nG_n}})$ the *n-centre of gravity* for G_n, where

$$\overline{x_{i,G_n}}:=\frac{1}{V_{G_n}}\int\cdots\int_{G_n}x_i dx_1...dx_n, \text{ for } i\in\{1,...,n\}.$$

Consider the function of n variables $f=f(x_1,...,x_n)$ and denote by $\frac{\partial f}{\partial x_i}$ the partial derivative with respect to the variable x_i for $i\in\{1,...,n\}$.

As examples, we can consider the *n-hyper box* $R_n:=[a_1,b_1]\times...\times[a_n,b_n]\subset\mathbb{R}^n$, for which

$$V_{R_n}=\prod_{i=1}^{n}(b_i-a_i) \text{ and } \overline{x_{i,R_n}}=\frac{b_i+a_i}{2} \text{ for } i\in\{1,...,n\}.$$

Also, if we consider the *n-hyper ball* centered in $C=(c_1,...,c_n)$ and with radius $R>0$ defined by

$$B_n(C,R):=\left\{(x_1,...,x_n)\mid \sum_{i=1}^{n}(x_i-c_i)^2\leq R^2\right\}\subset\mathbb{R}^n,$$

then the *n-volume of* $B_n(C,R)$ is

$$V_{B_n}=\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}R^n,\ n\geq 2$$

where Γ is Euler's *gamma function.*

Using explicit formulas for particular values of the gamma function at the integers and half integers gives formulas for the n-volume of the Euclidean ball as

$$V_{B_{2k}} = \frac{\pi^k}{k!} R^{2k} \text{ and } V_{B_{2k+1}} = \frac{2\,(k!)\,(4\pi)^k}{(2k+1)!} R^{2k+1},\ k \geq 1.$$

We also have $\overline{x_{i,B_n}} = c_i$ for all $i \in \{1, ..., n\}$.

2. The Main Results

We have:

Lemma 1. *If $f : G_n \to \mathbb{C}$ is differentiable on G_n, then for all $(x_1, ..., x_n)$, $(u_1, ..., u_n) \in G_n$ and $\lambda_i \in \mathbb{C}$, $i \in \{1, ..., n\}$ we have the equality*

$$\begin{aligned} f(x_1, ..., x_n) &= f(u_1, ..., u_n) + \sum_{i=1}^{n} (x_i - u_i)\,\lambda_i \\ &+ \sum_{i=1}^{n} (x_i - u_i) \int_0^1 \left(\frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \lambda_i \right) dt. \end{aligned} \tag{2.1}$$

Proof. By Taylor's multivariate theorem with integral remainder, we have

$$\begin{aligned} f(x_1, ..., x_n) &= f(u_1, ..., u_n) \\ &+ \sum_{i=1}^{n} (x_i - u_i) \int_0^1 \frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)]\, dt \end{aligned} \tag{2.2}$$

for all $(x_1, ..., x_n)$, $(u_1, ..., u_n) \in G_n$.

If $\lambda_i \in \mathbb{C}$, $i \in \{1, ..., n\}$, then

$$\begin{aligned} &(x_i - u_i) \int_0^1 \left(\frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \lambda_i \right) dt \\ &= (x_i - u_i) \int_0^1 \frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)]\, dt - (x_i - u_i)\,\lambda_i \end{aligned}$$

and by (2.2) we get the desired result (2.1). □

Suppose that $G_n \subset \mathbb{R}^n$ is a convex subset in $\mathbb{R}^n$. Now, for ϕ, $\Phi \in \mathbb{C}$, define the sets of complex-valued functions

$$\begin{aligned} &\bar{U}_{G_n}(\phi, \Phi) \\ &:= \Big\{ f : G_n \to \mathbb{C} |\, \operatorname{Re}\left[(\Phi - f(x_1, ..., x_n)) \left(\overline{f(x_1, ..., x_n)} - \overline{\phi} \right) \right] \geq 0 \\ &\qquad\qquad \text{for each } \ (x_1, ..., x_n) \in G_n \} \end{aligned}$$

and

$$\begin{aligned} &\bar{\Delta}_{G_n}(\phi, \Phi) \\ &:= \left\{ f : G_n \to \mathbb{C} |\ \left| f(x_1, ..., x_n) - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \text{ for each } \ (x_1, ..., x_n) \in G_n \right\}. \end{aligned}$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_{G_n}(\phi,\Phi)$ and $\bar{\Delta}_{G_n}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{G_n}(\phi,\Phi) = \bar{\Delta}_{G_n}(\phi,\Phi). \tag{2.3}$$

Proof. We observe that for any $w \in \mathbb{C}$ we have the equivalence

$$\left| w - \frac{\phi+\Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi - w)\left(\overline{w} - \overline{\phi}\right)\right] \geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^2 - \left| w - \frac{\phi+\Phi}{2}\right|^2 = \operatorname{Re}\left[(\Phi - w)\left(\overline{w} - \overline{\phi}\right)\right]$$

that holds for any $w \in \mathbb{C}$.

The equality (2.3) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\begin{aligned} &\bar{U}_{G_n}(\phi,\Phi) \\ &= \{ f : G_n \to \mathbb{C} \mid \left(\operatorname{Re}\Phi - \operatorname{Re} f(x_1,...,x_n)\right)\left(\operatorname{Re} f(x_1,...,x_n) - \operatorname{Re}\phi\right) \\ &\quad + \left(\operatorname{Im}\Phi - \operatorname{Im} f(x_1,...,x_n)\right)\left(\operatorname{Im} f(x_1,...,x_n) - \operatorname{Im}\phi\right) \geq 0 \\ &\qquad \textit{for each } (x_1,...,x_n) \in G_n \}. \end{aligned} \tag{2.4}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{G_n}(\phi,\Phi) := \{ f : G_n \to \mathbb{C} \mid\ & \operatorname{Re}(\Phi) \geq \operatorname{Re} f(x_1,...,x_n) \geq \operatorname{Re}(\phi) \\ & \text{and } \operatorname{Im}(\Phi) \geq \operatorname{Im} f(x_1,...,x_n) \geq \operatorname{Im}(\phi) \text{ for each } (x_1,...,x_n) \in G_n \}. \end{aligned} \tag{2.5}$$

One can easily observe that $\bar{S}_{G_n}(\phi,\Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_{G_n}(\phi,\Phi) \subseteq \bar{U}_{G_n}(\phi,\Phi). \tag{2.6}$$

We have:

Theorem 3. *Assume that $f : G_n \to \mathbb{C}$ is differentiable on G_n and $(u_1,...,u_n) \in G_n$. Let $(\phi_i,\Phi_i) \in \mathbb{C}$, $i \in \{1,...,n\}$ and assume that $\frac{\partial f}{\partial x_i} \in \bar{\Delta}_{G_n}(\phi_i,\Phi_i)$, $i \in \{1,...,n\}$, then*

$$\begin{aligned} &\left| \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1,...,x_n)\, dx_1...dx_n \right. \\ &\qquad \left. - f(u_1,...,u_n) - \sum_{i=1}^{n} \left(\overline{x_{i,G_n}} - u_i\right) \frac{\phi_i + \Phi_i}{2} \right| \\ &\qquad\qquad \leq \frac{1}{2} \sum_{i=1}^{n} |\Phi_i - \phi_i| \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - u_i|\, dx_1...dx_n. \end{aligned} \tag{2.7}$$

In particular,

$$\left|f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right)-\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f\left(x_1,...,x_n\right)dx_1...dx_n\right| \\ \leq\frac{1}{2}\sum_{i=1}^{n}\left|\Phi_i-\phi_i\right|\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|dx_1...dx_n. \tag{2.8}$$

Proof. From Lemma 1 we have

$$f\left(x_1,...,x_n\right)=f\left(u_1,...,u_n\right)+\sum_{i=1}^{n}\left(x_i-u_i\right)\frac{\phi_i+\Phi_i}{2} \\ +\sum_{i=1}^{n}\left(x_i-u_i\right)\int_0^1\left(\frac{\partial f}{\partial x_i}\left[t\left(x_1,...,x_n\right)+\left(1-t\right)\left(u_1,...,u_n\right)\right]-\frac{\phi_i+\Phi_i}{2}\right)dt \tag{2.9}$$

for all $\left(x_1,...,x_n\right),\left(u_1,...,u_n\right)\in G_n$.

By taking the integral mean $\frac{1}{V_{G_n}}\int\cdots\int_{G_n}$ over $\left(x_1,...,x_n\right)$ we get

$$\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f\left(x_1,...,x_n\right)dx_1...dx_n \\ =f\left(u_1,...,u_n\right)+\sum_{i=1}^{n}\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}x_idx_1...dx_n-u_i\right)\frac{\phi_i+\Phi_i}{2} \\ +\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}\left(x_i-u_i\right) \\ \times\left(\int_0^1\left(\frac{\partial f}{\partial x_i}\left[t\left(x_1,...,x_n\right)+\left(1-t\right)\left(u_1,...,u_n\right)\right]-\frac{\phi_i+\Phi_i}{2}\right)dt\right)dx_1...dx_n \tag{2.10}$$

for all $\left(u_1,...,u_n\right)\in G_n$.

By using the equality (2.10) we get

$$\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f\left(x_1,...,x_n\right)dx_1...dx_n-f\left(u_1,...,u_n\right)\right. \\ \left.-\sum_{i=1}^{n}\left(\frac{1}{V_{G_n}}\int\cdots\int_{G_n}x_idx_1...dx_n-u_i\right)\frac{\phi_i+\Phi_i}{2}\right| \\ \leq\frac{1}{V_{G_n}}\left|\int\cdots\int_{G_n}\sum_{i=1}^{n}\left(x_i-u_i\right)\right. \\ \left.\times\left[\int_0^1\left(\frac{\partial f}{\partial x_i}\left[t\left(x_1,...,x_n\right)+\left(1-t\right)\left(u_1,...,u_n\right)\right]-\frac{\phi_i+\Phi_i}{2}\right)dt\right]dx_1...dx_n\right| \tag{2.11}$$

$$\leq \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left| \sum_{i=1}^{n} (x_i - u_i) \right.$$
$$\left. \times \left[\int_0^1 \left(\frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \frac{\phi_i + \Phi_i}{2} \right) dt \right] dx_1 ... dx_n \right|$$
$$\leq \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \sum_{i=1}^{n} |x_i - u_i|$$
$$\times \left[\int_0^1 \left| \frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \frac{\phi_i + \Phi_i}{2} \right| dt \right] dx_1 ... dx_n =: I.$$

By the fact that $\frac{\partial f}{\partial x_i} \in \bar{\Delta}_{G_n}(\phi_i, \Phi_i)$, $i \in \{1, ..., n\}$, it follows that

$$\int_0^1 \left| \frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] - \frac{\phi_i + \Phi_i}{2} \right| dt \leq \frac{1}{2} |\Phi_i - \phi_i|.$$

Therefore

$$I \leq \frac{1}{2} \frac{1}{V_{G_n}} \sum_{i=1}^{n} |\Phi_i - \phi_i| \int \cdots \int_{G_n} |x_i - u_i| \, dx_1 ... dx_n$$

and by (2.11) we obtain the desired result (2.7). □

We also have:

Theorem 4. *Assume that $f : G_n \to \mathbb{C}$ is differentiable on G_n and $(u_1, ..., u_n) \in G_n$. Then*

$$\left| f(u_1, ..., u_n) - \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n) \, dx_1 ... dx_n \right| \tag{2.12}$$
$$\leq \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - u_i|$$
$$\times \left(\int_0^1 \left| \frac{\partial f}{\partial x_i} [t(x_1, ..., x_n) + (1-t)(u_1, ..., u_n)] \right| dt \right) dx_1 ... dx_n$$
$$\leq \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{G, \infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - u_i| \, dx_1 ... dx_n,$$

provided

$$\left\| \frac{\partial f}{\partial x_i} \right\|_{G_n, \infty} := \sup_{(x_1, ..., x_n) \in G_n} \left| \frac{\partial f}{\partial x_i} (x_1, ..., x_n) \right| < \infty.$$

In particular,

$$\left| f\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right) - \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n) \, dx_1 ... dx_n \right| \tag{2.13}$$
$$\leq \sum_{i=1}^{n} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left| x_i - \overline{x_{i,G_n}} \right|$$
$$\times \left(\int_0^1 \left| \frac{\partial f}{\partial x_i} \left[t(x_1, ..., x_n) + (1-t)\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right) \right] \right| dt \right) dx_1 ... dx_n$$
$$\leq \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{G_n, \infty} \frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left| x_i - \overline{x_{i,G_n}} \right| dx_1 ... dx_n.$$

Proof. We have from (2.1) that

$$f(x_1,...,x_n) = f(u_1,...,u_n) + \sum_{i=1}^{n}(x_i-u_i)\int_0^1 \frac{\partial f}{\partial x_i}[t(x_1,...,x_n)+(1-t)(u_1,...,u_n)]\,dt \tag{2.14}$$

for all $(x_1,...,x_n)$, $(u_1,...,u_n) \in G_n$.

By taking the integral mean $\frac{1}{V_{G_n}}\int\cdots\int_{G_n}$ over $(x_1,...,x_n)$ in (2.14) we get the following identity of interest

$$\begin{aligned}\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n &= f(u_1,...,u_n) \\ &+ \frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}(x_i-u_i) \\ &\times\left(\int_0^1 \frac{\partial f}{\partial x_i}[t(x_1,...,x_n)+(1-t)(u_1,...,u_n)]\,dt\right)dx_1...dx_n\end{aligned} \tag{2.15}$$

for all $(u_1,...,u_n) \in G_n$.

From (2.15) we get

$$\begin{aligned}&\left|\frac{1}{V_{G_n}}\int\cdots\int_{G_n} f(x_1,...,x_n)\,dx_1...dx_n - f(u_1,...,u_n)\right| \\ &\leq \frac{1}{V_{G_n}}\left|\int\cdots\int_{G_n}\sum_{i=1}^{n}(x_i-u_i)\right. \\ &\left.\times\left(\int_0^1 \frac{\partial f}{\partial x_i}[t(x_1,...,x_n)+(1-t)(u_1,...,u_n)]\,dt\right)dx_1...dx_n\right| \\ &\leq \frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|\sum_{i=1}^{n}(x_i-u_i)\right. \\ &\left.\times\left(\int_0^1 \frac{\partial f}{\partial x_i}[t(x_1,...,x_n)+(1-t)(u_1,...,u_n)]\,dt\right)\right|dx_1...dx_n \\ &\leq \frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}|x_i-u_i| \\ &\times\left(\int_0^1 \left|\frac{\partial f}{\partial x_i}[t(x_1,...,x_n)+(1-t)(u_1,...,u_n)]\right|dt\right)dx_1...dx_n =: J,\end{aligned} \tag{2.16}$$

which proves the first part of (2.12).

We also have that

$$\left|\frac{\partial f}{\partial x_i}[t(x_1,...,x_n)+(1-t)(u_1,...,u_n)]\right| \leq \left\|\frac{\partial f}{\partial x_i}\right\|_{G,\infty},\ i\in\{1,...,n\}$$

for all $(x_1,...,x_n)$, $(u_1,...,u_n) \in G_n$ and for all $t\in[0,1]$.

Therefore

$$J \leq \frac{1}{V_{G_n}}\int\cdots\int_{G_n}\sum_{i=1}^{n}\left|x_i-\overline{x_{i,G_n}}\right|\left\|\frac{\partial f}{\partial x_i}\right\|_{G_n,\infty},$$

which proves the last part of (2.12). □

Remark 1. *If we denote*

$$B_i(u_1,...,u_n) := \frac{1}{V_{G_n}} \int \cdots \int_{G_n} |x_i - u_i| \times \left(\int_0^1 \left| \frac{\partial f}{\partial x_i} [t(x_1,...,x_n) + (1-t)(u_1,...,u_n)] \right| dt \right) dx_1...dx_n$$

for $i \in \{1,...,n\}$, then by Hölder's integral inequalities for the multiple integral we have

$$B_i(u_1,...,u_n) \le \frac{1}{V_{G_n}} \begin{cases} \sup_{(x_1,...,x_n)\in G_n} |x_i - u_i| \\ \times \int \cdots \int_{G_n} \left(\int_0^1 \left| \frac{\partial f}{\partial x_i} [t(x_1,...,x_n) + (1-t)(u_1,...,u_n)] \right| dt \right) dx_1...dx_n, \\ \\ \left(\int \cdots \int_{G_n} |x_i - u_i|^p dx_1...dx_n \right)^{1/p} \\ \times \left[\int \cdots \int_{G_n} \left(\int_0^1 \left| \frac{\partial f}{\partial x_i} [t(x_1,...,x_n) + (1-t)(u_1,...,u_n)] \right| dt \right)^q dx_1...dx_n \right]^{1/q} \\ \text{for } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \sup_{(x_1,...,x_n)\in G} \left(\int_0^1 \left| \frac{\partial f}{\partial x_i} [t(x_1,...,x_n) + (1-t)(u_1,...,u_n)] \right| dt \right) \\ \times \int \cdots \int_{G_n} |x_i - u_i| dx_1...dx_n \end{cases}$$

$$=: M_i(u_1,...,u_n)$$

for all $(u_1,...,u_n) \in G_n$.

Therefore, by the first inequality in (2.12) we obtain

$$\left| f(u_1,...,u_n) - \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1,...,x_n) dx_1...dx_n \right| \le \sum_{i=1}^n M_i(u_1,...,u_n) \tag{2.17}$$

for all $(u_1,...,u_n) \in G_n$.

In particular, we have

$$\left| f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right) - \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1,...,x_n) dx_1...dx_n \right| \le \sum_{i=1}^n M_i\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right). \tag{2.18}$$

When the partial derivatives are convex in absolute value, we have:

Corollary 2. *With the assumptions of Theorem 4 and if* $\left|\frac{\partial f}{\partial x_i}\right|$ *are convex on* G_n*, for all for* $i \in \{1, ..., n\}$ *, then we have*

$$\left|f(u_1, ..., u_n) - \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n)\, dx_1 ... dx_n\right| \tag{2.19}$$

$$\leq \frac{1}{2}\frac{1}{V_{G_n}} \begin{cases} \sum_{i=1}^{n} \sup_{(x_1,...,x_n)\in G_n} |x_i - u_i| \\ \times \int \cdots \int_{G_n} \left[\left|\frac{\partial f}{\partial x_i}\right|(x_1, ..., x_n) + \left|\frac{\partial f}{\partial x_i}\right|(u_1, ..., u_n)\right] dx_1 ... dx_n, \\ \\ \sum_{i=1}^{n} \left(\int \cdots \int_{G_n} |x_i - u_i|^p\, dx_1 ... dx_n\right)^{1/p} \\ \times \left[\int \cdots \int_{G_n} \left[\left|\frac{\partial f}{\partial x_i}\right|(x_1, ..., x_n) + \left|\frac{\partial f}{\partial x_i}\right|(u_1, ..., u_n)\right]^q dx_1 ... dx_n\right]^{1/q} \\ \text{for } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \sum_{i=1}^{n} \sup_{(x_1,...,x_n)\in G_n} \left(\left[\left|\frac{\partial f}{\partial x_i}\right|(x_1, ..., x_n) + \left|\frac{\partial f}{\partial x_i}\right|(u_1, ..., u_n)\right]\right) \\ \times \int \cdots \int_{G_n} |x_i - u_i|\, dx_1 ... dx_n \end{cases}$$

for all $(u_1, ..., u_n) \in G_n$.

In particular,

$$\left|f\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right) - \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n)\, dx_1 ... dx_n\right| \tag{2.20}$$

$$\leq \frac{1}{2}\frac{1}{V_{G_n}} \begin{cases} \sum_{i=1}^{n} \sup_{(x_1,...,x_n)\in G_n} \left|x_i - \overline{x_{i,G_n}}\right| \\ \times \int \cdots \int_{G_n} \left[\left|\frac{\partial f}{\partial x_i}\right|(x_1, ..., x_n) + \left|\frac{\partial f}{\partial x_i}\right|\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right)\right] dx_1 ... dx_n, \\ \\ \sum_{i=1}^{n} \left(\int \cdots \int_{G_n} \left|x_i - \overline{x_{i,G_n}}\right|^p dx_1 ... dx_n\right)^{1/p} \\ \times \left[\int \cdots \int_{G_n} \left[\left|\frac{\partial f}{\partial x_i}\right|(x_1, ..., x_n) + \left|\frac{\partial f}{\partial x_i}\right|\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right)\right]^q dx_1 ... dx_n\right]^{1/q} \\ \text{for } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \sum_{i=1}^{n} \sup_{(x_1,...,x_n)\in G_n} \left(\left[\left|\frac{\partial f}{\partial x_i}\right|(x_1, ..., x_n) + \left|\frac{\partial f}{\partial x_i}\right|\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right)\right]\right) \\ \times \int \cdots \int_{G_n} \left|x_i - \overline{x_{i,G_n}}\right| dx_1 ... dx_n. \end{cases}$$

Remark 2. *From (2.20) we have*

$$\left|f\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right) - \frac{1}{V_{G_n}} \int \cdots \int_{G_n} f(x_1, ..., x_n)\, dx_1 ... dx_n\right| \tag{2.21}$$

$$\leq \frac{1}{2} \sum_{i=1}^{n} \sup_{(x_1,...,x_n)\in G_n} \left|x_i - \overline{x_{i,G_n}}\right|$$

$$\times \left[\frac{1}{V_{G_n}} \int \cdots \int_{G_n} \left|\frac{\partial f}{\partial x_i}\right|(x_1, ..., x_n)\, dx_1 ... dx_n + \left|\frac{\partial f}{\partial x_i}\right|\left(\overline{x_{1,G_n}}, ..., \overline{x_{n,G_n}}\right)\right]$$

and

$$\left|f\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right)-\frac{1}{V_{G_n}}\int\cdots\int_{G_n}f\left(x_1,...,x_n\right)dx_1...dx_n\right| \tag{2.22}$$
$$\leq\frac{1}{2}\sum_{i=1}^{n}\left(\sup_{(x_1,...,x_n)\in G_n}\left|\frac{\partial f}{\partial x_i}\right|(x_1,...,x_n)+\left|\frac{\partial f}{\partial x_i}\right|\left(\overline{x_{1,G_n}},...,\overline{x_{n,G_n}}\right)\right)$$
$$\times\frac{1}{V_{G_n}}\int\cdots\int_{G_n}\left|x_i-\overline{x_{i,G_n}}\right|dx_1...dx_n.$$

3. Examples for n-Dimensional Boxes

We can consider the *n-hyper box* $R_n:=[a_1,b_1]\times...\times[a_n,b_n]\subset\mathbb{R}^n$, $n\geq 2$ and assume that $f:R_n\to\mathbb{R}$ is a differentiable convex function on R_n. We have

$$V_{R_n}=\prod_{i=1}^{n}(b_i-a_i)\ \text{ and }\ \overline{x_{i,R_n}}=\frac{b_i+a_i}{2}\ \text{ for } i\in\{1,...,n\}.$$

Also for $i\in\{1,...,n\}$ we have

$$\begin{aligned}&\frac{1}{V_{R_n}}\int\cdots\int_{R_n}|x_i-u_i|\,dx_1...dx_n\\&=\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_i}^{b_i}...\int_{a_n}^{b_n}|x_i-u_i|\,dx_1...dx_i...dx_n\\&=\frac{1}{b_i-a_i}\int_{a_i}^{b_i}|x_i-u_i|\,dx_i=\frac{(u_i-a_i)^2+(b_i-u_i)^2}{2(b_i-a_i)}\\&=\left[\frac{1}{4}+\left(\frac{u_i-\frac{a_i+b_i}{2}}{b_i-a_i}\right)^2\right](b_i-a_i).\end{aligned}$$

Assume that $f:R_n\to\mathbb{C}$ is differentiable on R_n and $(u_1,...,u_n)\in R_n$. Let $(\phi_i,\Phi_i)\in\mathbb{C}$, $i\in\{1,...,n\}$ and assume that $\frac{\partial f}{\partial x_i}\in\bar{\Delta}_{R_n}(\phi_i,\Phi_i)$, $i\in\{1,...,n\}$, then by Theorem 3 we have

$$\left|\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}f(x_1,...,x_n)\,dx_1...dx_n\right. \tag{3.1}$$
$$\left.-f(u_1,...,u_n)-\sum_{i=1}^{n}\left(\frac{b_i+a_i}{2}-u_i\right)\frac{\phi_i+\Phi_i}{2}\right|$$
$$\leq\frac{1}{2}\sum_{i=1}^{n}|\Phi_i-\phi_i|\left[\frac{1}{4}+\left(\frac{u_i-\frac{a_i+b_i}{2}}{b_i-a_i}\right)^2\right](b_i-a_i).$$

In particular,

$$\begin{aligned}(3.2)\quad & \left| f\left(\frac{a_1+b_1}{2},...,\frac{a_n+b_n}{2}\right)\right. \\ & \left. -\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n} f(x_1,...,x_n)\,dx_1...dx_n\right| \\ & \leq \frac{1}{8}\sum_{i=1}^{n}|\Phi_i-\phi_i|(b_i-a_i).\end{aligned}$$

Assume that $f: R_n \to \mathbb{C}$ is differentiable on R_n and $(u_1,...,u_n) \in R_n$. Then by Theorem 4 we obtain the Ostrowski type inequality

$$\begin{aligned}(3.3)\quad & \left| f(u_1,...,u_n) - \frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n} f(x_1,...,x_n)\,dx_1...dx_n\right| \\ & \leq \sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|_{G,\infty}\left[\frac{1}{4}+\left(\frac{u_i-\frac{a_i+b_i}{2}}{b_i-a_i}\right)^2\right](b_i-a_i),\end{aligned}$$

provided

$$\left\|\frac{\partial f}{\partial x_i}\right\|_{R_n,\infty} := \sup_{(x_1,...,x_n)\in R_n}\left|\frac{\partial f}{\partial x_i}(x_1,...,x_n)\right| < \infty.$$

In particular,

$$\begin{aligned}(3.4)\quad & \left| f\left(\frac{a_1+b_1}{2},...,\frac{a_n+b_n}{2}\right)\right. \\ & \left. -\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n} f(x_1,...,x_n)\,dx_1...dx_n\right| \\ & \leq \frac{1}{4}\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|_{R_n,\infty}(b_i-a_i).\end{aligned}$$

Also, if $\left|\frac{\partial f}{\partial x_i}\right|$ are convex on R_n, for all for $i \in \{1,...,n\}$, then by Corollary 2 we have

$$\begin{aligned}(3.5)\quad & \left| f(u_1,...,u_n) - \frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n} f(x_1,...,x_n)\,dx_1...dx_n\right| \\ & \leq \frac{1}{2}\sum_{i=1}^{n}\left(\left\|\frac{\partial f}{\partial x_i}\right\|_{R_n,\infty}+\left|\frac{\partial f}{\partial x_i}\right|(u_1,...,u_n)\right)\left[\frac{1}{4}+\left(\frac{u_i-\frac{a_i+b_i}{2}}{b_i-a_i}\right)^2\right](b_i-a_i)\end{aligned}$$

for all $(u_1,...,u_n) \in R_n$.

In particular, we obtain

$$\left|f\left(\frac{a_1+b_1}{2},...,\frac{a_n+b_n}{2}\right)\right. \\ \left.-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n}f(x_1,...,x_n)\,dx_1...dx_n\right| \\ \leq \frac{1}{8}\sum_{i=1}^{n}\left(\left\|\frac{\partial f}{\partial x_i}\right\|_{R_n,\infty}+\left|\frac{\partial f}{\partial x_i}\right|\left(\frac{a_1+b_1}{2},...,\frac{a_n+b_n}{2}\right)\right)(b_i-a_i). \tag{3.6}$$

4. Examples for 3-Dimensional Balls

In this section we will point out some inequalities of Hermite-Hadamard's type for convex functions defined on a ball $B(C,R)$ where $C=(a,b,c)\in\mathbb{R}^3$, $R>0$ and

$$B(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\middle|\,(x-a)^2+(y-b)^2+(z-c)^2\leq R^2\right\}.$$

Let us consider the transformation $T_2:\mathbb{R}^3\to\mathbb{R}^3$ given by:

$$T_2(r,\psi,\varphi):=(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c).$$

It is well known that the Jacobian of T_2 is $J(T_2)=r^2\cos\psi$ and T_2 is a one-to-one mapping defined on the interval of $\mathbb{R}^3$, $[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$, with values in the ball $B(C,R)$ from $\mathbb{R}^3$. Thus we have the change of variable:

$$\iiint_{B(C,R)}f(x,y,z)\,dxdydz \\ =\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}f(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c)\,r^2\cos\psi drd\psi d\varphi.$$

We have

$$\overline{x_{B(C,R)}}=a,\ \overline{y_{B(C,R)}}=b\text{ and }\overline{z_{B(C,R)}}=c$$

and

$$\iiint_{B(C,R)}\left|x-\overline{x_{B(C,R)}}\right|dxdydz \\ =\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}|r\cos\psi\cos\varphi+a-a|\,r^2\cos\psi drd\psi d\varphi \\ =\int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}|\cos\varphi|\,r^3\cos^2\psi drd\psi d\varphi=\frac{R^4}{2}\pi.$$

Similarly

$$\iiint_{B(C,R)}\left|y-\overline{y_{B(C,R)}}\right|dxdydz=\iiint_{B(C,R)}\left|z-\overline{z_{B(C,R)}}\right|dxdydz=\frac{R^4}{2}\pi.$$

Let $f:B(C,R)\to\mathbb{C}$ be a differentiable function on $B(C,R)$. If the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ satisfy the conditions $\frac{\partial f}{\partial x}\in\bar{\Delta}_{B(C,R)}(\phi_1,\Phi_1)$, $\frac{\partial f}{\partial y}\in\bar{\Delta}_{B(C,R)}(\phi_2,\Phi_2)$ and $\frac{\partial f}{\partial z}\in\bar{\Delta}_{B(C,R)}(\phi_3,\Phi_3)$ then by (2.8) we have

$$\left|f(a,b,c)-\frac{1}{\frac{4\pi R^3}{3}}\iiint_{B(C,R)}f(x,y,z)\,dxdydz\right|\leq\frac{3}{16}R\sum_{i=1}^{3}|\Phi_i-\phi_i|. \tag{4.1}$$

Assume that $f : B(C,R) \to \mathbb{C}$ is differentiable on $B(C,R)$. Then by (2.13) we get

$$\left| f(a,b,c) - \frac{1}{\frac{4\pi R^3}{3}} \iiint_{B(C,R)} f(x,y,z)\, dxdydz \right| \leq \frac{3}{8} R \left[\left\| \frac{\partial f}{\partial x} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{B(C,R),\infty} + \left\| \frac{\partial f}{\partial z} \right\|_{B(C,R),\infty} \right], \tag{4.2}$$

provided

$$\left\| \frac{\partial f}{\partial x} \right\|_{B(C,R),\infty}, \ \left\| \frac{\partial f}{\partial y} \right\|_{B(C,R),\infty}, \ \left\| \frac{\partial f}{\partial z} \right\|_{B(C,R),\infty} < \infty.$$

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